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Optimal Assignment of Soldiers to Service-Units: A Supply and Demand Framework*

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Abstract This study proposes a method for improving the quality of the soldier-to-service-unit assignment of mandatory service soldiers in Korea by designing a proper pecuniary incentive. Based on soldiers' preferences (supply) and the demand of units, we calculate the optimal allocation and compensating wage differentials using the Hungarian algorithm, a commonly used method for solving linear programming problems. The calculated outcome of the optimal allocation and compensating wage differentials achieves Pareto efficiency, which maximizes the sum of utility from military service while minimizing wage expenditure. In addition, the presented compensatory wage differentials are market-clearing prices based on a general equilibrium approach, which is flexible and responds to the demand and supply of the military internal labor market.

Keywords Compensating wage differentials, linear programming, optimal assignment problem, fair allocation, internal labor market

JEL Classification J31, C60

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1. INTRODUCTION

Due to the national security threat imposed by its neighbors, Korea is one of the few countries in the world maintaining a mandatory military service regime. As a part of the conscription system, salaries paid to the soldiers had remained well below the market minimum wage for a long time. However, in response to the rising living standards in the overall economy, there has been an ongoing drive from the government to raise wages paid to soldiers, along with improving welfare and workplace condition of the mandatory service population. As a result, monthly salary earned by a typical soldier (nearing the end of mandatory service) rose sharply from about 200 USD in 2017 to 500 USD in 2022. Furthermore, according to the long-term budget allocation plan announced by the Korean Ministry of Defense, monthly wage paid to a soldier nearing the end of the military service is expected to reach over 2 million KRW (1700 in USD, this includes remuneration in a savings account) in 2025, which is more than a two-fold increase from the wage in 2022.

Table 1: Projected soldiers' wages (1000 KRW/month) by rank, years 2022 to 2025

Rank\Year	2022	2023	2024	2025
Private	510	600	720	860
Private First-Class	550	680	800	960
Corporal	610	800	1000	1200
Sergeant	680	1000	1250	1500

However, the current wage system for mandatory service soldiers exhibits very little or almost no compensating wage differentials. For instance, the current rule dictates extra pay to soldiers in 'unusual and difficult conditions' ranging from 18000 KRW to 118800 KRW a month (roughly 15 USD to 90 USD per month). This pay scheme is actually applicable to mandatory service soldiers, as they serve in various duties such as demolition duty, parachute duty, handling of high-voltage electricity, and operation of heavy equipment. Special location pay includes payments to service members in frontline DMZ (De-Militarized Zones), isolated islands, regions subject to hostile enemy actions, coastal guard areas, naval vessels, and posts in high elevation. The base special location pay for the service members varies from 15 to 20 USD per month, and may be pro-

[0] Source: Ministry of Defense long-term budget allocation plan (2022), figures exclude remunerations through savings account.

rated by 10 to 35 USD per month. (Refer to the Appendix for details.)

Despite the heated debate over the legitimacy of the steep wage increase and its possible role in distorting the incentive system of the economy[1], there is relatively little to almost no discussion regarding the appropriate compensation for variations in service conditions across different unit assignments. Despite the significant impact of this factor on the welfare of soldiers during their mandatory service, the current system heavily relies on the random assignment, primarily based on ex-ante fairness considerations. This reliance on randomness introduces distorted incentives, with a few individuals who have more information attempting to influence assignments through lobbying and other efforts, thereby complicating the equity of mandatory service.

There are many soldiers fulfilling their obligations in challenging and perilous assignments, including frontline posts (such as Guard Posts and General Out Posts situated in the De-Militarized Zone) and remote locations (such as island bases). According to a survey by Shin *et al.* (2021), these service units exhibit significantly worse living and working conditions compared to other units, and the assignment to these units/branches is strictly less preferred among the mandatory service soldiers. With the anticipated population decline in the near future, the challenges of short-staffing and the diminished quality of draftees in these units are expected to only exacerbate over time.

This paper outlines a method for aggregating soldiers' preferences for different types of units, resulting in a compensating wage gap and a more efficient assignment of soldiers to service units. We believe that this scheme is worth considering given the record-low fertility rate of Korea, and the projected steep decrease in the conscriptable population pool in the near future. First, the basic principle of supply-and-demand in economics tells us that, by introducing the market forces, one can expect a more efficient assignment by attracting more people into the less preferred but socially desirable assignment. From the policymaker point of view, this can help alleviate chronic short-staffing problem as well as the difficulty in commanding and managing such units. Second, we can achieve a more equitable and fair distribution of service based on the preference information submitted by the soldiers. The current assignment system relies heavily on the random allocation, which is fair ex-ante, but ensues a large discrepancy in the ex-post outcome. In the context of matching theory, the random allocation creates much justified envy, which is not a desirable feature for a good matching mechanism. In fact, with the mandatory service population expected to rapidly diminish over time, relying on the sacrifices of a few soldiers

[1] For a reference, see Min (2022)

who happened to draw unlucky assignments is neither desirable nor sustainable.

In detail, we applied the assignment game proposed in Shapley and Shubik (1971) to the context of deriving an appropriate compensating wage differential in the military internal labor market. Shapley and Shubik (1971) formulated the assignment game as a linear programming problem. We can employ Hungarian algorithm, which is a classic method for solving the allocation problem of minimizing the sum of costs subject to one-to-one matching constraint. The algorithm returns both the optimal assignment (a profile of matches between soldiers and units), and a vector of compensating wage differentials (posted by units) that incentivizes soldiers to self-select into the assignment. That is, the assignment and the prices (compensating wages) comprise a competitive market equilibrium. Accordingly, the vector of utilities that agents get corresponds to the core allocation in a cooperative game – no coaliton of players (a subset of units and soldiers) can obtain a higher utility than the vector. The allocation (assignment and prices) is also Pareto efficient and is in the Pareto frontier.

As the average salary for soldiers is projected to double in the coming years, our policy proposal suggests redirecting a portion of the planned universal wage increase into elevating the compensating wage gap between different service units. This adjustment will make the assignment more resemble the competitive labor market outcome, thereby enhancing efficiency in equilibrium. We first show that it is possible to implement the optimal service-unit and soldier assignment (dictated by the central planner) by introducing an appropriate level of compensating wage differentials. The problem of the central planner is known as the 'assignment problem', and the economics literature has long emphasized that a market clearing wage can support this assignment. Second, we solve for and simulate the optimal assignment/compensating wage level pair using the linear programming method. Our approach fully considers the supply and demand constraints of the service unit-soldier assignment problem and can account for the soldiers' correlated preferences over the service units. Through a series of simulations, we highlight the difference between our result and the work by Shin *et al.* (2021), which uses survey data to calculate the compensating wage differentials. Lastly, we consider extending our baseline proposal to a reduced-form second best assignment, which only requires information on the soldiers' ordinal preferences. The algorithm is called random serial dictatorship, and is widely used as a means for implementing a fair and incentive-compatible assignment. We discuss the efficiency and fairness considerations of our algorithm and conclude.

2. LITERATURE REVIEW

2.1. THE DATA: DISCRETE CHOICE EXPERIMENT ON SOLDIERS' PREFERENCES (SHIN ET AL. (2021))

The main data motivating simulation exercises in this paper come from the survey of soldiers' service conditions in the report by Shin *et al.* (2021). The report categorized military service units into four segments along two dimensions: the location of the unit (front or rear), and the main duty (combat or non-combat). The survey of active duty soldiers reveals that, taking into account various factors, such as the location, task, living condition, peers and non-pecuniary benefits, the soldiers on average rank rear-combat units the best, rear-noncombat units the second, frontline-noncombat units the third, and frontline-combat units the worst. The frontline units include examples of less preferred units mentioned in Section 1, such as: frontline posts (GP-GOP), DMZ (De-Militarized Zone), coastal or riverine defense, and islands located near the Northern Limit Line (NLL). Services in these units are shown to be significantly more challenging, and the working conditions worse than average so that the soldiers find the assignment in such units strictly less desirable.

