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Platform Competition in Two-sided Markets and Welfare Implications*

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Abstract We study the behavior of competing platforms in a two-sided market. A user derives two types of utility from joining a platform: the intrinsic membership benefit, which varies across users, and the network benefit, which is determined by the number (measure) of users from the other side joining the same platform. When multiple platforms compete in price to attract users, in each symmetric equilibrium, each platform earns a zero profit. We also construct the unique equilibrium in the case of platform monopoly. Our comparison of welfare levels attained in oligopoly and monopoly shows that when there are at least two platforms in the market, social welfare decreases as the number of platforms increases; and that social welfare is maximized when two platforms or a single platform operates, depending on the magnitudes of network externality and marginal cost.

Keywords Two-sided market, platform competition, network externality, social welfare

JEL Classification D4, D43, L1

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1. INTRODUCTION

Platforms facilitate interactions among users in two-sided (or multi-sided) markets. Online commerce companies (e.g., Amazon and eBay) connect buyers and sellers. Media platforms (e.g., social networking services, newspapers, and portals) bring viewers to advertisers. Ride-hailing services (e.g., Uber and Lyft) match drivers and riders. Video-sharing services (e.g., YouTube and TikTok) gather content creators, viewers, and advertisers to one place. A key feature demonstrated by these industries is network externality: as more users from one side of the market join a platform, users from the other side will see greater value in joining the same platform.

The presence of network externality brings about an effect that challenges the conventional relationship between competition and social welfare (i.e., social welfare rises with more competition). Offering its service at a lower price to one side, a platform may attract more users from both sides and increase its profit. In the case of symmetric platforms, the latter competition will bring the equilibrium price down to marginal cost, with all platforms getting an equal share of the market. Yet this does not necessarily mean that users are better off. Dispersed across several platforms, users will derive less utility from network externality and if this reduction is substantial enough, it may outweigh benefits from the lower price.

We study a model of platform competition in two-sided markets that formalizes the above intuition. Following seminal papers on two-sided markets (Rochet and Tirole, 2003, Armstrong, 2006), we assume that users on each side earn two types of utility by joining a platform: (i) the intrinsic benefit arising from consuming services provided by the platform, which is heterogeneous across users; and (ii) the (cross-side) network externality, which depends on the number (measure) of users from the other side on board the same platform. Platforms compete by simultaneously setting prices (membership fees for entering a platform) for two sides and users subsequently choose which platform to join, if any.

To analyze this game, we adopt a refinement of subgame perfect equilibrium as an equilibrium concept. The refinement says that users on either side coordinate in their platform choice, so that a platform with lower prices attracts more users. We show that in each symmetric equilibrium, each platform earns a zero profit. This zero-profit result applies to any equilibrium where all platforms charge the same prices (for each side, all platforms set the same price but the price for one side need not equal the price for the other side). The zero-profit result is new and quite unexpected, given what we know from the literature. Heterogeneity of the intrinsic benefit users derive makes platforms differentiated products. The literature on Bertrand competition for differentiated goods suggests that profits will be positive. Our analysis shows that the latter lesson is not valid in our model. Underlying the difference is our refinement of subgame perfection. The coordination requirement that more users are attracted to platforms with lower prices brings about intense price competition among platforms. In our model, its effect turns out sufficiently strong to drive platforms' profits to zero, despite their advantage arising from product differentiation.Concerning welfare implications, we focus on the equilibrium where all platforms charge marginal cost and ask how welfare levels vary with the number of platforms in the market. It turns out that social welfare decreases as the number of platforms in oligopoly increases, regardless of the values of parameters for marginal cost and network externality. As long as multiple firms operate, the price reduction effect due to competition is achieved in full. Yet with platforms splitting the market, network effects weaken and a welfare loss follows. Thus, among all oligopolies involving different numbers of platforms, duopoly yields the highest level of social welfare.

Next, we shift attention to platform monopoly, characterize the equilibrium, and compare its outcomes to duopoly outcomes. The monopolist platform's equilibrium pricing strategy depends on the relative magnitudes of marginal cost and network externality. The most interesting case is one in which marginal cost is moderate, so that the monopolist sells to a positive yet less than full share of users on each side of the market. In that case, each side is charged the average of marginal cost and 1. The share of users from the other side joining the platform enters a user's utility function and 1 is the maximum of those shares. While monopoly incurs the usual deadweight loss, the level of social welfare in monopoly may be larger than that in duopoly. More precisely, for each value of marginal cost, we find a threshold for network externality such that monopoly yields a higher welfare level than duopoly if and only if network externality is stronger than the threshold. Further, the threshold is non-decreasing in the cost parameter.

The proposition that competition may not translate to a welfare gain is not new and was previously noted in the following theoretical studies. Correia-da-Silva *et al.* (2019) consider a model where platforms supply a homogeneous service and compete in quantity (which differs from our assumption of price competition). They find that a merger between platforms may benefit or harm consumers on both sides of the market depending on the extent of network effects. Consumers are better off with fewer platforms if network effects are strong enough. In a two-sided market model that allows sellers to price their products, Ko and Shen (2021) study conditions under which market dominance arises and its impacts on welfare. With strong network effects, competition gives rise to a dominant (monopolist) platform that captures all sellers and buyers and the resulting social welfare is higher than in a symmetric equilibrium with evenly split market shares. In multi-sided markets, Tan and Zhou (2021) also show that the effects of competition can be at odds with the received wisdom. As platform competition intensifies, market prices and platform profits can rise and consumer surplus can fall. In particular, there exists a cutoff for the number of platforms in the market, above which the perverse pattern about prices and platform profits materializes. Our results on welfare implications of platform competition are in line with these studies. The impact of competition on price and welfare has been investigated from an empirical angle too. Analyzing data from the Canadian newspaper industry in the late 1990s, which underwent a series of mergers, Chandra and Collard-Wexler (2009) observe that increased concentration did not yield higher prices for subscription or advertising.[1] Mergers may have disparate impacts on two sides of the market. Based on data from the 1996-2006 U.S. radio industry, Jeziorski (2014) finds that consolidation of radio stations led to a small increase in listener welfare and a substantial decrease in advertiser welfare. In an investigation of data from the German TV magazine market, Song (2021) shows that even with higher post-merger ad prices, advertisers are not necessarily worse off and the effects on readers are negligible.

It should also be noted that some papers report mixed welfare effects of platform competition, unlike the studies mentioned above. Anderson and Peitz (2020) study media markets consisting of multi-homing advertisers and single-homing viewers. They establish a "see-saw" relationship between advertisers and viewers: if entry decreases total platform profits, as would often be the case, advertiser surplus falls whereas viewer surplus rises. In a two-sided market involving buyers and sellers, Teh *et al.* (2023) show that increased platform competition results in a lower total fee although the fee structure may change in several directions after entry. Whether the shift in the fee structure favors buyers or sellers is determined by the fraction of buyers multi-homing and only one side may gain. In terms of modeling, our approach follows seminal contributions by Rochet and Tirole (2003, 2006) and Armstrong (2006), which are further developed recently by Tan and Zhou (2021) and Teh *et al.* (2023). Similarly to these papers, we take prices as the platforms' key instrument for strategizing and let them engage in Bertrand competition. Our results on platform competition in oligopoly

[1] Chandra and Collard-Wexler (2009) also provide a theoretical model that shows the effects of a merger on prices for readers and advertisers to be ambiguous.

turn out starker than the existing results by Correia-da-Silva *et al.* (2019), Ko and Shen (2021), and Tan and Zhou (2021). This is due largely to some simplifying assumptions we work with, which renders analysis very tractable. The assumptions include the distribution of the intrinsic benefit users derive from platforms, the homogeneous technology of platforms, the single-homing behavior of users, and the membership-fee feature of prices.

While we are concerned with Bertrand competition, several papers investigate the effects of Cournot competition among platforms on social welfare. An early contribution by Schiff (2003) compares three market structures: monopoly and Cournot duopolies with single- and multi-homing. If we confine attention to single-homing, monopoly achieves higher social welfare than a Cournot duopoly with single-homing does. As discussed above, Correia-da-Silva *et al.* (2019) find conditions under which a merger between platforms benefits or harms users, assuming that platforms compete in quantity. Tremblay *et al.* (2023) permit various homing possibilities in Cournot platform competition and show that social welfare decreases as the number of platforms increases for the most commonly considered homing allocations. Our findings on welfare consequences of platform competition echo main messages of these papers under the different assumption of Bertrand competition.

The rest of the paper is organized as follows. We set up a model for two-sided markets in Section 2 and analyze platforms' equilibrium behavior in oligopoly in Section 3. To assess welfare implications of the latter, we study platform monopoly as a benchmark in Section 4 and draw a comparison of the two in Section 5. For a smooth passage, all proofs and additional observations are relegated to Appendix A.

2. THE MODEL

Consider a market consisting of two sides. Let $\boldsymbol{S} \equiv \{1,2\}$ be the set of sides. Each side $s \in S$ has a unit mass of users (customers), denoted by $\boldsymbol{I}_s \equiv [0,1]$. Given the symmetry we assume below for the two sides, it is most plausible to interpret users as consumers of match-making service (e.g., subscription to dating platforms), seeking to interact with agents on the other side. Users derive utility from the interaction and platforms can facilitate it. Suppose that $\boldsymbol{K} \geq 2$ identical platforms operate in the market. In a slight abuse of notation, we denote the set of platforms by $\boldsymbol{K} \equiv \{1,\cdots,K\}$ too. To accommodate a unit mass of users from side $s \in S$, each platform $k \in K$ incurs a zero fixed cost and a constant marginal cost $\boldsymbol{c} \in \mathbb{R}_+$ and charges a price $\boldsymbol{p_{ks}} \in \mathbb{R}_+$ to them for the service it provides.

By joining a platform, a user on side $s \in S$ derives two types of utility. One is the **intrinsic membership benefit**, denoted $\boldsymbol{X_s}$, that the user earns from consuming services provided by the platform. We assume that X_s is uniformly distributed on $[0,1]$, with its distribution function denoted by $\boldsymbol{F}$. The realization of X_s is privately known to an individual user while its distribution is common knowledge among all users and platforms. The second is **network effects** that arise from interacting, on the chosen platform, with users on the other side and its magnitude is determined in part by the mass of users joining that platform. For each $k \in K$ and each $t \in S\backslash\{s\}$, let $\boldsymbol{n_{kt}} \in [0,1]$ be the measure of users on side t joining platform k. Let $\boldsymbol{\alpha} \in (0,1)$ be the parameter for network effects. When a user on side s adopts platform k, he earns the network benefit of αn_{kt}. The term αn_{kt} captures the network externality that is inherent to user experience on platforms. In sum, if a user on side s joins platform k that has measure n_{kt} of users from side t and pays p_{ks}, his payoff is $X_s + \alpha n_{kt} - p_{ks}$; the payoff from not joining at all is zero. Let $\boldsymbol{n_s} \equiv \sum_{k \in K} n_{ks}$ be the total measure of users on side s joining some platform.