The report conducted a discrete selection experiment which measures appropriate compensation level if the soldiers were to accept services in the less preferred units. In this experiment, the authors offered subjects the choices of hypothetical unit assignment, along with compensations such as: additional monthly leave (0 to 2 days per month), additional monthly salary (0 to 40,000 KRW per month), and early promotion (0 to 2 months). The authors estimate that serving in frontline units leads to a reduction in utility ranging from approximately 0.25 to 0.4, while additional 10,000 KRW monthly income corresponds to a utility increase of 0.055. In conclusion, they calculate that the assignment to frontline units should be compensated with a monthly income ranging from 45 to 75 thousand KRW.

The trade-off relationship between the frontline assignment and wage compensation, as proposed by Shin *et al.* (2021), is meaningful in that it provides specific estimates of compensatory wages for some units. However, the study is limited in that the figures only reflect the average compensatory wages based on the preference levels of all discrete-choice subjects. In order to calculate a budget-efficient compensation wage scheme, it is necessary we calculate the reservation wage of the marginal soldier who otherwise would not serve in frontline units. This requires taking into account both the supply side (soldiers' preference) and the demand side (service units) constraints and finding the market clearing wage.

2.2. THEORY: ECONOMICS OF ASSIGNMENT AND MATCHING

A centralized organization, such as a military or a firm, tries to allocate its human resources efficiently by solving an optimal assignment problem. The assignment problem is a special case of the mathematical optimization problem, which can be formulated using linear programming. When the matching is one-to-one, one can employ the Hungarian algorithm to solve the assignment problem given by linear programming.

Matching theory is a branch of economics that studies the efficient function and the economic incentives within a matching market. An early study by Gale and Shapley (1962) has suggested the Deferred-Acceptance algorithm which solves for a stable matching in one-to-one bipartite matching problems (marriage market). Later studies such as Adachi (2000) have shown that if there are multiple stable matchings, then the set of stable matchings forms a mathematical structure called a lattice.

Kelso and Crawford (1982) studied one-to-many matching problems with wages in the labor market context. They identify a condition for one-to-many matching to be stable –the gross substitutes condition– and show that a salary adjustment process can achieve the stable matching. Hatfield and Milgrom (2005) generalized this result to matching problems with the set of contracts. They identify two conditions for the existence of stable matching in this context –the substitutes condition and the law of aggregate demand– and suggest a cumulative offer process that converges to a stable outcome.

Shapley and Shubik (1971) have worked on the assignment problem between sellers and buyers when the agents can be compensated by money. They show that the problem can be formulated as a linear programming problem and its solution corresponds to the core allocation in the cooperative game theory. Shapley and Scarf (1974) extended this problem to allocating indivisible goods when some agents cannot be compensated by money. Similarly, existence of the core in an assignment problem where there are both indivisible goods and money (this is called housing allocation) is proven in Quinziii (1984). The paper invokes theorems by Scarf (1967) on the existence of the core in non-transferrable utility cooperative games.

3. THEORETICAL BACKGROUND

3.1. PROBLEM: *N*-PERSON, *N*-SERVICE UNIT ASSIGNMENT

We first define the linear programming problem for the optimal soldier-to-service-unit assignment. The assignment problem is a linear programming problem where the objective is to minimize the sum of the total disutility (denoted C_{ij}) of the soldier assignment, subject to no duplicate assignment constraints (one-to-one matching).

Let I be the set of soldiers and J be the set of service units. For each soldier $i \in I$, the social cost incurred from assigning him to unit $j \in J$ is denoted by C_{ij}. The utilities of the soldiers and service units are assumed to be quasilinear, implying that the disutility level C_{ij} is cardinal and comparable across individuals. The element C_{ij} incorporates various factors contributing to a soldier's disutility, such as workload, the nature of tasks, how short-staffed the unit is, and the work environment, aggregating them into a monetary cost.

Our focus is on promoting soldiers' voluntary choice which leads us to suppress the impact of unit preferences from the model. Incorporating variations in unit preferences into the model does not fundamentally alter the structure of the problem, as we can encode the unit preferences into C_{ij}. However, this approach may not result in compensating wage differentials that encourage soldiers to voluntarily sort themselves into an optimal assignment. Hence, we assume minimal variance in the unit preferences, which also mirrors the current system's reliance on random distribution.

We solve the balanced matching problem of assigning n soldiers to n service units. In fact, by assigning duplicate service units, any one-to-many assignment problem can be transformed into a balanced assignment problem with an equal number of soldiers and units. Then the linear programming problem of finding the optimal assignment of n soldiers to n service-units minimizing the total sum of soldiers' disutility is defined as follows:

Definition 1 (One-to-one matching problem)**.**

$$\min_{\{X_{ij}\}} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

$$s.t. \sum_{j=1}^{n} X_{ij} = 1,\ i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} X_{ij} = 1,\ j = 1, 2, \ldots, n, \quad X_{ij} \in \{0, 1\}.$$

The decision variables X_{ij} are 0, 1 integer which takes the value of 1 if soldier i is assigned to unit j. The objective function is the total disutility of the assignment, and the constraints reflect one-to-one matching, meaning that any soldier i is matched (assigned) to at most 1 service unit, while any service unit j is matched with at most 1 soldier. Even if we ignore the constraint that X_{ij} is either 0 or 1, it is known that all candidate solutions with one-to-one matching constraints take values of either 0 or 1. [2]

3.2. SOLUTION: A TWO-STEP APPROACH

The problem is one of the assignment model proposed by Shapley and Shubik (1971), where the solution can be found using linear programming. Furthermore, Roth and Postlewaite (1977) shows that the solution is Pareto efficient and is an allocation induced by a competitive market equilibrium.

This paper takes a two-step approach: first, we solve for the optimal allocation of the soldiers to service units (Step 1), and second, we derive the minimum compensating wage level of service units that induce such assignment (Step 2). This allocation and the wage level comprise the competitive market equilibrium.

Both the optimal allocation and the compensating wage level can be obtained using the Hungarian algorithm, which is one of the well-known solution methods for the assignment problem (linear programming problem). It is a method that simultaneously solves for both the assignment that minimizes the total cost and the price (compensating wage) vector that induces agents to choose the optimal allocation. However, the algorithm may return multiple price vectors that induce the same optimal allocation; therefore, instead of relying on the algorithm alone, we formulate a second linear programming problem that solves for the optimal wage scheme (Step 2).

3.2.1 Step 1: Find the optimal assignment using Hungarian algorithm

The process of Hungarian algorithm for solving the one-to-one matching problem works as follows.

1. Write down the cost matrix C_{ij}. For example, assume that the problem is to minimize the cost (disutility) of assigning five soldiers to five units, with the cost matrix given as in Table 2.

2. Note that the solution (optimal assignment) is the same even if we alter the matrix by adding or subtracting uniformly from a row or a column. In order to simplify the matrix, within each row, find the smallest element

[2]The constraint matrix is an example of a class of matrices that are called totally unimodular.

Table 2: An example of the cost matrix

	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
Person 1	400	150	150	140	130
Person 2	400	350	300	350	300
Person 3	600	300	500	300	400
Person 4	350	300	300	250	250
Person 5	500	400	400	300	250

and subtract the element equally from all elements in each row. Record the subtracted number and keep track of the optimal value.

		Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
Person 1	130	270	20	20	10	0
Person 2	300	100	50	0	50	0
Person 3	300	300	0	200	0	100
Person 4	250	100	50	50	0	0
Person 5	250	250	150	150	50	0

3. Do the same for columns, if some columns consist only of nonzero elements. In our example, this applies only to the first column.

		Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
		100				
Person 1	130	170	20	20	10	0
Person 2	300	0	50	0	50	0
Person 3	300	200	0	200	0	100
Person 4	250	0	50	50	0	0
Person 5	250	150	150	150	50	0

4. In the modified matrix, value 0's corresponds to the minimal cost optimal assignment. Since we are looking for an optimal one-to-one matching, there must be at least one zero in every row and column. A collection of coordinates with zero is said to *cover* all rows and columns if it includes

all soldiers and units. If the set of 0's in the matrix fully covers the rows and columns of the matrix, then we have found the optimal assignment.

5. In the current matrix, it is not possible to cover all rows with zeros because both rows 1 and 5 contain their unique zeros in column 5. In this case, we need to generate additional zeros in the matrix by subtracting more elements. Find the smallest nonzero element in the matrix, and subtract it uniformly from its row. If this process generate negative numbers, we add them back in columns. In our example, since 10 is the smallest number, subtracting it from row 1 and adding it to column 5 yields:

		Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
		100				-10
Person 1	140	160	10	10	0	0
Person 2	300	0	50	0	50	10
Person 3	300	200	0	200	0	110
Person 4	250	0	50	50	0	10
Person 5	250	150	150	150	50	10

Iterate until we have enough 0's left in the matrix to fully cover the rows and columns. This process does not necessarily increase the zeros in the matrix, but does decrease the sum of the total matrix elements, and the algorithm surely terminates. In our example, subtracting 10 from row 5 yields:

		Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
		100				-10
Person 1	140	160	10	10	0*	0
Person 2	300	0	50	0*	50	10
Person 3	300	200	0*	200	0	110
Person 4	250	0*	50	50	0	10
Person 5	260	140	140	140	40	0*

Here, we have found enough 0 elements which cover all rows and columns. They are marked with stars and correspond to the optimal assignment.

6. The result from applying Hungarian algorithm to the original cost matrix is given as follows (Table 3). Again the stars correspond to the optimal

assignment, while the zeros in the previous matrix were marked with underlines.

Table 3: (Example) A solution derived by Hungarian algorithm

comp. wage ↘		Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
net disutil. ↓		100	0	0	0	-10
Person 1	140	400	150	150	140*	130
Person 2	300	400	350	300*	350	300
Person 3	300	600	300*	500	300	400
Person 4	250	350*	300	300	250	250
Person 5	260	500	400	400	300	250*

The numbers assigned to each row/column of the Table 3 is obtained by the algorithm's subtraction/addition process. As a result, for the optimal assignment, the cost in the matrix is the sum of its corresponding row and column elements. For instance, the overall cost of Person 4-Unit 1 match is 350, which is split into 250 in Person 4, and 100 in Unit 1. This bears an interpretation that the column number is the price (compensating wage) posted by the unit, while the row number is the soldiers' net disutility from the assignment. For example, since the gross disutility of Person 4 from each unit is a vector $(350, 300, 300, 250, 250)$, subtracting compensation vector $(100, 0, 0, 0, -10)$ from it yields net disutility of Person 4 from each unit, which is a vector of 250, 300, 300, 250 and 260.

One can also easily check that a soldier's net disutility is the lowest if he chooses the unit with underlined cost. Therefore, when the units post the wages (column numbers), the soldiers find it in their best interests to choose the units dictated by the optimal assignment. Overall, the Hungarian algorithm returns both the optimal assignment and the price vector that induces such assignment. While running the algorithm, adding or subtracting from each column can be thought of as a price-adjustment process in order for the market clearance.

3.2.2 Step 2: Deriving the optimal compensating wage differential

The Hungarian algorithm described in the previous section returns both the optimal assignment and the compensating wages. The compensating wages incentivize soldiers to self-select into the socially desirable assignment. But as any two wage vectors that differ by a constant show, there are infinitely many compensating wage vectors that support the same assignment. For example, the optimal assignment in the previous section can equally be induced as follows.

comp. wage ↘		Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
net disutil. ↓		110	20	20	10	0
Person 1	130	400	<u>150</u>	<u>150</u>	<u>140</u>*	<u>130</u>
Person 2	280	400	350	<u>300</u>*	350	300
Person 3	280	600	<u>300</u>*	500	300	400
Person 4	240	<u>350</u>*	300	300	<u>250</u>	250
Person 5	250	500	400	400	300	<u>250</u>*

In this outcome, compared with the previous example, all soldiers are strictly better off as all net disutilities are strictly smaller, while all units pay a strictly higher set of compensating wages. In fact, as long as the relative magnitude of compensating wages is held constant, there is a trade-off relationship between the overall soldiers' disutility and the wages paid by the units.

Therefore, we follow a separate approach in calculating the optimal compensating wage vector. Having found the optimal assignment that minimizes the sum of military service disutility (Step 1), we find the compensating wage vector for all service units that achieves the assignment, with the minimal government expenditure (Step 2). This additional requirement of budget constraint does not seem unrealistic because any additional compensating wage gap must be funded by the government. The appropriate compensating wage differential for each unit should be derived through the linear programming once again, in the goal of minimizing the total sum of compensating wages expenditure, subject to the soldiers' incentive constraints and the minimal wage guarantee.

Denote by $\sigma^*(i)$, $i \in I$, the optimal assignment $\sigma^* : I \to J$ which is a solution that minimizes the total cost in step 1.($X_{i\sigma^*(i)} = 1$, $X_{ij} = 0$ if $j \neq \sigma^*(i)$.) In Step 2, we solve for the vector of compensating wage differentials (w_j) of service units $j = 1, 2, \ldots, n$. Although the wages are allowed to differ across units, each unit treats all soldiers equally, and posts a single wage w_j. We assume that there is an anchoring wage for a particular service-unit (call it j') to be $w_{j'} = 0$. The j' can be thought of as the 'standard' unit paying the base wage. The wage scheme is set to support the assignment derived in Step 1 while minimizing the sum of wage expenditure. Then the linear programming problem is defined as follows:

Definition 2 (Solving for the compensating wage scheme)**.**

$$\begin{aligned}
&\min_{\{w_j\}} \sum_{j=1}^{n} w_j \\
&s.t.\ C_{i\sigma^*(i)} - w_{\sigma^*(i)} \leq C_{ij} - w_j, \ \text{ for all } (i,j)\text{-pairs} \\
&\quad w_{j'} = 0 \ \text{ (base wage for unit } j')
\end{aligned}$$

When each service unit posts wage w_j, $j = 1, 2, \ldots, n$, the constraints imply that each soldier i prefers his optimal assignment $\sigma^*(i)$ and receiving the net disutility $C_{i\sigma^*(i)} - w_{\sigma^*(i)}$ as opposed to being matched with any other units j. With the wage scheme, the soldiers are induced to select $\sigma^*(i)$ and the wages 'support' the original assignment.

Comment. In case a service-unit demands multiple soldiers, we introduced duplicate dummy service-units and transformed the problem into a balanced matching problem. One may worry that these duplicate service-units may post different wages.

However, we can show that all duplicate service-units post the same wage in the solution to the above problem. For i whose assignment is $\sigma(i)$, the equilibrium level of disutility satisfies

$$C_{i\sigma(i)} - w_{\sigma(i)} \leq C_{ij} - w_j, \ \text{ for all } j \neq \sigma(i).$$

This is also true for j' that is a duplicate of $\sigma(i)$. Since $C_{i\sigma(i)} = C_{ij'}$, this implies that $-w_{\sigma(i)} \leq -w_{j'}$. The same logic applies to i' whose assignment $\sigma(i')$ is j', and $C_{i'j'} = C_{i'\sigma(i)}$ implies $-w_{j'} \leq -w_{\sigma(i)}$. Overall, $w_{\sigma(i)} = w_{j'}$.

3.3. PROPERTIES OF THE OPTIMAL ASSIGNMENT

3.3.1 Optimal wage and dual linear programming

In the context of linear programming, the dual problem to the primal problem (optimal one-to-one assignment) is defined as follows:

Definition 3 (Dual problem)**.**

$$\begin{aligned}
&\max_{\{s_i\}_{i=1}^n, \{w_j\}_{j=1}^n} \sum_{i=1}^{n} s_i + \sum_{j=1}^{n} w_j \\
&s.t.\quad s_i + w_j \leq C_{ij} \ \text{ for all } (i,j) \text{ pairs}
\end{aligned}$$

The decision variables s_i and w_j in the dual problem corresponds to the lagrange multipliers of the constraints in the primal problem:

$$\sum_{j=1}^{n} X_{ij} = 1, \sum_{i=1}^{n} X_{ij} = 1.$$

The row and column values associated with the Hungarian algorithm are the solutions, which also have a natural interpretation of the soldiers' net disutility (s_i, $i \in I$) and compensating wage levels (w_j, $j \in J$).