Our game proceeds in two stages. First, platforms simultaneously set prices. Next, observing the price profile, users on both sides simultaneously choose a platform to join, if any. For simplicity, we do not allow the possibility of multi-homing.

Before introducing an equilibrium notion, let us describe strategies for all players. For each $k \in K$, **platform *k*'s (pure) strategy** is a pair $(p_{k1}, p_{k2}) \in \mathbb{R}^S_+$ of prices. Let $\boldsymbol{p} \equiv (p_{k1}, p_{k2})_{k \in K} \in \mathbb{R}^{S \times K}_+$ be the profile of prices set by all platforms. For each $i \in I_1 \cup I_2$, **user *i*'s (pure) strategy** is a mapping $\boldsymbol{z_i} : \mathbb{R}^{S \times K}_+ \to K \cup \{0\}$, associating with each price profile $p \in \mathbb{R}^{S \times K}_+$ his choice of a platform $z_i(p) \in K \cup \{0\}$, where for convenience, we denote by 0 the option of not joining any platform. Let $\boldsymbol{z} \equiv (z_i)_{i \in I_1 \cup I_2}$ be the profile of user strategies. Given p and z, for each $k \in K$ and each $s \in S$, let $\boldsymbol{m_{ks}(p,z)} \equiv \int_{i \in I_s} \mathbf{1}_{\{z_i(p)=k\}}\, di$ be the measure of users on side s joining platform k.[2]

As a solution concept, we consider a refinement of subgame perfect equilibrium. Let us call a strategy profile $\boldsymbol{(p,z)} \equiv ((p_{k1}, p_{k2})_{k \in K}, (z_i)_{i \in I_1 \cup I_2})$ an **equilibrium** if the following conditions hold:

(i) (*User payoff maximization*) For all distinct $s, t \in S$, each $i \in I_s$, and each

[2]Here $\mathbf{1}_{\{z_i(p)=k\}}$ is the indicator function whose value is 1 if $z_i(p) = k$ and 0 otherwise.

$q \in \mathbb{R}_+^{S\times K}$,

$$z_i(q) \in \arg\max_{k\in K\cup\{0\}} X_s + \alpha m_{kt}(q, z_i', z_{-i}) - q_{ks}, \tag{1}$$

where $z_i' : \mathbb{R}_+^{S\times K} \to K \cup \{0\}$ satisfies $z_i'(q) = k$ and for notational simplicity, we take the objective function to be zero if $k = 0$.

(ii) (*Platform profit maximization*) For each $k \in K$,

$$(p_{k1}, p_{k2}) \in \arg\max_{q_{k1}, q_{k2}\in\mathbb{R}_+} (q_{k1} - c)\, m_{k1}\left((q_{k1}, q_{k2}), (p_{\ell 1}, p_{\ell 2})_{\ell\in K\setminus\{k\}}, z\right) + (q_{k2} - c)\, m_{k2}\left((q_{k1}, q_{k2}), (p_{\ell 1}, p_{\ell 2})_{\ell\in K\setminus\{k\}}, z\right). \tag{2}$$

(iii) (*User coordination*) For all $k, \ell \in K$, all $s, t \in S$, and each $q \in \mathbb{R}_+^{S\times K}$, if $q_{ks} < q_{\ell s}$, $q_{kt} \le q_{\ell t}$, and $m_{\ell s}(q,z) > 0$, then $m_{ks}(q,z) > m_{\ell s}(q,z)$ and $m_{kt}(q,z) \ge m_{\ell t}(q,z)$.

Conditions (i) and (ii) are the usual optimality condition for users and platforms required by subgame perfection. On the other hand, condition (iii), the user coordination condition, refines the concept of subgame perfection by steering users to platforms with lower prices. It comes into play only when one platform is more price-competitive than another and leads users to hold the expectation that cheaper platforms will attract more users.

The user coordination condition is a reasonable restriction on user expectation and captures the prominent role played by price in markets with network externality. Particularly in an early stage of competition, price is a key criterion guiding users' platform choice. If network effects are presumed to be relatively small or their magnitudes are hard to anticipate, a user would determine which platform to join by comparing the price a platform charges and the intrinsic membership benefit it offers to him. Thus, platforms with lower prices are likely to have more users on board. The user coordination condition is also a common feature of consumer behavior that platforms recognize and exploit in real-life two-sided markets. In digital industries where the marginal cost of serving an additional user is small, platforms often provide services for free or at negligible prices. The objective of this pricing scheme is to expand the platforms' reach and market share first, allow network effects to grow, and then charge fees for premium services and advertising once it can assert dominance in the market. In our model, the user coordination condition has the effect of escalating price competition among platforms. A consequence, shown in Theorem 1 below, is that for no platform can a positive profit be supported in equilibrium.

We search for a **symmetric equilibrium** $(\boldsymbol{p}, \boldsymbol{z})$ where all platforms charge the same prices and users are evenly divided across the platforms; i.e., for all

$k,\ell \in K$ and each $s \in S$, $p_{ks} = p_{\ell s}$ and $m_{ks}(p,z) = m_{\ell s}(p,z)$). This is a natural refinement of equilibrium because all platforms are completely symmetric. Also, it should be noted that our definition of symmetric equilibrium requires the platforms to choose the same price profile (i.e., for all $k,\ell \in K$, $(p_{k1}, p_{k2}) = (p_{\ell 1}, p_{\ell 2})$), but not the same prices for the two sides (i.e., for any $k \in K$, $p_{k1} = p_{k2}$ is not required).

3. EQUILIBRIUM ANALYSIS

First, we show that all platforms earn a zero profit in symmetric equilibria. All proofs, including the one for the following theorem, are in Appendix A.

Theorem 1. *In each symmetric equilibrium, each platform earns a zero profit.*

Our model contains elements of Bertrand competition with differentiated goods. Users earn the intrinsic benefit (X_s for each side $s \in S$) from platforms, which are heterogeneous across users. The conventional wisdom from the literature on Bertrand competition with differentiated goods says that firms should have positive profits. By contrast, Theorem 1 shows that competition drives platfroms' profits to zero, regardless of the level of network effects (α). This zero-profit result hinges critically on the user coordination condition we impose as a part of the definition of equilibrium in Section 2. The condition represents intense price competition in which platforms operate to win more users in the market and increase network effects. While it is not too demanding a requirement on consumer behavior in theory, users may not coordinate on platforms as systematically and neatly in reality. Users may respond to small price differentials differently or be led to different expectations on the level of network externality they can enjoy on each platform. Therefore, we interpret Theorem 1 as providing a theoretical limit to which platforms tend when price competition intensifies and users, unsure of which platform will emerge dominant, can base their platform choice solely on prices.

Some techniques used to prove Theorem 1 are also interesting. In showing that if a symmetric equilibrium were to give positive profits, a platform would have a profitable deviation, we draw on theorems on the comparative statics of fixed points by Villas-Boas (1997) and characterize measures of users joining a platform when a deviation arises. See Appendix A.2 for details.

Next, we turn to welfare implications of equilibrium outcomes. In a symmetric equilibrium, all platforms charge the same price to each side and there is an unambiguous way of measuring consumer surplus. Suppose that platforms choose prices (ρ_1, ρ_2) in the equilibrium, resulting in measures (n_1, n_2)

Table 1: Solution to system (3) of equations

Cases	Required conditions		Solution (n_1,n_2)
1	$0 \le \psi_1 \le K^2-\alpha^2$	$0 \le \psi_2 \le K^2-\alpha^2$	$\left(\frac{K^2(1-\rho_1)+K\alpha(1-\rho_2)}{K^2-\alpha^2}, \frac{K^2(1-\rho_2)+K\alpha(1-\rho_1)}{K^2-\alpha^2}\right)$
2	$\rho_1 \ge 1$	$\rho_2 \ge 1$	$(0,0)$
3	$\rho_1 \le \frac{\alpha}{K}$	$\rho_2 \le \frac{\alpha}{K}$	$(1,1)$
4	$\psi_1 \le 0$	$\rho_2 \le 1$	$(0,1-\rho_2)$
5	$\rho_1 \le 1$	$\psi_2 \le 0$	$(1-\rho_1,0)$
6	$\psi_1 \ge K^2-\alpha^2$	$\frac{\alpha}{K} \le \rho_2 \le 1+\frac{\alpha}{K}$	$\left(1,1+\frac{\alpha}{K}-\rho_2\right)$
7	$\frac{\alpha}{K} \le \rho_1 \le 1+\frac{\alpha}{K}$	$\psi_2 \ge K^2-\alpha^2$	$\left(1+\frac{\alpha}{K}-\rho_1,1\right)$

of users joining some platform. Then the **side-1 users' demand** for platform service is $\boldsymbol{D}_1\left(\boldsymbol{\rho}_1,\frac{n_2}{K}\right) \equiv \Pr\left[X_1 \ge \rho_1 - \alpha\frac{n_2}{K}\right] = 1-F\left(\rho_1-\alpha\frac{n_2}{K}\right)$; i.e., $n_1 = 1 - F\left(\rho_1-\alpha\frac{n_2}{K}\right)$. Thus, the **consumer surplus for side 1** is defined as $\boldsymbol{V}_1\left(\boldsymbol{\rho}_1,\frac{n_2}{K}\right) \equiv \int_{\rho_1}^{\infty} D_1\left(x,\frac{n_2}{K}\right)dx$. The latter depends not only on the price charged to consumers on side 1 but the measure of consumers joining the platform from side 2. For side 2, the demand $\boldsymbol{D}_2\left(\boldsymbol{\rho}_2,\frac{n_1}{K}\right)$ and consumer surplus $\boldsymbol{V}_2(\boldsymbol{\rho}_2,\frac{n_1}{K})$ are defined similarly. Since all platforms earn a zero profit in a symmetric equilibrium, the consumer surpluses for the two sides constitute social welfare.

To find equilibrium welfare levels, it is important to study the behavior of the following system of equations in (n_1,n_2):

$$\begin{cases} n_1 = 1-F\left(\rho_1-\frac{\alpha}{K}n_2\right); \\ n_2 = 1-F\left(\rho_2-\frac{\alpha}{K}n_1\right). \end{cases} \tag{3}$$

As we show below, the above system has a unique solution for all values of ρ_1 and ρ_2 (see Appendix A.1).