Denote by $(s_i^*, w_j^*)_{i\in I, j\in J}$ the solution to the dual problem. The strong duality theorem of linear programming states that the values of the primal and the dual problem are equal when the primal problem has feasible and bounded solution. For our problem, a feasible solution exists for the primal problem, which, from the strong duality theorem, implies that its dual problem also has a solution, and the values of the two problems coincide:

$$\sum_{i=1}^{n} C_{i\sigma^*(i)} = \sum_{i=1}^{n} s_i^* + \sum_{j=1}^{n} w_j^*.$$

This leads to the following observation on the problem defined in Definition 2:

Proposition 1. The compensating wage scheme $(w_j^\star)_{j\in J}$ obtained in (Step 2) is a solution to the dual problem of the linear programming in (Step 1).

Proof. The proof relies on the following two observations on solutions to the dual problem. First, for each i, any dual solution $(s_i^*, w_{\sigma^*(i)}^*)_{i=1}^n$ must satisfy:

$$C_{i\sigma^*(i)} = s_i^* + w_{\sigma(i)}^*, \ \ i \in \{1, 2, \ldots, n\}.$$

If not, the constraint of the dual problem ($s_i + w_j \leq C_{ij}$ for all i, j) implies that the only possibility is $C_{i\sigma^*(i)} > s_i^* + w_{\sigma(i)}^*$. Summing up over all i's, the conclusion from the duality theorem is violated, a contradiction. Therefore, we can interpret s_i^* and $w_{\sigma^*(i)}^*$ as a sharing of the optimal cost ($C_{i\sigma^*(i)}$) split into soldier i's net disutility $(s_i^*)_{i=1}^n$ and the matched service unit's compensating wage level $(w_j^*)_{j=1}^n$.

Second, combining the above equality with the dual constraints, the following holds for the solution of the dual program (s_i^*, w_j^*):

$$s_i^* = C_{i\sigma^*(i)} - w_{\sigma^*(i)} \leq C_{ij} - w_j^*. \tag{1}$$

That is, service member i's net disutility (disutility minus the wage) from the assignment $\sigma^*(i)$ is smaller than any other net disutility derived from matching with other (j) units. By setting the compensating wage schedule as $(w_j^*)_{j\in J}$, the duty member i is incentivized to select its optimal assignment, $\sigma^*(i)$.

Note that the condition in (1) is exactly the same as the constraint we used to solve for the compensating wage scheme in (Step 2). Denote by $(w_j^\star)_{j=1}^n$ the solution to the problem in (Step 2). By letting $s_i^\star = \min_{j\in J}(C_{ij} - w_j^\star)$ for each $i \in I$, we find that $s_i^\star = C_{i\sigma^*(i)} - w^\star_{\sigma^*(i)}$, and the pair $(s_i^\star, w_j^\star)_{i\in I, j\in J}$ sum to $\sum_{i,j}(s_i^\star + w_j^\star) = \sum_i C_{i\sigma^*(i)}$. This is to say that $(s_i^\star, w_j^\star)_{i\in I, j\in J}$ satisfies (1) (it is included in the constraint) and achieves the value of the dual program, which is $\sum_{i,j}(s_i^* + w_j^*) = \sum_i C_{i\sigma^*(i)}$. Hence, $(s_i^\star, w_j^\star)_{i\in I, j\in J}$ is a solution to the dual problem. □

3.3.2 Pareto efficiency

Given that we are addressing the internal labor market within the military, the compensating wage gap requires an additional government expenditure. In constructing our optimal assignment through Steps 1 and 2, we seek a solution that minimizes the total expenditure, assuming a ‘standard’ unit pays a base compensating wage of 0. This constraint can be considered as ensuring a minimal living standard, serving as an individual rationality constraint for the soldiers.

In fact, given that we are devising a wage scheme within a conscription regime, one might argue that a soldier’s individual rationality constraint is not pertinent in this particular example. In that line of reasoning, there is no need to concern ourselves with the government budget constraint, as a compensating wage gap can be achieved by reducing pay in units where many soldiers express a preference for such a scheme. Nevertheless, as emphasized in the introduction of this paper, there has been a persistent and ongoing demand for the enhancement of soldiers’ living standards, and wages are projected to increase significantly in the coming years. Therefore, we find it neither reasonable nor realistic to reverse this trend. This raises the question of how much burden such a compensating wage scheme will impose on the government.

Our conclusion is that, with a fixed government budget, the optimal assignment is the most efficient in terms of outcomes. Given the projected rapid increase in salaries until 2025, we believe it would be beneficial to allocate some of these funds towards compensating wage gaps, thereby enhancing overall efficiency.

An assignment, along with the wage scheme, is *Pareto efficient* if no group of individuals in the economy can unilaterally improve from it. In this context, Pareto efficiency is defined by soldiers’ disutilities and the overall government expenditure.

Definition 4. 22eAn assignment $\sigma^* : I \to J$ and a compensating wage scheme $(w_j^*)_{j=1}^n$ are *Pareto efficient* if no group of soldiers can decrease their net disutility through swaps and side payments, without additional government expenditure, exceeding $\sum_{j=1}^n w_j^*$.

This definition deviates from Pareto efficiency for both soldiers and service-units. In fact, some service units may incur losses from the introduction of a compensating wage scheme, as they end up paying more. This does not constitute a Pareto improvement from the perspective of the units.

If an assignment is inefficient, a group of soldiers can improve from it through assignment swaps and side payments, without requiring additional budget input from the government. The current system, randomly allocating soldiers to service units, is Pareto inefficient, as significant gains can be achieved through voluntary exchanges. In contrast, Shapley and Shubik (1971) demonstrated that the outcome of the assignment game is in the core and is indeed Pareto efficient. Thus, no group of soldiers can improve upon the optimal assignment. This result can be summarized in the following statement:

Proposition 2. Any pair of an optimal assignment ($\sigma^* : I \to J$ that minimizes $\sum_{i=1}^n \sum_{j=1}^n C_{ij} X_{ij}$) and the wage schedule $(w_j^*)_{j=1}^n$ that solves its dual problem, is Pareto efficient.

Proof. The optimal-assignment primal program and its dual program jointly solve for the optimal soldier-service unit matching, $\sigma^* : I \to J$, as well as the compensating wage schedule, $(w_j^*)_{j=1}^n$, which induces the soldiers to self-select into the optimal assignment. The constraint in the dual problem states that:

$$s_i^* = C_{i\sigma^*(i)} - w_{\sigma^*(i)} \leq C_{ij} - w_j^* \text{ for all } i,\ j.$$

s_i^* is the net disutility of soldier i in the assignment σ^*. As a result, if we consider an alternative assignment $\sigma' : S \to \sigma^*(S)$ among a subset of soldiers $S = \{i_1, i_2, \ldots, i_k\}$, all soldiers i in S also satisfy:

$$s_i^* \leq C_{i\sigma'(i)} - w_{\sigma'(i)}, \text{ for all } i \in S.$$

Summing over all i's in S yields $\sum_k s_{i_k}^* \leq \sum_k C_{i\sigma'(i_k)} - \sum_k w_{\sigma'(i_k)}$. This implies that the sum of net disutilities in group S is at least the sum of net disutilities in the optimal assignment. Therefore, unless the government is willing to spend more on the wage of group S, $\sum_k w_{\sigma'(i_k)}$, it is not possible to achieve an improvement in the total net disutility of group S. Given fully transferrable utilities (disutilities) with quasilinear preferences and money, this implies that no group S can improve upon the optimal assignment. □

3.3.3 Incentive compatibility of the soldiers' report

Thus far, we have solved the model under the assumption of having complete information about the level of disutility that soldiers receive from each unit assignment $(C_{ij})_{i\in I, j\in J}$. However, if we are to actually calculate the optimal assignment and the compensation wage, we must rely on the reported subjective disutility levels from the soldiers. Accordingly, one may worry that the soldiers might misreport their subjective disutility levels in an attempt to secure additional compensation wages or a more favorable assignment. Surprisingly, an application of the result from Roth and Sotomayor (1992), which states the incentive compatibility of a particular stable matching, suggests that the mentioned concern may not be as significant as one might initially assume. Given a social-cost minimizing optimal assignment, $\sigma^* : I \to J$, consider the following modification of the problem in Definition 2.

Definition 5 (The soldier-optimal stable matching under budget constraint)**.**

$$\begin{aligned}
&\min_{\{w_j\}} \sum_{i=1}^{n} C_{i\sigma^*(i)} - w_{\sigma^*(i)} \\
&s.t.\ \ C_{i\sigma^*(i)} - w_{\sigma^*(i)} \leq C_{ij} - w_j, \ \ \text{for all } (i,j)\text{-pairs} \\
&\qquad \sum_{j=1}^{n} w_j \leq M \ \ \text{(government budget constraint)}
\end{aligned}$$

In words, the problem identifies a compensating wage scheme $\{w_j^*\}$ which induces the soldiers to self-select into the optimal assignment, while the total expenditure on the compensating wages is capped at M. The obtained matching is stable in the sense that no soldier wants to deviate to another unit, and no soldier can be made better off without additional government expenditure. This program solves for the optimal stable matching from the soldiers' perspectives, under the budget feasibility constraint. In the context of the Hungarian algorithm introduced previously, its solution is a shift of the compensating wage vector by an appropriate constant, such that the total expenditure sum to M.

Proposition 3 (Roth and Sotomayor, 1992)**.** Consider a mechanism that receives as input the soldiers' report of cost levels $(C_{ij})_{i\in I, j\in J}$, and returns the soldier-optimal stable matching found in Definition 5. The dominant strategy for soldiers is to truthfully report their cost levels $(C_{ij})_{i\in I, j\in J}$.

Intuitively, the Proposition implies that if the soldiers understand that any false reporting cannot improve upon the already optimal final outcome, they would choose to tell the truth. One premise is that soldiers generally recognize the budget feasibility constraint outlined in the problem, which imposes a limit on the attainable wage. Moreover, the compensating wage scheme is determined by the marginal soldier willing to shift to a less desirable unit, implying that in a large market, the likelihood of a few individuals consistently influencing the wage outcome is small.

Another concern involves the soldiers' difficulty of processing and reporting detailed information (such as monetary disutility) to the mechanism. This challenge could be mitigated by having the soldiers report ordinal rankings of the units instead. We explore this alternative later in greater detail in the second-best policy recommendation.

3.3.4 Fairness of the optimal matching

As briefly mentioned in the introduction section, military units inevitably have varying working conditions, and to ensure ex-ante fairness, the military utilizes a computerized random system when assigning soldiers to units. In contrast, we have been discussing an approach that allows conscripted soldiers to select their preferred units, taking into account both their wages and service utility. Transitioning from an ex-ante fair allocation to a willingness-based allocation represents a shift in the philosophical foundation of mandatory military service and may give rise to concerns regarding fairness.

In the economics of matching theory, fairness is closely related to the concept of *envy*, which is an important criterion when evaluating the performance of a matching algorithm. An agent i is said to *envy* the assignment of j, $\sigma(j)$, if i prefers $\sigma(j)$ over the assignment of i himself: $\sigma(j) \succ_i \sigma(i)$. With the presence of randomization, it is defined with the agents' expected utilities over the space of lotteries. An agent i with ordinal preference $\succ_i$ over the outcomes of a lottery is said to *sd-envy* the lottery of j, x_j, if his lottery x_i is first-order stochastically dominated by x_j with respect to $\succ_i$. An allocation of the lotteries is said to be *sd-envy-free* if no agent sd-envies another agent's lottery. An assignment is *ex-post envy-free* if no agent possesses envy for other's assignment for any ex-post outcome of the lottery. A weakening of the sd-envy-freeness is *ex-ante envy-freeness*, in which no agent obtains a higher expected utility from the lottery of another agent.

The ex-post envy-free condition is stricter than the other conditions because, the other concepts only require that the allocation of objects to be fair in ex-ante probabilities, whereas the ex-post envy-freeness means that the allocation must be fair for every realizations of lotteries. Returning to our discussion, using pure random allocation can be justified on the grounds that it generates identical lotteries for all soldiers. However, it inevitably leads to many instances of ex-post envy, and it is not ex-ante envy-free either, unless everyone has the same preferences. For example, if there is a soldier who prefers being assigned to a frontline unit, the random allocation introduces a positive probability that the soldier may end up being assigned elsewhere; if he does get assigned to a non-frontline unit, it is better that he swap his assignment with another who prefers the unit over the frontline. In contrast, following the adjustment of each soldier's utility with the implementation of an appropriate compensating wage scheme, there is no ex-post envy among soldiers concerning their assignments, as they favor their assignments over any other alternatives.

Proposition 4. A completely random assignment is ex-ante envy-free only if all agents share identical preferences, and it always creates ex-post envy. On the other hand, the optimal assignment facilitated by a compensating wage scheme is free from ex-post envy.

Even if we must incorporate some randomization due to the inability to fully adjust the wage to the optimal level, it is evident that introducing compensation for commonly less-preferred units substantially diminishes ex-post envy. We explore this possibility in the policy exercise example provided below.

3.3.5 Extension to non-transferrable utility models

Our linear programming formulation is built upon the assumption of quasi-linearity in the disutilities experienced by soldiers. The existence and the applicability of a compensating wage scheme carries through for a general utility formulation, although determining it will no longer be as straightforward as in the previous linear programming formulation.

Shapley and Scarf (1974) identified the algorithm for finding competitive prices when the goods to be allocated are indivisible and there is no money. Start from an arbitrary assignment of soldiers to units.

Definition 6. Define a *top trading cycle (TTC)* for I to be any set $S \subseteq I$ whose members are indexed in such a way that its i-th member ranks the assignment of its $(i+1)$-th member at least as well as any other units in I. The last member in S, call him i_s, ranks the assignment of the 1st member in S the best among all units in I.

Since we can construct such cycles for $I - S$ and so on, I can be partitioned into a sequence of such cycles, $\{S_1, S_2, \ldots, S_p\}$. Then, any arbitrary prices

$\pi_1 > \pi_2 > \ldots > \pi_p$ to the goods in S_1, S_2, ..., S_p, respectively, form a vector of competitive equilibrium prices. Any k-th indexed member in S_i, who receives π_i by selling his good (assignment), can only purchase from the cycles with lower prices: $S_i \cup S_{i+1} \cup \ldots \cup S_p$. However, among the goods, he prefers the best the $(k+1)$-th indexed member's assignment in S_i. Hence, there is no trade outside the cycle, and there is no need for money anyway. The outcome of this trade is Pareto efficient.

However, the top trading cycle (TTC) and its resulting price vector lack the compensating wage interpretation, as prices are formed within cliques. Quinziii (1984) extended the model to allow for side-payments between the members and showed that there is a competitive equilibrium that supports the optimal allocation. The utility of soldier i for amount of money y_i and unit h_r is denoted by $u_i(y_i, h_r)$ and u_i is strictly increasing in y_i. Let $x_{ij} = 1$ if i gets the unit of j. The vector of x's satisfy the usual one-to-one matching constraint.

Definition 7 (Quinziii, 1984)**.** A price vector $p \in \mathbb{R}^n_+$ along with a feasible allocation $((m_i)_{i\in I}, (x_{ij})_{i\in I, j\in J})$ is a competitive equilibrium if:

- $m_i + \sum_{j\in J} p_j x_{ij} \leq w_i + p_i$ (budget constraint)
- $u_i(y_i, h_r) > u_i(m_i, \sum_{j\in N} x_{ij} h_j)$ implies $y_i + p_r > w_i + p_i$. (utility maximization)

This approach is more satisfactory as it generates a single price for each unit. However, it requires personalized side-payments to compensate soldiers, even if they are assigned to the same unit. This practical implementation might pose challenges in real-world applications.