A concrete expression for the solution (n_1,n_2) depends on (ρ_1,ρ_2) and can be categorized into seven cases, as in Table 1. Let $\psi_1(\rho_1,\rho_2) \equiv K^2(1-\rho_1)+K\alpha(1-\rho_2)$ and $\psi_2(\rho_1,\rho_2) \equiv K^2(1-\rho_2)+K\alpha(1-\rho_1)$. The values of ψ_1 and ψ_2 play a key role in determining the nature of a solution. Figure 1 illustrates conditions required for each case in Table 1. The conditions rely on when the inequality constraint $0 \le n_s \le 1$ for $s \in S$ turns out to be binding. For instance, the solution (n_1,n_2) falls in the interior of the unit squre $[0,1]^2$ if only if $\psi_1(\rho_1,\rho_2),\psi_2(\rho_1,\rho_2) \in \left(0,K^2-\alpha^2\right)$ (observe that $\psi_1(\rho_1,\rho_2)$ and $\psi_2(\rho_1,\rho_2)$ are the numerators of the expression for the solution (n_1,n_2) in Case 1). According to system (3), as the (common) price ρ_1 for side 1 decreases to 0, n_1 increases to the upper bound 1 but when exactly it hits the upper bound hinges on the the price ρ_2 for side 2. At a high value of ρ_2, a relatively small measure of users on side 2 joins the platforms (n_2 decreases), so that n_1 reaches the upper

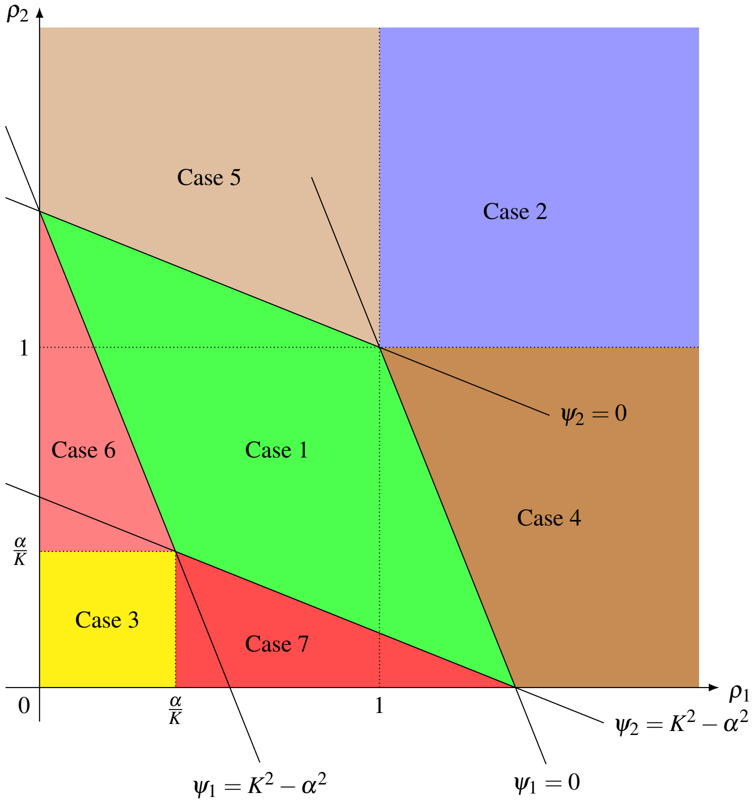

Figure 1: Illustrating the required conditions of Table 1 in the (ρ_1, ρ_2) space.

bound only when ρ_1 is quite low. As ρ_2 decreases, the threshold for ρ_1 at which the upper bound binds n_1 increases. This is why the border between Cases 1 and 6 slopes downward in Figure 1. The other three borders limiting Case 1 can be explained similarly. On the other hand, in Cases 2–7, at least one of n_1 and n_2 hits the upper bound 1 or the lower bound 0. For example, if $\rho_1, \rho_2 \geq 1$ as in Case 2, $\psi_1(\rho_1, \rho_2), \psi_2(\rho_1, \rho_2) \leq 0$, placing n_1 and n_2 at the lower bound 0. Here the threshold 1 is the maximum of the intrinsic benefit X_s ($s \in S$). If ρ_1 and ρ_2 are sufficiently small as in Case 3 ($\rho_1, \rho_2 \leq \frac{\alpha}{K}$), then facing low prices on both sides, all users optimally join the platforms, i.e., $n_1 = n_2 = 1$.

Now consider a symmetric equilibrium (p, z) such that for each $k \in K$,

$$(p_{k1}, p_{k2}) = (c, c).$$

This can be a benchmark equilibrium because the two sides of the market are treated symmetrically and all platforms make a zero profit (as Theorem 1 shows). In equilibrium (p,z), the total measures $(n_1,n_2) \equiv (\sum_{k\in K} m_{k1}(p,z), \sum_{k\in K} m_{k2}(p,z))$ of users joining any platform on the equilibrium path solve a version of system (3) where (ρ_1,ρ_2) is replaced by (c,c). Its welfare consequences are summarized in the following proposition (the proof is simple and is therefore omitted).

Proposition 1. *Consider a symmetric equilibrium in which each platform charges c to each side.*

(1) *If* $\frac{\alpha}{K} \leq c \leq 1$*, then the consumer surplus for each side is* $\frac{1}{2}\left[\frac{K(1-c)}{K-\alpha}\right]^2$ *and the social welfare is* $\left[\frac{K(1-c)}{K-\alpha}\right]^2$.

(2) *If* $c < \frac{\alpha}{K}$*, then the consumer surplus for each side is* $\frac{1}{2}+\frac{\alpha}{K}-c$ *and the social welfare is* $1+2\left(\frac{\alpha}{K}-c\right)$.

(3) *If* $c > 1$*, then the consumer surplus for each side and the social welfare are both zero.*

We may interpret the above proposition in conjunction with Table 1 and Figure 1. The equilibrium considered in the proposition has $(\rho_1,\rho_2) = (c,c)$. In Figure 1, this means that (ρ_1,ρ_2) lies on the 45-degree line through the origin, falling in one of Cases 1, 2, and 3. If $\frac{\alpha}{K} \leq \rho_1 = \rho_2 = c \leq 1$, then Case 1 applies and the total measures of users are $\left(\frac{K^2(1-\rho_1)+K\alpha(1-\rho_2)}{K^2-\alpha^2}, \frac{K^2(1-\rho_2)+K\alpha(1-\rho_1)}{K^2-\alpha^2}\right)$, resulting in the consumer surplus of $\frac{1}{2}\left[\frac{K(1-c)}{K-\alpha}\right]^2$ for each side. In line with our intuition, the latter quantity is decreasing in the marginal cost (c) and increasing in the network effect parameter (α). Next, if $\rho_1 = \rho_2 = c > 1$, i.e., the marginal cost exceeds the maximum intrinsic benefit from joining a platform, then Case 2 applies and the platforms' prices are too high to attract any user, leading to zero welfare gain for platforms and users. Finally, if $\rho_1 = \rho_2 = c < \frac{\alpha}{K}$, i.e., the marginal cost is smaller than the maximum level of externality ($\alpha \cdot \frac{1}{K}$) when all users join and are split evenly across all platforms, then Case 3 applies and all users adopt some platform, yielding the consumer surplus of $\frac{1}{2}+\frac{\alpha}{K}-c$ for each side. As long as $\rho_1 = \rho_2 = c < \frac{\alpha}{K}$, decreasing (ρ_1,ρ_2) further will not affect the equilibrium measures (n_1,n_2) of users on platforms as they have already reached the upper bound 1. However, when the prices $\rho_1 = \rho_2 = c$ fall by $\Delta c > 0$, the surplus for each user rises by Δc. This explains why the consumer surplus $\frac{1}{2}+\frac{\alpha}{K}-c$ for each side is linear in c. A similar interpretation is valid for linearity in $\frac{\alpha}{K}$.

4. THE CASE OF PLATFORM MONOPOLY

To better assess implications of platform competition, we analyze the case of platform monopoly in parallel. Suppose that a single platform operates in the market, with the same technology as in Section 2. Since there is only one platform, in this section we drop the subscript k that indicates the platform's identity and write $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ for its prices and $(\boldsymbol{n}_1, \boldsymbol{n}_2)$ for the measures of users joining it.

We continue to work with the solution concept introduced in Section 2 but it reduces to standard subgame perfection in this section. Further, user strategies need not enter analysis as prominently as before because their optimality is quite obvious. Suppose that the monopolist platform has set prices (p_1, p_2). Users decide only on whether or not to join the platform. Their strategies can be collectively represented by measures (n_1, n_2) of users on the two sides adopting the platform. At prices (p_1, p_2), the equilibrium measures (n_1, n_2) of joining users are determined by the following system of equations (this follows from an argument similar to that in Section 3):

$$\begin{cases} n_1 = 1 - F(p_1 - \alpha n_2) \\ n_2 = 1 - F(p_2 - \alpha n_1) \end{cases} \tag{4}$$

As we show in Appendix A.1, system (4) has a unique solution for all values of p_1 and p_2 and we can denote it by $\boldsymbol{n}_1(\boldsymbol{p}_1, \boldsymbol{p}_2)$ and $\boldsymbol{n}_2(\boldsymbol{p}_1, \boldsymbol{p}_2)$. This uniqueness alone determines which user should optimally join the platform. Therefore, we may leave out user strategies when describing equilibria.

The platform seeks to maximize its profit. That is, it chooses prices to solve:

$$\max_{p_1, p_2 \in \mathbb{R}_+} \pi(p_1, p_2) \equiv (p_1 - c) n_1(p_1, p_2) + (p_2 - c) n_2(p_1, p_2). \tag{5}$$

Assuming that user strategies are given by $n_1(p_1, p_2)$ and $n_2(p_1, p_2)$, we refer to the platform's strategy (p_1, p_2) as an **equilibrium** if it solves (5).

To search for equilibria, we need to solve (4) first. For completeness, we provide solution $(n_1(p_1, p_2), n_2(p_1, p_2))$ to (4) although it is similar to what we found in Section 3. Let $\phi_1(p_1, p_2) \equiv 1 - p_1 + \alpha(1 - p_2)$ and $\phi_2(p_1, p_2) \equiv 1 - p_2 + \alpha(1 - p_1)$. The types of the solution vary with values of ϕ_1 and ϕ_2 and are as in Table 2. Figure 2 illustrates conditions required for the seven cases in Table 2. Note that system (4) is similar to system (3). The only difference is that $\frac{\alpha}{K}$ in (3) is replaced by α (and the price variables ρ_1 and ρ_2 are replaced by p_1 and p_2). Consequently, the required conditions and the solution in Table 2 can be

Table 2: Solution to system (4) of equations

Cases	Required conditions		Solution (n_1, n_2)
1	$0 \le \phi_1 \le 1-\alpha^2$	$0 \le \phi_2 \le 1-\alpha^2$	$\left(\frac{1-p_1+\alpha(1-p_2)}{1-\alpha^2}, \frac{1-p_2+\alpha(1-p_1)}{1-\alpha^2}\right)$
2	$p_1 \ge 1$	$p_2 \ge 1$	$(0,0)$
3	$p_1 \le \alpha$	$p_2 \le \alpha$	$(1,1)$
4	$\phi_1 \le 0$	$p_2 \le 1$	$(0, 1-p_2)$
5	$p_1 \le 1$	$\phi_2 \le 0$	$(1-p_1, 0)$
6	$\phi_1 \ge 1-\alpha^2$	$\alpha \le p_2 < 1+\alpha$	$(1, 1+\alpha-p_2)$
7	$\alpha \le p_1 \le 1+\alpha$	$\phi_2 \ge 1-\alpha^2$	$(1+\alpha-p_1, 1)$

obtained by substituting $K = 1$ in Table 1. In Figure 2, this change shows in the form of a narrower diamond-shaped region for Case 1 relative to Figure 1. The interpretation we gave for $(\psi_1(\rho_1,\rho_2), \psi_2(\rho_1,\rho_2))$ in the discussion following system (3) can easily be adapted to $(\phi_1(p_1,p_2), \phi_2(p_1,p_2))$.