3.3.6 A second-best alternative

Indeed, in one-sided matching scenarios where the Top Trading Cycle (TTC) is implemented, it is common to use a random assignment for the initial allocation. This approach helps mitigate potential biases that might arise from a predetermined initial assignment. Abdulkadiroğlu and Sönmez (1998) have shown that the approach is equivalent to randomly assigning priorities, followed by the soldiers selecting the best available alternative in sequence, and is referred to as a 'random serial dictatorship.' Implementing a random serial dictatorship requires only the soldiers' rankings of the units, making it much less informationally burdensome than reporting their willingness-to-accept. However, if all soldiers share a common ordinal preference over the units, the random serial dictatorship is equivalent to the current system of completely random assignment. Therefore, we first address the preference orderings by introducing appropriate compensation, which can be obtained from an experimental study, as in Shin *et al.* (2021), and then apply the Top Trading Cycle (TTC).[3] This two-step approach is summarized as follows:

- Initially, we identify a suitable compensating wage scheme using a sample of soldiers.
- Subsequently, we suggest the compensating wage scheme to the soldiers and implement a random serial dictatorship (TTC with random endowment) for the actual matching.

The method relies on a single round of comprehensive research into the soldiers' willingness-to-accept, and it is evidently suboptimal when contrasted with calculating compensating wages for the entire set of soldiers. However, once this initial step is completed, the second matching (random serial dictatorship outcome) can be implemented using the soldiers' submitted rankings over the units, without the need for actual random assignment and without the formation of trading cycles. This approach significantly reduces the information processing burden on the soldiers, as they only need to submit ordinal preferences. In the next section, we refer to this method as the second-best solution.

4. SIMULATION

4.1. SIMULATION PROCEDURE

We simulate the distribution of compensating wage differentials by repeatedly assigning 40 soldiers to four types of service-units. The service-units are categorized into 'frontline units' [4] versus 'rear area units' based on the locations, and 'combat' and 'non-combat' on the missions. Overall there are four possible combinations, or types of service-units.

Units' demand We assume that 6 (out of 40) soldiers need to be assigned in 'frontline–combat' units, 26 (out of 40) in 'rear–combat' units, 2 (out of 40) in 'frontline–noncombat' units, and 6 (out of 40) in 'rear–noncombat' units.

[3] We thank a referee for pointing to this alternative.

[4] GP-GOP, DMZ, coastal or riverine defense unit, West-5-Islands, NLL adjacent islands

Table 4: Estimated marginal/relative disutility reported in Shin *et al.* (2021)

	Marginal Utility	Standard Error	Relative Utility (Disutility)
Frontline Combat	-0.220	0.093	-0.365 (0.365)
Frontline Noncombat	-0.103	0.082	-0.248 (0.248)
Rear Combat	+0.177	0.084	+0.032 (-0.032)
Rear Noncombat	+0.145	-	-

Distribution of the soldiers' utility We set the level of disutility for a 'rear area-noncombat' unit to be zero for all soldiers. The type corresponds to the 'standard' unit in the dual constraint. With this assumption, we analyze the compensating wage differentials based on the simulated disutility levels for the other three types. Therefore, the final value derived from this analysis is not the absolute level of compensatory wages but the 'gap' in compensatory wages.

In order to simulate the soldiers' preferences (disutility levels), we relied on the figures reported in Shin *et al.* (2021). The first two columns of Table 4 presents the soldiers' average marginal utility from the four types of assignment, and corresponding standard errors. On average, 'frontline-combat' units are the least preferred, while 'rear-combat' units exhibit the highest average marginal utility among the four choices. [5] The last column reports the relative level of utility (disutility in parenthesis) compared to the 'rear-noncombat' unit. The relative utility of the 'rear-combat' unit is actually positive, which means as for service in the rear areas, average soldiers derive a positive($+$) value of the relative utility for combat missions compared to non-combat, and demand a negative($-$) compensatory wage. Although Shin *et al.* (2021) does not offer a convincing explanation for this phenomenon, it is plausible that soldiers consider factors such as staff shortages or living conditions in rear-noncombat units.

We incorporate the aforementioned elements into a model as follows. Unlike Shin *et al.* (2021), our model and simulation can accommodate a more diverse class of soldiers' preference structures while aligning with the figures reported in Table 4. We match the marginal distribution of disutilities, (X_0, X_1, X_2) which stands for rear-combat, frontline-noncombat, and frontline-combat, respectively: $X_0 \sim N(-0.032, 0.084^2)$, $X_1 \sim N(0.248, 0.082^2)$, $X_2 \sim N(0.365, 0.093^2)$.

[5] This is rather peculiar because combat missions are shown to deliver negative utility for frontline units, while the effect is the opposite for rear units. We suspect that the rear-combat units are preferable in terms of workload, living conditions, or other amenities, despite the nature of the task.

- A soldier's disutility from assignment to a unit i comprises two components: the inherent difficulty of the task denoted as μ_i ($i \in \{0,1\}$ for combat and non-combat missions) and the impact of other amenities e_i ($i \in \{0,1,2\}$ for the three units). This latter component encompasses factors such as additional days of leave, the degree of short-staffing in the unit, living conditions, and the quality of peers. It is possible that the expected impact of amenities, denoted as $E[e_i]$, outweighs the expected innate disutility, $E[\mu_i]$, resulting in an overall negative disutility from assignment (net positive utility), as in the 'rear-combat' units which exhibit a positive utility expectation (+0.032) compared to rear-noncombat units. This could be attributed to a high realization of amenity factors (e) in combat units, even though a combat unit is expected to be more challenging in its task dimension (μ). Assume no systematic errors in soldiers' decision-making so that they are well aware of the various dimensions of the units (μ_i and e_i) to be assigned.

- Assume that μ_i and e_i, as well as e_i and e_j are realized independently. The first assumption is justified by considering that μ_i is a preference realization, while e_i is closely tied to government policy, such as the investment priorities of the Ministry of Defense. The second assumption implies that, beyond the average component, the random part of e_i's combines idiosyncratic preferences and policy shocks.

- To accommodate correlations, we allow for the possibility that μ_i and μ_j are correlated. For instance, it is plausible that μ_0, the innate disutility from being assigned to a combat mission, is more related to the physical strength of the soldier, while μ_1, the counterpart from being assigned to a frontline unit, is primarily influenced by locational disamenities. If there are other factors jointly determining the realization of μ_0 and μ_1, this correlation will be reflected in the correlation between the two random variables.

  Note 1. The exact direction and magnitude of the correlation would require careful study of soldiers' preferences, which is beyond the scope of our current analysis. However, it is important to note that our model and simulation are flexible enough to account for various scenarios.

- To streamline the focus on the policy experiment, especially in examining how the dispersion of amenities (e_i) affects soldiers' welfare from an assignment, and considering data limitations, we make a simplifying assumption about the structure of preference variables. We assume that the variance of μ_i's is identical, denoted as $Var[\mu_0] = Var[\mu_1] = \sigma_\mu$, implying a covariance between them of $Cov(\mu_0, \mu_1) = \rho_{01}\sigma_\mu^2$. Sacrificing this assumption is possible but would involve more parameters and result in a less straightforward comparative statics.

Under the stated assumptions, we have $X_0 = \mu_0 + e_0$, $X_1 = \mu_1 + e_1$, and $Cov(X_0, X_1) = Cov(\mu_0, \mu_1)$. The innate disutility from a frontline-combat unit, X_2, is the sum of μ_0 and μ_1, while the amenities e_2 are determined independently from other elements. Let σ_i denote the standard deviation of the random variable e_i. To match the marginal distribution of (X_0, X_1, X_2), the following restrictions are necessary:

$$V(X_0) = V(\mu_0 + e_0) = \sigma_\mu^2 + \sigma_0^2 = 0.084^2 \tag{1}$$

$$V(X_1) = V(\mu_1 + e_1) = \sigma_\mu^2 + \sigma_1^2 = 0.082^2 \tag{2}$$

$$V(X_2) = V(\mu_0 + \mu_1 + e_2) = 2\sigma_\mu^2 + 2\rho_{01}\sigma_\mu^2 + \sigma_2^2 = 0.093^2 \tag{3}$$

This implies that the admissible range of parameters is: $\sigma_\mu < 0.082$ and $2(1+\rho_{01})\sigma_\mu^2 < 0.093^2$. The joint distribution of (X_0, X_1, X_2) is: [6]

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} -0.032 \\ 0.248 \\ 0.365 \end{bmatrix}, \begin{bmatrix} 0.084^2 & \rho_{01}\sigma_\mu^2 & (1+\rho_{01})\sigma_\mu^2 \\ \rho_{01}\sigma_\mu^2 & 0.082^2 & (1+\rho_{01})\sigma_\mu^2 \\ (1+\rho_{01})\sigma_\mu^2 & (1+\rho_{01})\sigma_\mu^2 & 0.093^2 \end{bmatrix} \right).$$

The formula suggests that if a significant portion of the dispersion in disutilities is attributed to soldiers' preferences (assumed to be correlated across units), the realizations of X's are likely to move together. However, if the dispersion is primarily due to idiosyncratic unit conditions, there is less correlation among X's.

When translating the level of disutility into monetary value, we apply the trade-off relationship proposed in Shin *et al.* (2021), where utility increases by 0.055 for each additional 10,000 KRW in monthly salary.

[6] Note:

$$Cov(X_0, X_2) = Cov(\mu_0 + e_0, \mu_0 + \mu_1 + e_2) = (1+\rho_{01})\sigma_\mu^2 \tag{4}$$

$$Cov(X_1, X_2) = Cov(\mu_1 + e_1, \mu_0 + \mu_1 + e_2) = (1+\rho_{01})\sigma_\mu^2, \tag{5}$$

Estimation of the compensating wage differentials through simulation Given the data limitations, we initially set $\sigma_\mu = 0.05$ and $\rho_{01} = 0.25$ as approximate values. While simulation results may show some variability depending on the assumed correlation coefficients, the qualitative implications remain consistent. Additional simulations in Section 4.2 offer further insight into how correlation coefficients influence simulation outcomes.

In our simulations, we randomly sample disutility levels for 40 soldiers from a multivariate normal distribution and convert them into monetary values. Using these simulated disutility levels and the unit demand (capacity) as inputs, we execute our two-step linear programming approach to determine the optimal compensating wage differentials. Table 5 presents the results of 50 repetitions of these simulations, alongside the calculations from Shin *et al.* (2021). On average, a compensating wage differential of 63,804 KRW is required for 'frontline-combat,' 5,677 KRW for 'rear-combat,' and 28,208 KRW for 'frontline-noncombat' compared to 'rear-noncombat,' which serves as the base unit with a compensating wage of 0.

	Capacity	Simulated Average	Shin *et al.* (2021)
Frontline Combat	6	63,804	66,364
Rear Combat	26	5,677	-5,818
Frontline Noncombat	2	28,208	45,091
Rear Noncombat	6	0	0

Table 5: Simulation results, an average of 50 times repetition

The negative number reported by Shin *et al.* (2021) for 'rear-combat' units is a result of the positive utility gain reported by the soldiers. In contrast, our calculation shows a positive compensating wage differential for these units. The underlying rationale for this result lies in the high demand (capacity) for 'rear-combat' units. To find the market-clearing price, our calculation responds to the substantial demand by elevating the compensating wage differentials. In other simulations with different unit capacities (Table 7), the wage differential for 'rear-combat' units exhibits negative averages, aligning with the findings of Shin *et al.* (2021).

4.2. COMPARATIVE STATICS

Correlation between disutilities The study conducted by Shin *et al.* (2021) presented only average and standard error values for soldiers' disutilities, omitting any mention of correlations among the different types. In response, our current focus is on examining how compensating wage differentials vary based on the correlations between each type. This analysis aims to shed light on the relationships between variables and their impact on compensating wages, considering the potential interdependencies within the dataset. Our analysis aims to understand both the magnitude and direction of compensating wages as we manipulate these variables.

For explanatory purposes, in this section, we simulated compensating wages using the following distribution of the disutilities vector (X_0, X_1, X_2):

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} -0.032 \\ 0.248 \\ 0.365 \end{bmatrix}, \begin{bmatrix} \sigma_0^2 & \rho\sigma_0\sigma_1 & \rho\sigma_0\sigma_2 \\ \rho\sigma_0\sigma_1 & \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_0\sigma_2 & \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right),$$

where $\sigma_0 = 0.084$, $\sigma_1 = 0.082$, and $\sigma_2 = 0.093$. The outcomes of these simulations, with variations in ρ ranging from -0.5 to 1, are outlined in Table 6.

Table 6: Compensating wage differentials while varying parameter values ρ

	Frontline Combat (cap:6)	Rear Combat (cap:26)	Frontline Noncombat (cap:2)	Rear Noncombat (cap:6)
ρ =-0.5	48,412	2,682	18,021	0
ρ =-0.25	51,762	4,064	23,007	0
ρ =0	55,147	5,040	27,506	0
ρ =0.25	58,381	5,395	31,888	0
ρ =0.5	62,488	6,306	37,008	0
ρ =0.75	67,369	6,871	43,450	0
ρ =1	78,258	7,919	58,573	0

With an increasing correlation, compensating wage differentials for all types uniformly rise. Intuitively, a high correlation among soldiers' preferences implies that a larger number of soldiers demand higher wages for all three types. Consequently, to clear the market, more substantial compensating wage differentials are needed to attract these 'marginal' soldiers with high disutility. On the contrary, a strong negative correlation indicates that a soldier's disutility levels in each unit are highly idiosyncratic. In such a case, only small compensatory wage differentials are required to attract them to different units.

Demand for each unit type In this section, we set the correlation coefficient at 0.25 and fix the capacity for 'rear-combat' and 'frontline-noncombat' at five indi-

Table 7: Compensating wage differentials for various capacity, $\sigma_{\mu} = 0.05$ and $\rho_{01} = 0.25$

	Frontline Combat	Rear Combat	Frontline Noncombat	Rear Noncombat
Cap	10	5	5	20
Wage Differential	59,074	-18,394	29,899	0
Cap	15	5	5	15
Wage Differential	66,305	-15,301	33,501	0
Cap	20	5	5	10
Wage Differential	73,508	-11,441	37,576	0
Cap	25	5	5	5
Wage Differential	81,166	-6,477	42,531	0

viduals each. Subsequently, we adjust the capacity vector for 'frontline-combat' and 'rear-noncombat' to take values from (10 and 20), (15 and 20), (20 and 10), and (25 and 5) to observe how compensatory wages vary in response to changes in demand within the military internal labor market. All other assumptions, except those mentioned, remain consistent with what was presented before. The results of these simulations can be found in Table 7.

As demonstrated in Table 7, an increase in military demand for 'frontline-combat' and a decrease in demand for 'rear-noncombat', possibly due to security needs or organizational changes, result in an escalation of compensatory wages for all three units, including 'frontline-combat'. This is because when the demand for 'frontline-combat' increases, the wage needed to attract the last marginal soldier to join the position becomes higher than before. Additionally, this creates a chain effect on the supply of 'rear-combat' and 'frontline-noncombat,' leading to increases in compensatory wages for those positions as well. In contrast to the study conducted by Shin *et al.* (2021), which suggests a fixed price regardless of changes in military demand, this study presents different prices that vary based on these demand changes.

Meanwhile, in this simulation, we estimate a negative compensating wage differential for 'rear-combat,' indicating that a lower wage is required to attract individuals to that position compared to the reference group ('rear-noncombat').

This is because, as mentioned above, some army soldiers have a positive relative utility for combat missions compared to non-combat missions in rear-area units and are attracted with a negative compensatory wage. In particular, when the demand for 'rear-noncombat' is small and can be sufficiently covered by soldiers willing to accept negative compensatory wages, the compensatory wage differential is calculated as negative.