Next, we turn attention to the platform's problem, namely (5). As one can see from Table 2, the objective function is not differentiable at some points (whose measure is zero), which warrants some care in solving (5). Let $\boldsymbol{C} \equiv \{(p_1,p_2) \in \mathbb{R}^2_+ : 0 \le \phi_1(p_1,p_2) \le 1-\alpha^2$ and $0 \le \phi_2(p_1,p_2) \le 1-\alpha^2\}$. This is the diamond-shaped region in Figure 2, on the interior of which the objective function of (5) is differentiable. For the purpose of profit maximization, we can restrict attention to (p_1,p_2)'s in C because there always exists $(p_1,p_2) \in C$ that yields the maximum profit. This observation allows us to take the usual first-order approach when the solution lies in the interior of C. Appendix A.3 shows the first-order conditions for (5) and derives a variant of the Lerner formula from them. In general, the nature and value of the solution, i.e., the monopolistic equilibrium, are determined by the relative magnitudes of α and c. Our next result gives the exact relationship.

Theorem 2. *In the platform monopoly case, the following holds for equilibria.*

(1) *If* $2\alpha - 1 \le c \le 1$, *then the unique equilibrium is* $(p_1^*, p_2^*) = \left(\frac{c+1}{2}, \frac{c+1}{2}\right) \in [\alpha, 1]^2$.

(2) *If* $c < 2\alpha - 1$, *then the unique equilibrium is* $(p_1^*, p_2^*) = (\alpha, \alpha)$.

(3) *If* $c > 1$, *then the set of equilibria consists of* $(p_1^*, p_2^*) \in \mathbb{R}^2_+$ *with* $p_1^*, p_2^* \ge 1$, *all of which are outcome-equivalent.*

Remark 1. In the above equilibria, the resulting measures $(n_1(p_1^*,p_2^*), n_2(p_1^*,p_2^*))$ of users joining the platform are $\left(\frac{1-c}{2(1-\alpha)}, \frac{1-c}{2(1-\alpha)}\right)$ if $2\alpha - 1 \le c \le 1$, $(1,1)$ if $c < 2\alpha - 1$, and $(0,0)$ if $c > 1$. Also, although the above equilibria are of simple

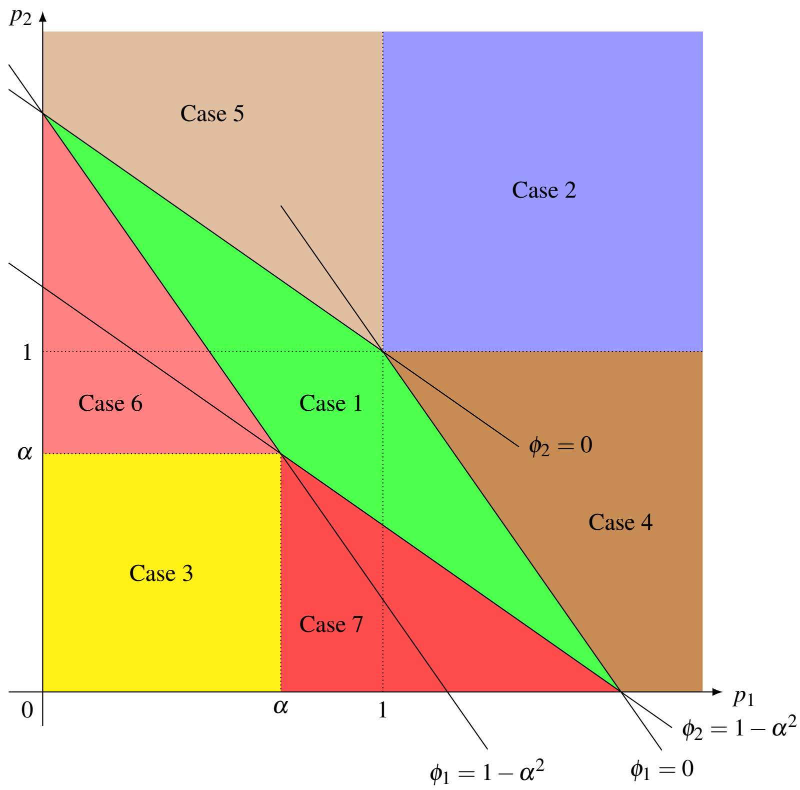

Figure 2: Illustrating the required conditions of Table 2 in the (p_1, p_2) space.

form, establishing them is not trivial, as our proof indicates. The main difficulty concerns non-differentiability of the objective function in (5) at some points in the constraint set, possibly at or around profit maximizers. $\triangle$

The equilibria characterized in Theorem 2 depend on c and α in an intuitive manner. When c is neither too high nor too low ($2\alpha - 1 \leq c \leq 1$), the monopolist platform's prices (p_1^*, p_2^*), belonging to the region C representing Case 1 in Figure 2, are increasing in c and hence the equilibrium measures of users are decreasing in c. As c falls below $2\alpha - 1$, however, the platform attracts all users on both sides and fixes the prices at α, corresponding to the point in Figure 2 where the region for Case 1 is tangent to the region for Case 3. When c exceeds 1, the maximum intrinsic benefit, optimality requires the platform to charge at least 1 to both sides, i.e., to choose any point in the region for Case 2 in Figure 2. This induces no user to join the platform and the resulting profit is zero regardless of which (p_1^*, p_2^*) with $p_1^*, p_2^* \geq 1$ is chosen.

Next, we derive welfare implications of monopoly pricing. With side 1's demand for platform service given by $D_1(p_1, n_2) = 1 - F(p_1 - \alpha n_2)$, the **consumer surplus for side 1** can be defined as $\boldsymbol{V_1(p_1, n_2)} \equiv \int_{p_1}^{\infty} D_1(x, n_2)\, dx$. The **consumer surplus for side 2**, $\boldsymbol{V_2(p_2, n_1)}$, is defined similarly. As usual, **social welfare** is the sum of the consumer surpluses for the two sides and the platform's profit. With Theorem 2 and Remark 1 at hand, it is simple to find consumer surplus and the platform's profit. Therefore, we state them below without a proof.

Proposition 2. *In the platform monopoly case, equilibria have the following welfare implications.*

(1) *If* $2\alpha - 1 \leq c \leq 1$*, then the platform's profit is* $\frac{(1-c)^2}{2(1-\alpha)}$ *and the consumer surplus for each side is* $\frac{(1-c)^2}{8(1-\alpha)^2}$*, so that the social welfare is* $\frac{(1-c)^2(3-2\alpha)}{4(1-\alpha)^2}$.

(2) *If* $c < 2\alpha - 1$*, then the platform's profit is* $2(\alpha - c)$ *and the consumer surplus for each side is* $\frac{1}{2}$*, so that the social welfare is* $2(\alpha - c) + 1$.

(3) *If* $c > 1$*, then the platform's profit and the consumer surplus for each side are both zero (and so is the social welfare).*

As was the case with Theorem 2, the welfare levels in the above proposition depend on the relative magnitudes of c and α. For a moderate value of c with $2\alpha - 1 \leq c \leq 1$, the platform charges $\frac{c+1}{2}$ to both sides. Thus, a decrease in c raises the equilibrium measures of users on the platform, their consumer surplus, and the platform's profit. If c is too low in the sense of satisfying $c < 2\alpha - 1$, the prices are fixed at $p_1^* = p_2^* = \alpha$ and the full measures of users adopt the platform. Thus, the platform appropriates any further decrease in c as its profit (and hence the functional form $2(\alpha - c)$) and the consumer surplus remains constant.

5. WELFARE COMPARISON

Let us now compare welfare levels achieved when multiple or a single platform operates in the market. Denote by $\boldsymbol{SW(K, \alpha, c)}$ the level of social welfare when there are $K \in \mathbb{N}$ platforms in the market, the network effect parameter is $\alpha \in (0,1)$, and the cost parameter is $c \geq 0$. Our next result says that for oligopolies, social welfare decreases as more platforms are in the market; and which among duopoly and monopoly yields higher social welfare depends on the magnitudes of network externality and marginal cost.

Theorem 3. (1) *For each* $c \in [0,1)$ *and each* $\alpha \in (0,1)$*,* $SW(K, \alpha, c)$ *is decreasing in* K *when* $K \geq 2$.

(a)

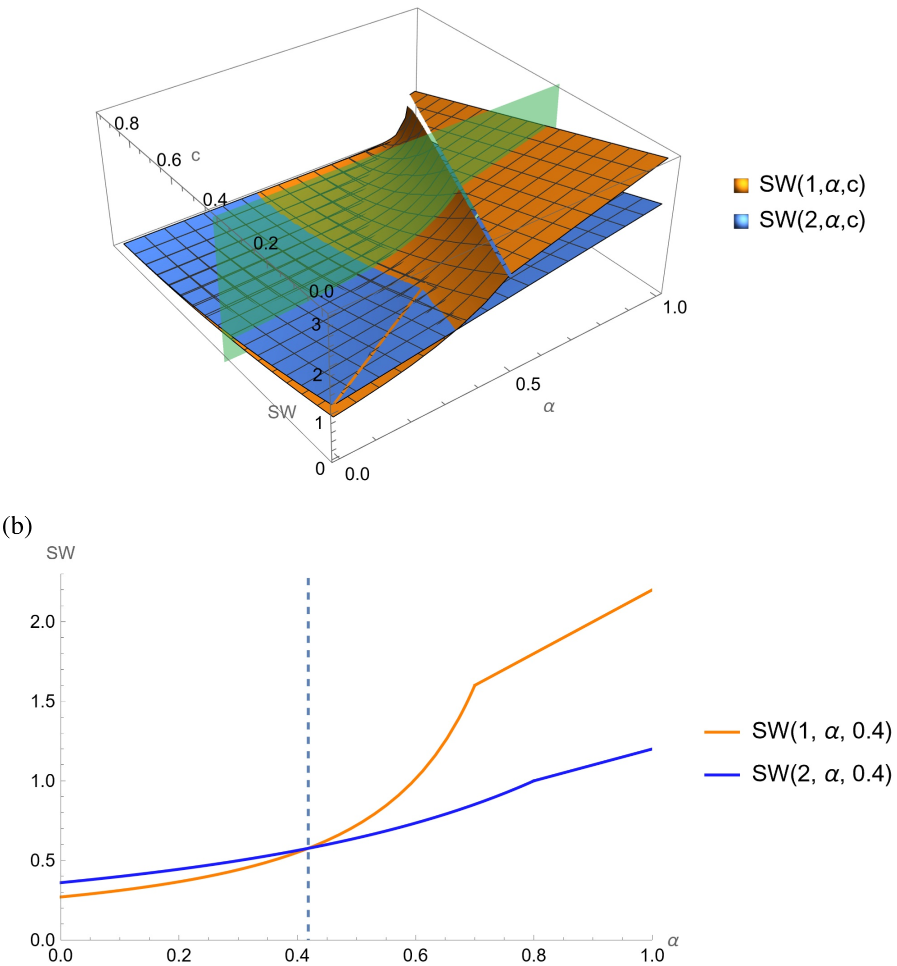

Figure 3: Plotting the graphs of $SW(K,\alpha,c)$ for $K \in \{1,2\}$

(2) *For each* $c \in [0,1)$*, there is a threshold* $\hat{\alpha}(c) \in (0,1)$ *such that* $SW(1,\alpha,c) > SW(2,\alpha,c)$ *if and only if* $\alpha > \hat{\alpha}(c)$*. In fact, letting* $\overline{\alpha} \in (0,1)$ *be the solution to* $\frac{3-2\alpha}{4(1-\alpha)^2} = \frac{4}{(2-\alpha)^2}$,[3] *if* $c \geq \frac{\overline{\alpha}}{2}$*, then* $\hat{\alpha}(c) = \overline{\alpha}$*; and if* $c < \frac{\overline{\alpha}}{2}$*, then* $\hat{\alpha}(c)$ *is the solu-*

[3]Numerically, $\overline{\alpha} \approx 0.418545\cdots$.

tion to $\frac{(1-c)^2(3-2\alpha)}{4(1-\alpha)^2} = 1+\alpha-2c$*,* $\hat{\alpha}(c) < \overline{\alpha}$*, and* $\hat{\alpha}(c)$ *is increasing in* c*.*

Remark 2. When $c \geq 1$, welfare comparison becomes trivial because consumer surplus and the platforms' profit all fall to zero regardless of how many platforms operate in the market. △

In view of Proposition 1, part (1) of Theorem 3 is quite intuitive. We already know that for $K \geq 2$, $SW(K,\alpha,c)$ is $\left[\frac{K(1-c)}{K-\alpha}\right]^2$ if $\frac{\alpha}{K} \leq c \leq 1$, $1+2\left(\frac{\alpha}{K}-c\right)$ if $c < \frac{\alpha}{K}$, and 0 if $c > 1$. The expressions $\left[\frac{K(1-c)}{K-\alpha}\right]^2$ and $1+2\left(\frac{\alpha}{K}-c\right)$ are decreasing in K. Using this, one can show that for any fixed $c \in [0,1)$ and $\alpha \in (0,1)$, $SW(K,\alpha,c) > SW(K+1,\alpha,c)$ although the threshold $\frac{\alpha}{K}$ also varies as K increases.

Concerning part (2) of Theorem 3, we provide a visual illustration. Panel (a) of Figure 3 plots the graphs of $SW(K,\alpha,c)$ for $K \in \{1,2\}$ (the vertical axis measures the social welfare level $SW(\cdot)$ and the other two axes measure α and c). The **orange** and **blue** surfaces represent $SW(1,\alpha,c)$ and $SW(2,\alpha,c)$, respectively. As hinted in Propositions 1 and 2, the two surfaces are continuous but not smooth. For relatively small values of α and c, the blue surface is above the orange surface. However, as α and c grow, the orange surface gradually rises above the blue surface. At what value of α this reversal takes place depends on c. If c is large enough ($c \geq \frac{\overline{\alpha}}{2}$), the two surfaces intersect at the same $\hat{\alpha}(c) = \overline{\alpha}$ but otherwise, $\hat{\alpha}(c)$ decreases as c decreases. To better understand the behavior of $SW(1,\alpha,c)$ and $SW(2,\alpha,c)$ at a fixed c, say $c = 0.4$, we place a vertical plane in transparent **green** in panel (a). Cutting through the orange and blue surfaces at $c = 0.4$, the green plane emphasizes how the orange surface overtakes the blue surface as α increases. Panel (b) is a two-dimensional view of the intersection of the green plane with the orange and blue surfaces. It plots $SW(1,\alpha,c)$ and $SW(2,\alpha,c)$ while fixing c at 0.4 (which exceeds $\frac{\overline{\alpha}}{2}$) and varying only α. The **orange** curve representing $SW(1,\alpha,0.4)$ and the **blue** curve representing $SW(2,\alpha,0.4)$ are both increasing in α. The orange curve crosses the blue curve from below at $\hat{\alpha}(0.4) = \overline{\alpha} \approx 0.418545\cdots$, as indicated by Theorem 3.

The message of Theorem 3 that competition can decrease social welfare is in line with some earlier theoretical (Correia-da-Silva *et al.*, 2019; Ko and Shen, 2021; Tan and Zhou, 2021) and empirical (Chandra and Collard-Wexler, 2009; Song, 2021) contributions. We are able to provide a complete answer to which market structure achieves higher social welfare because the stylized nature of our model renders welfare comparison tractable. It should also be noted that Theorem 3 relies fundamentally on Theorem 1, which is quite different from the typical insight from the literature on Bertrand competition with differentiated goods.

Another interesting point of comparison pertains to equilibrium prices in different market structures. Let us compare the monopoly equilibrium in Theorem 2 to the oligopoly equilibrium in Proposition 1 where all platforms set their prices for both sides equal to the marginal cost c. If $2\alpha - 1 \leq c \leq 1$, then the monopoly price $\frac{c+1}{2}$ exceeds the oligopoly price c (since $c < 1$). A similar observation is valid when $c < 2\alpha - 1$. Therefore, users on both sides face higher prices in monopoly than in oligopoly (for any $K \geq 2$). This consequence differs from Chandra and Collard-Wexler (2009). The latter study proves that mergers in a two-sided market do not necessarily raise the price for either side of the market and supports the findings empirically based on data from the Canadian newspaper industry. The difference between our results and Chandra and Collard-Wexler (2009) can be attributed to the fact that with Bertrand competition in effect in our model, each platform in oligopoly earns a zero profit and any price above c is not realized in equilibrium.

6. CONCLUDING REMARKS

In a simple model of two-sided markets, we studied the behavior of platforms competing in price and compared its welfare implications with those of platform monopoly. Our first result concerned platforms' profit in oligopoly. In each symmetric equilibrium, the profit for each platform is zero. This zero-profit result is quite unexpected because firms engaged in Bertrand competition with differentiated goods usually earn positive profits. Then we turned attention to monopoly and characterized the optimal pricing for the monopolist platform. Our last result drew a comparison of the welfare levels attained in oligopoly and monopoly. Unlike in markets without network effects, social welfare is decreasing in the number of platforms. This leaves only two market structures, duopoly and monopoly, as candidates for welfare maximization. Which of the latter two is welfare-dominant depends on the magnitudes of network effects and the platforms' marginal cost. If network effects are weak relative to the marginal cost, duopoly maximizes social welfare. With strong network effects in place, monopoly achieves the highest welfare level.

Our findings are established in a model with several simplifying assumptions. The assumptions limit relevance and applicability of the model. However, the resulting increase in tractability allows us to obtain cleaner results. These results are qualitatively consistent with earlier papers on platform competition, such as Correia-da-Silva *et al.* (2019), Ko and Shen (2021), and Tan and Zhou (2021). With regard to the issue of regulating two-sided markets, our findings serve as a cautionary tale and call for a nuanced approach to evaluating consequences of platform mergers and anti-trust policies. A broad message emerging from our results is that network externality should enter any complete analysis of platform industries as a key factor since it heavily influences which market structures or policy measures enhance social welfare.

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A. APPENDIX: PROOFS AND FURTHER OBSERVATIONS

This appendix contains proofs and observations that were omitted in Sections 3–5. We first establish uniqueness of the solution to system (4) because it is needed to prove Theorem 1. After proving Theorem 1, we derive the first-order conditions for the monopolist equilibrium. Finally, we prove Theorems 2 and 3

A.1. UNIQUENESS OF THE SOLUTION TO SYSTEM (4)

As a first step toward identifying equilibria, we study system (4) of equations in detail. Define a mapping $\Phi : [0,1]^2 \to [0,1]^2$ as for each $x \equiv (x_1, x_2) \in [0,1]^2$, $\Phi(x) = (1 - F(p_1 - \alpha_1 x_2), 1 - F(p_2 - \alpha_2 x_1)) \in [0,1]^2$. Notice that (n_1, n_2) is a solution to (4) if and only if it is a fixed point of Φ. Now we establish that (4) has a unique solution by showing that Φ is a contraction mapping.

Lemma 1. *For each* $(p_1, p_2) \in \mathbb{R}^2_+$ *and each* $\alpha \in (0,1)$*, system* (4) *of equations has a unique solution.*

Proof. It is enough to show that the mapping Φ is a contraction. Let $x, y \in [0,1]^2$. Note that

$$\|\Phi(x) - \Phi(y)\|^2 = [F(p_1 - \alpha x_2) - F(p_1 - \alpha y_2)]^2 + [F(p_2 - \alpha x_1) - F(p_2 - \alpha y_1)]^2.$$

We proceed in two steps.

Step 1: *Showing that* $[F(p_1 - \alpha x_2) - F(p_1 - \alpha y_2)]^2 \le [\alpha(x_2 - y_2)]^2$.

Case 1: $p_1 - \alpha x_2 \in (0,1)$ and $p_1 - \alpha y_2 \in (0,1)$.

$[F(p_1 - \alpha x_2) - F(p_1 - \alpha y_2)]^2 = (\alpha x_2 - \alpha y_2)^2$, so that the claim holds.

Case 2: $p_1 - \alpha x_2 \in (0,1)$ and $p_1 - \alpha y_2 \ge 1$.

This case holds if and only if $y_2 \le \frac{p_1 - 1}{\alpha} < x_2 < \frac{p_1}{\alpha}$, which, in particular, implies that $0 < x_2 - \frac{p_1 - 1}{\alpha} \le x_2 - y_2$. Then $[F(p_1 - \alpha x_2) - F(p_1 - \alpha y_2)]^2 = (\alpha x_2 - p_1 + 1)^2 \le [\alpha(x_2 - y_2)]^2$.

Case 3: $p_1 - \alpha x_2 \in (0,1)$ and $p_1 - \alpha y_2 \le 0$.

This case holds if and only if $\frac{p_1 - 1}{\alpha} < x_2 < \frac{p_1}{\alpha} \le y_2$, which, in particular, implies that $0 < \frac{p_1}{\alpha} - x_2 \le y_2 - x_2$. Then

$$[F(p_1 - \alpha x_2) - F(p_1 - \alpha y_2)]^2 = (p_1 - \alpha x_2)^2 \le [\alpha(y_2 - x_2)]^2.$$

Case 4: $p_1 - \alpha x_2 \ge 1$ and $p_1 - \alpha y_2 \in (0,1)$.