4.3. POLICY IMPLICATIONS

As mentioned in the introduction, the Korean government plans to significantly increase the wage paid to mandatory service soldiers during the 2023-2027 period. Based on our previous simulations, we calculate the gains from an alternative policy that utilizes the planned salary increase for soldiers as compensating wage differentials. The current system admits allowances to frontline units of at most 45,000 KRW per month. [7] Assume that the current system of allowances is maintained throughout, and that the total available monthly budget for operating four units comprising 40 soldiers is 60 million KRW per month. The scheme is indicated in the left column of Table 8. Overall, the Table 8 compares three compensating wage schemes that exhaust the same budget:

1. Increasing the base salary and maintaining the current salary and allowance system, as in the mid-term defense plan, while also maintaining the random allocation of soldiers to units.

2. Adjustment of allowance payments to the level suggested by Shin *et al.* (2021), while maintaining the random allocation.

3. An average of simulated wage schemes obtained from our two-step process.

While the last scheme relies on the incentives and voluntary assignment, the first two schemes are assumed to keep the current system that randomly assigns soldiers to service-units. Using the figures from the "2023-2027 Defense Mid-term Plan" of Korea, we assume that 1.491 million KRW per month is the target base salary (excluding savings payment) for soldiers nearing the end of mandatory service by 2027. Based on this assumption, and the reported disutility levels from Shin *et al.* (2021), we calculated the sum total of pecuniary disutility levels of the 40 soldiers for the three schemes, respectively. Our calculations show that, as long as the soldiers are assigned randomly, both plans - no discrimination and

[7] The figures are based on assignment to DMZ units.

Table 8: A comparison of three scenarios

		Mid-term Defense Plan		Shin *et al.* (2021)	
	Cap	Base salary	Allowance	Base salary	Allowance
Frontline-Combat	6	1,491,000	45,000	1,491,573	66,364
Rear-Combat	26	1,491,000	0	1,491,573	-5,818
Frontline-Noncombat	2	1,491,000	45,000	1,491,573	45,091
Rear-Noncombat	6	1,491,000	0	1,491,573	0
Total		59,640,000	360,000	59,662,902	337,098
		60,000,000		60,000,000	

		This study	
	Cap	Base salary	Comp. wage
Frontline-Combat	6	1,485,329	63,804
Rear-Combat	26	1,485,329	5,677
Frontline-Noncombat	2	1,485,329	28,208
Rear-Noncombat	6	1,485,329	0
Total		59,413,158	586,842
		60,000,000	

the proposal of Shin *et al.* (2021) show no improvement in the overall disutility of military service. The average values of the total disutility for all random combinations of matching is 337,098 KRW.

In comparison, the average value of total disutility for the optimal matchings proposed in this study was simulated to be 56,703 KRW. This implies that our optimal assignment derived from the linear programming method (Hungarian algorithm) increases soldiers' utility by 280,395 KRW with the same budget. This result relies on the fact that, as discussed in Section 3, our algorithm finds an optimal matching that is always Pareto-efficient, and that the derived appropriate compensating wage differential is a price (wage) system that guarantees Pareto efficiency by facilitating the optimal matching.

We also simulated our second-best solution outlined in Section 3.3.6. The compensatory wage differentials derived from the previous simulations were applied to another set of 40 randomly selected samples. In this round of matching, each individual's disutility levels, considering the compensating wage, were mapped to their ordinal preference rankings, and the Top Trading Cycle (TTC) method was subsequently applied. Recognizing that the random serial dictatorship yields different matching outcomes based on the initial allocation, we addressed this variability by obtaining an average from repeating the process 50 times. The average value of total disutility for the second-best matchings was simulated to be 78,869 KRW. This implies that the second-best allocation, while not as optimal as the best solution, still has the advantage of increasing utility by a substantial 258,229 KRW.

5. CONCLUSION

This paper proposes an algorithm for the assignment of soldiers to service units that achieves higher efficiency without additional budget expenses. The key idea is to design an appropriate compensating wage scheme to incentivize soldiers to self-select into the most efficient assignments with service units. Specifically, we applied the Shapley and Shubik (1971) assignment game to find the optimal soldier-service unit matching and derive an appropriate compensating wage differential in the military internal labor market. The Shapley and Shubik (1971) assignment game is a linear programming problem, and we used the Hungarian algorithm to solve this problem.

The Hungarian algorithm not only calculates the optimal allocation that maximizes the utility (or minimizes costs) of participants but can also derive a price system (compensatory wage level) that attracts participants to the ideal assignment. However, the values derived through the Hungarian algorithm may not be unique, implying that there can be multiple solutions. Therefore, in this paper, after calculating the optimal allocation (Step 1), we suggested compensatory wage levels that encourage the optimal assignment of each soldier while satisfying minimum expenditure requirements (Step 2).

Compared with the current system of wage differentials, our approach is similar to algorithms adopted by ride-hailing services (such as Uber or Lyft), where prices adjust readily in response to demand and supply. As we rely on the market mechanism, the optimal allocation found through linear programming is always Pareto efficient. With the quasi-linear preference assumption, the optimal assignment method (Hungarian algorithm) enables improvement in military service satisfaction within the same budget. Even if we relax the quasi-linear assumption, the results hold, although relying on the random serial dictatorship dampens the efficiency gain.

In the latter part of the paper, we conducted an optimal placement simulation using the actual utility level of army soldiers, presenting a specific figure for the appropriate compensating wage differential. Our algorithm can account for variations in unit capacities or changes in underlying soldiers' preferences (correlation among assignments). As long as appropriate inputs are provided, the algorithm is bound to find a market-clearing wage. One concern is the possibility that soldiers awaiting assignment might provide incorrect or distorted information about their preferences. Theory suggests that additional budget allocations may be necessary to incentivize soldiers to report truthfully. However, we conclude that this policy alternative is worth considering, especially given the projected steep increase in soldiers' salaries in the near future.

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6. APPENDIX

This section provides a selection of current special-pay and incentive-pay schedules implemented in the Republic of Korea Army, applicable to mandatory service members.

Table 9: Current payment schedule for special duties in the Army

Category	Duty	Monthly Payment
(Category A)	Person in charge of handling explosives and ammunition testing	118,800
(Category B)	Person engaged in parachute descent from aircraft or other equipments more than twice a year	85,000
	Person trained more than twice a year for Helicopter Rappel or Fast Rope	55,000
	Person performing underwater or waterborne operations at least once a month	50,000
(Category C)	Person in charge of managing electric current of 3,300 volts or more Person engaged in reparing ammunition Person who, having been in Category A or B, engages in parachute jumps from a model tower at least once a month instead of jumping from an aircraft, due to weather conditions or shortages in equipment Person operating a container crane Person engaged in the excavation and inspection of hazardous materials Person driving a heavy equipment transport vehicle (K-915)	18,000

Table 10: Location pay scheme applicable to mandatory service soldiers, authors' excerpt from relevant regulations (amended in 2022)

This rule is applicable to sergeants, soldiers, and police in their mandatory service
(Area A) prorated by 25,000 KRW per month, (Area B) prorated by 20,000 KRW per month. (Additional allowance) 40,000 KRW for the frontier islands in West Sea, 20,000 KRW for 4 provinces adjacent to the demilitarized zone and northern limit line, 10,000 KRW for personnel working in coastal posts adjacent to the demilitarized zone and maritime boundary areas.
(Remarks) Individuals in (Area A) include those working in the demilitarized zone, Ulleungdo and Dokdo, and those stationed for maritime operations in the areas adjacent to the western 5 islands. Individuals in (Area B) include those stationed at coastal posts adjacent to the demilitarized zone or those working at high-altitude areas above 800 meters for counterintelligence operations. The additional allowance is provided to individuals in (Area A) who work in the demilitarized zone, western 5 islands in the West Sea, and 4 provinces adjacent to the northern limit line, and to individuals in (Area B) stationed at coastal posts adjacent to the demilitarized zone or those stationed for counterintelligence operations.