This case is symmetric to Case 2.

Case 5: $p_1 - \alpha x_2 \geq 1$ and $p_1 - \alpha y_2 \geq 1$.
Trivially, $[F(p_1 - \alpha x_2) - F(p_1 - \alpha y_2)]^2 = 0$.
Case 6: $p_1 - \alpha x_2 \geq 1$ and $p_1 - \alpha y_2 \leq 0$.
This case holds if and only if $x_2 \leq \frac{p_1-1}{\alpha} < \frac{p_1}{\alpha} \leq y_2$, which, in particular, implies $0 < \frac{1}{\alpha} = \frac{p_1}{\alpha} - \frac{p_1-1}{\alpha} < y_2 - x_2$. Thus, $[F(p_1 - \alpha x_2) - F(p_1 - \alpha y_2)]^2 = 1 \leq [\alpha(y_2 - x_2)]^2$.
Case 7: $p_1 - \alpha x_2 \leq 0$ and $p_1 - \alpha y_2 \in (0,1)$.
This case is symmetric to Case 3.
Case 8: $p_1 - \alpha x_2 \leq 0$ and $p_1 - \alpha y_2 \geq 1$.
This case is symmetric to Case 6.
Case 9: $p_1 - \alpha x_2 \leq 0$ and $p_1 - \alpha y_2 \leq 0$.
Trivially, $[F(p_1 - \alpha x_2) - F(p_1 - \alpha y_2)]^2 = 0$.

Step 2: *Showing that Φ is a contraction with Lipschitz constant $\alpha < 1$.*
An argument similar to that in Step 1 implies that

$$[F(p_2 - \alpha x_1) - F(p_2 - \alpha y_1)]^2 \leq [\alpha(x_1 - y_1)]^2.$$

Then

$$\|\Phi(x) - \Phi(y)\| \leq \left[\alpha^2(x_2 - y_2)^2 + \alpha^2(x_1 - y_1)^2\right]^{\frac{1}{2}} = \alpha \|x - y\|.$$

□

A.2. PROOF OF THEOREM 1

Before proving Theorem 1, we make some preliminary observations about the following system of equations:

$$\begin{cases} x_1 = 1 - F(p_1 - \alpha x_2) \\ x_2 = 1 - F(p_2 - \alpha x_1) \end{cases} \tag{6}$$

Denote the solution to (6) by $x(p_1, p_2, \alpha) \equiv (x_1(p_1, p_2, \alpha), x_2(p_1, p_2, \alpha))$. In the next two lemmas, we show that $x_1(\cdot)$ and $x_2(\cdot)$ respond to changes in (p_1, p_2, α) in an intuitive manner.

Lemma 2. *For each $s \in S$, $x_s(\cdot)$ is non-decreasing in α.*

Proof. Let $s \in S$. Let $\alpha, \tilde{\alpha} \in (0,1)$ satisfy $\alpha < \tilde{\alpha}$. Define $\tilde{\Phi}$ in the same way as Φ is defined, except that α is replaced by $\tilde{\alpha}$. Both Φ and $\tilde{\Phi}$ are contraction mappings. Further, $\Phi \leq \tilde{\Phi}$ (i.e., for each $x \in [0,1]^2$, $\Phi(x) \leq \tilde{\Phi}(x)$). Finally, $\tilde{\Phi}$ is non-decreasing in x. Thus, by Theorem 2 in Villas-Boas (1997), the fixed points $x(p_1, p_2, \alpha)$ and $x(p_1, p_2, \tilde{\alpha})$ of Φ and $\tilde{\Phi}$, respectively, satisfy $x(p_1, p_2, \alpha) \leq x(p_1, p_2, \tilde{\alpha})$. That is, each of $x_1(\cdot)$ and $x_2(\cdot)$ is non-decreasing in α. □

Lemma 3. *For all $s, t \in \{1, 2\}$, $x_s(\cdot)$ is non-increasing in p_t.*

Proof. Let $s, t \in \{1, 2\}$. Let $p_t, \hat{p}_t \in \mathbb{R}_+$ satisfy $p_t < \hat{p}_t$. Define $\hat{\Phi}$ in the same way as Φ is defined, except that p_t is replaced by $\hat{p}_t$ (but p_s remains unchanged). Both Φ and $\hat{\Phi}$ are contraction mappings. Further, $\Phi \geq \hat{\Phi}$ (i.e., for each $x \in [0,1]^2$, $\Phi(x) \geq \hat{\Phi}(x)$). Finally, Φ is non-decreasing in x. Thus, by Villas-Boas (1997, Theorem 2), the fixed points $x(p_s, p_t, \alpha)$ and $x(p_s, \hat{p}_t, \alpha)$ of Φ and $\hat{\Phi}$, respectively, satisfy $x(p_s, p_t, \alpha) \geq x(p_s, \hat{p}_t, \alpha)$. That is, each of $x_1(\cdot)$ and $x_2(\cdot)$ is non-increasing in p_t. □

With the aid of the above two lemmas, we now prove Theorem 1. Let (p, z) be a symmetric equilibrium. Denote the common prices in (p, z) by ρ_1 and ρ_2, respectively. Consider platform $k \in K$. For each $q_k \in \mathbb{R}^2_+$, let $\pi(q_k)$ be platform k's profit from charging q_k when all the other platforms charge $p_{-k} = (\rho_1, \rho_2)_{\ell \in K \setminus \{k\}}$. To show $\pi(\rho_1, \rho_2) = 0$, suppose, by contradiction that $\pi(\rho_1, \rho_2) > 0$. Clearly, $\rho_1 > c$ or $\rho_2 > c$ should hold. Without loss of generality, assume that $\rho_1 > c$. Distinguishing the following two cases, we show that platform k has a profitable deviation.

Case 1: $\rho_2 > 0$.

Let $\rho'_1 \in (c, \rho_1)$ and $\rho'_2 \in (0, \rho_2)$. Assume that platform k deviates to $p'_k \equiv (\rho'_1, \rho'_2)$. For each $(\ell, s) \in K \times S$, let $n_{\ell s} \equiv m_{\ell s}(p, z)$, $n'_{\ell s} \equiv m_{\ell s}(p'_k, p_{-k}, z)$, $n_s \equiv \sum_{\ell \in K} n_{\ell s}$, and $n'_s \equiv \sum_{\ell \in K} n'_{\ell s}$. At the price profile p, the total measures (n_1, n_2) of users joining any platform solve the following system of equations:

$$\begin{cases} n_1 = 1 - F\left(\rho_1 - \frac{\alpha}{K} n_2\right); \\ n_2 = 1 - F\left(\rho_2 - \frac{\alpha}{K} n_1\right). \end{cases} \tag{7}$$

Because (p, z) is a symmetric equilibrium, $n_{k1} = \frac{n_1}{K}$ and $n_{k2} = \frac{n_2}{K}$, so that platform k's profit is $\pi(\rho_1, \rho_2) = \frac{n_1}{K}(\rho_1 - c) + \frac{n_2}{K}(\rho_2 - c)$.

Now we show that for each $\ell \in K \setminus \{k\}$, $n'_{\ell 1} = n'_{\ell 2} = 0$. Let $\ell \in K \setminus \{k\}$. Suppose, by contradiction, that $n'_{\ell 1} > 0$; the case $n'_{\ell 2} > 0$ is similar. Since $p'_{k1} < p_{\ell 1}$ and $p'_{k2} < p_{\ell 2}$, the user coordination condition implies that $n'_{k1} > n'_{\ell 1}$. Together with $p'_{k2} < p_{\ell 2}$, this means that no user on side 2 would join platform ℓ, i.e., $n'_{\ell 2} = 0$. Now since $n'_{k2} \geq 0 = n'_{\ell 2}$ and $p'_{k1} < p_{\ell 1}$, no user on side 1 would join platform ℓ, i.e., $n'_{\ell 1} = 0$, a contradiction.

By the claim proved in the previous paragraph, $(n'_1, n'_2) = (n'_{k1}, n'_{k2})$ and $\pi(\rho'_1, \rho'_2) = n'_1(\rho'_1 - c) + n'_2(\rho'_2 - c)$. Further, (n'_1, n'_2) solves the following system of equations:

$$\begin{cases} n'_1 = 1 - F(\rho'_1 - \alpha n'_2); \\ n'_2 = 1 - F(\rho'_2 - \alpha n'_1). \end{cases} \tag{8}$$

Let $(\hat{n}_1, \hat{n}_2)$ be the solution to

$$\begin{cases} \hat{n}_1 = 1 - F(\rho_1 - \alpha \hat{n}_2); \\ \hat{n}_2 = 1 - F(\rho_2 - \alpha \hat{n}_1). \end{cases} \tag{9}$$

The parameters $\left(\frac{\alpha}{K}, \rho_1, \rho_2\right)$, $(\alpha, \rho'_1, \rho'_2)$, and (α, ρ_1, ρ_2) of (7), (8), and (9), respectively, satisfy $\frac{\alpha}{K} < \alpha$ and $(\rho_1, \rho_2) \gg (\rho'_1, \rho'_2)$. Applying Lemmas 2 and 3, it follows that $(n_1, n_2) \leqq (\hat{n}_1, \hat{n}_2) \leqq (n'_1, n'_2)$. Also, $(n'_1, n'_2) \to (\hat{n}_1, \hat{n}_2)$ as $(\rho'_1, \rho'_2) \to (\rho_1^-, \rho_2^-)$. Now to show that platform k gains from deviating to p'_k (i.e., $\pi(\rho_1, \rho_2) < \pi(\rho'_1, \rho'_2)$), we distinguish two subcases.

Case 1.1: $\rho_2 \geq c$.

Since $\pi(\rho_1, \rho_2) > 0$, either $n_1 > 0$ or $n_2 > 0$. Without loss of generality, assume that $n_1 > 0$.[4] Then $n_{k1} = \frac{n_1}{K} < n_1 \leq \hat{n}_1$. Observe that

$$\lim_{(\rho'_1, \rho'_2) \to (\rho_1^-, \rho_2^-)} n'_1(\rho'_1 - c) = \hat{n}_1(\rho_1 - c) \geq n_1(\rho_1 - c) > n_{k1}(\rho_1 - c); \text{ and}$$
$$\lim_{(\rho'_1, \rho'_2) \to (\rho_1^-, \rho_2^-)} n'_2(\rho'_2 - c) = \hat{n}_2(\rho_2 - c) \geq n_2(\rho_2 - c) \geq n_{k2}(\rho_2 - c).$$

Thus, for (ρ'_1, ρ'_2) with $c < \rho'_1 < \rho_1$ and $0 < \rho'_2 < \rho_2$ close enough to (ρ_1, ρ_2), $\pi(\rho'_1, \rho'_2) > \pi(\rho_1, \rho_2)$, so that (p, z) is not an equilibrium, a contradiction.

Case 1.2: $0 < \rho_2 < c$**.**

Since $\pi(\rho_1, \rho_2) = \frac{n_1}{K}(\rho_1 - c) + \frac{n_2}{K}(\rho_2 - c) > 0$ and $\rho_2 < c$, $n_1 > 0$. Also, as $(\rho'_1, \rho'_2) \to (\rho_1^-, \rho_2^-)$, $\pi(\rho'_1, \rho'_2) = n'_1(\rho'_1 - c) + n'_2(\rho'_2 - c) \to \hat{n}_1(\rho_1 - c) + \hat{n}_2(\rho_2 - c)$. For the claim that for (ρ'_1, ρ'_2) close enough to (ρ_1, ρ_2), $p'_k = (\rho'_1, \rho'_2)$ is a profitable deviation (i.e., $\pi(\rho'_1, \rho'_2) > \pi(\rho_1, \rho_2)$), it is enough to show that

$$\hat{n}_1(\rho_1 - c) + \hat{n}_2(\rho_2 - c) > n_1(\rho_1 - c) + n_2(\rho_2 - c). \tag{10}$$

The rest of the argument in Case 1.2 proves (10).

[4] Suppose that $n_1 = 0$. Then $\pi(\rho_1, \rho_2) > 0$ implies $n_2 > 0$ and $\rho_2 > c$. Together with $\rho_1 > c$, this case is similar to the case of $n_1 > 0$.

For each $(x_1,x_2) \in [0,1]^2$ and each $v \in (0,1)$, let $\varphi_1(x_1,x_2;v) \equiv x_1 + F(\rho_1 - vx_2) - 1$ and $\varphi_2(x_1,x_2;v) \equiv x_2 + F(\rho_2 - vx_1) - 1$, where we treat v as a parameter. By Lemma 1, for each $v \in (0,1)$, the system of equations $\varphi_1(x_1,x_2;v) = 0$ and $\varphi_2(x_1,x_2;v) = 0$ has a unique solution; call it $x(v) \equiv (x_1(v),x_2(v))$. Clearly, $x\left(\frac{\alpha}{K}\right) = (n_1,n_2)$ and $x(\alpha) = (\hat{n}_1,\hat{n}_2)$. Also, φ_1 and φ_2 are differentiable almost everywhere and continuous. Whenever the partial derivatives of φ_1 and φ_2 exist, $\frac{\partial \varphi_1}{\partial x_1} = \frac{\partial \varphi_2}{\partial x_2} = 1$, $\frac{\partial \varphi_1}{\partial x_2}, \frac{\partial \varphi_2}{\partial x_1} \in \{0,-v\}$, $\frac{\partial \varphi_1}{\partial v} \in \{0,-x_2\}$, and $\frac{\partial \varphi_2}{\partial v} \in \{0,-x_1\}$. The exact values of these partial derivatives depend on whether $F(\cdot)$ is constant on a neighborhood around $\rho_1 - vx_2$ and on a neighborhood around $\rho_2 - vx_1$. The function $x(v)$ is continuous in v and we may obtain its derivatives (when they exist) by applying the implicit function theorem:

$$\begin{pmatrix} x_1'(v) \\ x_2'(v) \end{pmatrix} = -\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_1}{\partial v} \end{pmatrix} = \frac{1}{1 - \frac{\partial \varphi_1}{\partial x_2}\frac{\partial \varphi_2}{\partial x_1}} \begin{pmatrix} \frac{\partial \varphi_2}{\partial v}\frac{\partial \varphi_1}{\partial x_2} - \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_1}{\partial v}\frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_2}{\partial v} \end{pmatrix}. \quad (11)$$

First, we show that $\hat{n}_1 - n_1 \geq \hat{n}_2 - n_2$. Observe that for each $s \in \{1,2\}$,

$$\hat{n}_s = x_s(\alpha) = x_s\left(\frac{\alpha}{K}\right) + \int_{[\frac{\alpha}{K},\alpha]} x_s'(v)\,dv = n_s + \int_{[\frac{\alpha}{K},\alpha]} x_s'(v)\,dv.$$

Thus, the claim is equivalent to $\int_{[\frac{\alpha}{K},\alpha]} x_1'(v)\,dv \geq \int_{[\frac{\alpha}{K},\alpha]} x_2'(v)\,dv$. For the latter, it suffices to show that for each $v \in \left[\frac{\alpha}{K},\alpha\right]$ at which x is differentiable, $x_1'(v) \geq x_2'(v)$. Let $v \in \left[\frac{\alpha}{K},\alpha\right]$. We distinguish the following four cases. **(a)** First, assume that $F(\cdot)$ is constant around $\rho_1 - vx_2$ and around $\rho_2 - vx_1$. By (11), $(x_1'(v),x_2'(v)) = (0,0)$. **(b)** Second, assume that $F(\cdot)$ is not constant around $\rho_1 - vx_2$ and is constant on around $\rho_2 - vx_1$. By (11), $(x_1'(v),x_2'(v)) = (x_2(v),0)$. **(c)** Third, assume that $F(\cdot)$ is not constant around $\rho_1 - vx_2$ and is not constant around $\rho_2 - vx_1$. By (11), $(x_1'(v),x_2'(v)) = \left(\frac{vx_1(v)+x_2(v)}{1-v^2}, \frac{x_1(v)+vx_2(v)}{1-v^2}\right)$. Under the assumption $\rho_1 > c > \rho_2$, for each $v \in (0,1)$, $x_1(v) \leq x_2(v)$, so that $x_1'(v) \geq x_2'(v)$. **(d)** Finally, the case where $F(\cdot)$ is constant around $\rho_1 - vx_2$ and is not constant around $\rho_2 - vx_1$ never arises. To see this, note that since $x_1(v) \geq x_1\left(\frac{\alpha}{K}\right) = n_1 > 0$ (recall that $x_1(v)$ is non-decreasing in v), $F(\cdot)$ can be constant around $\rho_1 - vx_2$ only if $F(\rho_1 - vx_2) = 0$, so that $x_1(v) = 1$. Under the assumption $\rho_1 > c > \rho_2$, $x_2(v) \geq x_1(v) = 1$, which means that $F(\rho_2 - vx_1) = 0$, a contradiction.

Since $\rho_1 > c > \rho_2$, $n_2 \geq n_1 > 0$. Further, since $\pi(\rho_1,\rho_2) > 0$, $\frac{c-\rho_2}{\rho_1-c} < \frac{n_1}{n_2} \leq 1$, so that $\rho_1 - c > c - \rho_2$. Combining the latter with the claim $\hat{n}_1 - n_1 \geq \hat{n}_2 - n_2$ proved in the previous paragraph yields (10).

Case 2: $\rho_2 = 0$.

Let $\rho_1' \in (c,\rho_1)$. Assume that platform k deviates to $p_k' \equiv (\rho_1',0)$. For each $(\ell,s) \in K \times S$, define $n_{\ell s}, n_{\ell s}'$, n_s, and n_s' as in Case 1. At the price profile p, the total measures (n_1,n_2) of users joining any platform solve the following system of equations[5]:

$$\begin{cases} n_1 = 1 - F\left(\rho_1 - \frac{\alpha}{K} n_2\right); \\ n_2 = 1. \end{cases} \tag{12}$$

That is, $n_1 = 1 - F\left(\rho_1 - \frac{\alpha}{K}\right)$ and $n_2 = 1$. Further, $n_{k1} = \frac{n_1}{K}$ and $n_{k2} = \frac{1}{K}$. Since $\pi(\rho_1,0) > 0$, $n_1 > 0$.

An argument similar to that in Case 1 shows that for each $\ell \in K \backslash \{k\}$, $n_{\ell 1}' = 0$. Thus, $n_1' = n_{k1}'$ and (n_1', n_2') solves:

$$\begin{cases} n_1' = 1 - F\left(\rho_1' - \alpha n_{k2}'\right); \\ n_2' = 1. \end{cases} \tag{13}$$

Next, we show that for each $\ell \in K \backslash \{k\}$, $n_{\ell 2}' = 0$. Let $\ell \in K \backslash \{k\}$. By the user coordination condition, $\forall h \in K \backslash \{k\}$, $n_{k2}' \geq n_{h2}'$. Thus, since $1 = n_2' = n_{k2}' + \sum_{h \in K \backslash \{k\}} n_{h2}'$, it follows that $n_{k2}' \geq \frac{1}{K}$. Then $0 < n_1 = 1 - F\left(\rho_1 - \frac{\alpha}{K}\right) \leq 1 - F\left(\rho_1' - \alpha n_{k2}'\right) = n_1'$. Since $n_{k1}' = n_1' > 0$, $n_{\ell 1}' = 0$, and $p_{k2}' = 0 = p_{k2}$, no user on side 2 would join platform ℓ, i.e., $n_{\ell 2}' = 0$.

In sum, $n_{k1}' = 1 - F(\rho_1' - \alpha_1)$ and $n_{k2}' = 1$, so that $\pi(\rho_1',0) = [1 - F(\rho_1' - \alpha)](\rho_1' - c) - c$. Now we distinguish two subcases.

Case 2.1: $F\left(\rho_1 - \frac{\alpha}{K}\right) = 0$.

This case implies $F(\rho_1' - \alpha) = 0$. Then $\pi(\rho_1,0) = \frac{1}{K}(\rho_1 - 2c) > 0$ and $\pi(\rho_1',0) = \rho_1' - 2c$. For $\rho_1' \in (c,\rho_1)$ close enough to ρ_1, $\pi(\rho_1',0) > \pi(\rho_1,0)$, so that (p,z) is not an equilibrium, a contradiction.

Case 2.2: $F\left(\rho_1 - \frac{\alpha_1}{K}\right) > 0$.

First, we show that there is $\rho_1' \in (c,\rho_1)$ such that

$$\left[1 - F(\rho_1' - \alpha)\right](\rho_1' - c) \geq \left[1 - F\left(\rho_1 - \frac{\alpha}{K}\right)\right](\rho_1 - c).$$

Suppose not. Then for each $\rho_1' \in (c,\rho_1)$,

$$\begin{aligned} \left[1 - F\left(\rho_1 - \frac{\alpha}{K}\right)\right](\rho_1' - c) &\leq \left[1 - F(\rho_1' - \alpha)\right](\rho_1' - c) \\ &< \left[1 - F\left(\rho_1 - \frac{\alpha}{K}\right)\right](\rho_1 - c), \end{aligned}$$

[5]Because $\rho_2 = 0$, all users on side 2 will join some platform.

where the first inequality follows from $F\left(\rho_1 - \frac{\alpha}{K}\right) \geq F(\rho_1' - \alpha)$. Letting $\rho_1' \to \rho_1^-$ in the above inequalities yields $F\left(\rho_1 - \frac{\alpha}{K}\right) = F(\rho_1 - \alpha)$. Since $\alpha > 0$ and $F\left(\rho_1 - \frac{\alpha}{K}\right) < 1$ (because $n_1 > 0$), this is possible only if $F\left(\rho_1 - \frac{\alpha}{K}\right) = 0$, a contradiction.

Now by the claim proved in the previous paragraph,

$$\begin{aligned}\pi(\rho_1', 0) &= \left[1 - F(\rho_1' - \alpha)\right](\rho_1' - c) - c \\ &\geq \left[1 - F\left(\rho_1 - \frac{\alpha}{K}\right)\right](\rho_1 - c) - c \\ &= K\pi(\rho_1, 0) \\ &> \pi(\rho_1, 0),\end{aligned}$$

so that (p, z) is not an equilibrium, a contradiction.

A.3. THE FIRST-ORDER CONDITIONS FOR THE MONOPOLIST EQUILIBRIUM

In this subsection, we provide the first-order conditions for the monopolist's profit maximization problem (5) when the solution lies in the interior of the region C (defined in Section 4). Then we turn them into a variant of the Lerner formula.

Let (p_1^*, p_2^*) be a solution to (5) lying in the interior of C. Since the objective function in (5) is differentiable, (p_1^*, p_2^*) is characterized by the following first-order conditions:

$$n_1 + (p_1^* - c)\frac{\partial n_1}{\partial p_1} + (p_2^* - c)\frac{\partial n_2}{\partial p_1} = 0; \tag{14}$$

$$n_2 + (p_2^* - c)\frac{\partial n_2}{\partial p_2} + (p_1^* - c)\frac{\partial n_1}{\partial p_2} = 0, \tag{15}$$

where $n_1 = n_1(p_1^*, p_2^*)$ and $n_2 = n_2(p_1^*, p_2^*)$. By the implicit function theorem,

$$\begin{pmatrix} \frac{\partial n_1}{\partial p_1} & \frac{\partial n_1}{\partial p_2} \\ \frac{\partial n_2}{\partial p_1} & \frac{\partial n_2}{\partial p_2} \end{pmatrix} = -\frac{1}{1 - \alpha^2}\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}.$$

With these expressions for $\frac{\partial n_s}{\partial p_t}$ ($s, t \in S$), Equations (14) and (15) can be written more concisely as

$$n_1 - \frac{1}{1 - \alpha^2}\left[p_1^* - c + \alpha(p_2^* - c)\right] = 0$$

and

$$n_2 - \frac{1}{1-\alpha^2}\left[p_2^* - c + \alpha(p_1^* - c)\right] = 0.$$

We can turn conditions (14) and (15) into a variant of the Lerner formula. For all $s,t \in S$, let $\eta_{st} \equiv -\frac{\partial n_s}{\partial p_t} \cdot \frac{p_t}{n_s}$ be side s's demand elasticity with respect to price p_t. The first-order conditions are equivalent to the following:

$$n_1 = n_1 \frac{p_1^* - c}{p_1^*}\eta_{11} + n_2 \frac{p_2^* - c}{p_1^*}\eta_{21};$$
$$n_2 = n_2 \frac{p_2^* - c}{p_2^*}\eta_{22} + n_1 \frac{p_1^* - c}{p_2^*}\eta_{12}.$$

A.4. PROOF OF THEOREM 2

In this subsection, we prove Theorem 2 by means of several lemmas. First, we show that in solving the profit maximization problem (5), it is enough to consider only (p_1, p_2)'s in C.

Lemma 4. *There exists* $(p_1, p_2) \in C$ *solving problem* (5).[6]

Proof. We show that for each $p \in \mathbb{R}^2_+ \backslash C$, there exists $p' \in C$ with $\pi(p') \geq \pi(p)$.
Case 1: $p_1, p_2 \geq 1$.
Clearly, $\pi(p_1, p_2) = 0 = \pi(1,1)$ and $(1,1) \in C$.
Case 2: $p_1 \leq 1$ and $\phi_2(p_1, p_2) < 0$.
At p, $n_1(p) = 1 - p_1$ and $n_2(p) = 0$. Let p_2' satisfy $\phi_2(p_1, p_2') = 0$. Then $p' \equiv (p_1, p_2') \in S$. Further, $n_1(p') = 1 - p_1$ and $n_2(p') = 0$, so that $\pi(p_1, p_2) = \pi(p_1, p_2')$.
Case 3: $p_2 \leq 1$ and $\phi_1(p_1, p_2) < 0$.
This case is symmetric to Case 2.
Case 4: $p_1, p_2 \leq \alpha$.
Then $n_1 = 1 = n_2$. Thus, $\pi(p) \leq \pi(\alpha, \alpha)$ and $(\alpha, \alpha) \in S$.
Case 5: $p_1 < \alpha$, $p_2 \geq \alpha$, and $\phi_1(p_1, p_2) > 1 - \alpha^2$.
Then $n_1(p) = 1$ and $n_2(p) = 1 + \alpha - p_2$. Let p_1' satisfy $\phi_1(p_1', p_2) = 1 - \alpha^2$. Then $p' \equiv (p_1', p_2) \in S$. Further, $n_1(p') = 1$ and $n_2(p') = 1 + \alpha - p_2$, so that $\pi(p_1, p_2) \leq \pi(p_1, p_2')$.
Case 6: $p_2 < \alpha$, $p_1 \geq \alpha$, and $\phi_2(p_1, p_2) > 1 - \alpha^2$.
This case is symmetric to Case 5. □

[6] This lemma does not exclude the possibility that there is another profit-maximizing choice $(p_1', p_2') \in \mathbb{R}^2_+ \backslash C$.

As indicated in Theorem 2, the values of α and c can be categorized into three cases: (i) $2\alpha-1 \le c \le 1$; (ii) $c < 2\alpha-1$; and (iii) $c > 1$. The next lemma identifies the solution to (5) for the first case.

Lemma 5. *Assume that* $2\alpha-1 \le c \le 1$. *Then* $(p_1^*, p_2^*) = (\frac{c+1}{2}, \frac{c+1}{2})$ *is the unique solution to* (5).

Proof. It suffices to show that for each $p \in C$ with $p \neq (\frac{c+1}{2}, \frac{c+1}{2})$, there is $p' \in C$ such that $\pi(p') > \pi(p)$. Let $p \in C \setminus \left\{(\frac{c+1}{2}, \frac{c+1}{2})\right\}$. Recall that $n_1(p) = \frac{1-p_1+\alpha(1-p_2)}{1-\alpha^2}$ and $n_2(p) = \frac{1-p_2+\alpha(1-p_1)}{1-\alpha^2}$. Let us distinguish two cases.

Case 1: $p_1 > p_2$ or $p_1 < p_2$.

Assume that $p_1 > p_2$; the case $p_1 < p_2$ is similar. Consider a small change from (p_1, p_2) to $(p_1 + dp_1, p_2 + dp_2) \in C$, where $dp_2 = -dp_1$ and $dp_1 < 0$ (i.e., move away from p along a line of slope -1 in the north-west direction).[7] Then

$$
\begin{aligned}
d\pi(p_1,p_2) &= \frac{\partial \pi(p_1,p_2)}{\partial p_1} dp_1 + \frac{\partial \pi(p_1,p_2)}{\partial p_2} dp_2 \\
&= -\frac{2(p_1-p_2)}{1+\alpha} dp_1 \\
&> 0.
\end{aligned}
$$

Thus, for $dp_1 < 0$ close enough to zero, $\pi(p_1 + dp_1, p_2 - dp_1) > \pi(p_1, p_2)$.

Case 2: $p_1 = p_2 > \frac{c+1}{2}$ or $p_1 = p_2 < \frac{c+1}{2}$.

Assume that $p_1 = p_2 > \frac{c+1}{2}$; the case $p_1 = p_2 < \frac{c+1}{2}$ is similar. Let $\rho \equiv p_1 (= p_2)$. Consider a small change from (p_1, p_2) to $(p_1 + dp_1, p_2 + dp_2) \in C$, where $dp_1 = dp_2 < 0$ (i.e., move away from p along a line of slope 1 in the south-west direction).

$$
\begin{aligned}
d\pi(p_1,p_2) &= \frac{\partial \pi(p_1,p_2)}{\partial p_1} dp_1 + \frac{\partial \pi(p_1,p_2)}{\partial p_2} dp_2 \\
&= -\frac{2(2\rho-c-1)}{1-\alpha} dp_1 \\
&> 0,
\end{aligned}
$$

where the inequality follows from $\rho > \frac{c+1}{2}$. Thus, for $dp_1 < 0$ close enough to zero, $\pi(p_1 + dp_1, p_2 + dp_1) > \pi(p_1, p_2)$. □

[7]Clearly, for any $p \in C$ with $p_1 < p_2$, if $dp_1 < 0$ is close enough to zero, then $(p_1 + dp_1, p_2 + dp_2) \in C$.

In view of the other two cases, the case $2\alpha - 1 \le c \le 1$ is one where the value of c is neither too small nor too big, so that the solution lies on the "diagonal" of region C. If c is small enough, in the sense of satisfying $c < 2\alpha - 1$, the platform lowers the prices to the extent that all users on both sides are on board. The following lemma makes this point.

Lemma 6. *Assume that* $c < 2\alpha - 1$. *Then* $(p_1^*, p_2^*) = (\alpha, \alpha)$ *is the unique solution to* (5).

Proof. Since uniqueness is simple to check, we only show that $(p_1^*, p_2^*) = (\alpha, \alpha)$ solves (5). For the latter claim, it suffices to prove that for each $(p_1, p_2) \in C$, if $\phi_1(p) < 1 - \alpha^2$ or $\phi_2(p) < 1 - \alpha^2$, then p is not a solution to (5). Assume that $\phi_1(p) < 1 - \alpha^2$; the case $\phi_2(p) < 1 - \alpha^2$ is similar. We distinguish two cases.

Case 1: $\phi_2(p) < 1 - \alpha^2$.

Then it is possible to decrease p_1 slightly without leaving C (and without affecting p_2). Further, using $n_1(p) = \frac{1-p_1+\alpha(1-p_2)}{1-\alpha^2}$ and $n_2(p) = \frac{1-p_2+\alpha(1-p_1)}{1-\alpha^2}$,

$$\begin{aligned} \frac{\partial \pi(p)}{\partial p_1} &= n_1(p) - \frac{1}{1-\alpha^2}(p_1 - c) - \frac{\alpha}{1-\alpha^2}(p_2 - c) \\ &= \frac{1}{1-\alpha^2}\left[2(1-p_1) + 2\alpha(1-p_2) + (1+\alpha)(c-1)\right] \\ &< \frac{c - 2\alpha + 1}{1-\alpha} \\ &< 0, \end{aligned}$$

where the first inequality follows from $\phi_1(p) < 1 - \alpha^2$. Thus, decreasing p_1 slightly will increase π, so that p is not profit-maximizing.

Case 2: $\phi_2(p) = 1 - \alpha^2$.

First, note that $\phi_2(p) = 1 - \alpha^2$ is possible only if $p_1 > \alpha$. Consider a small change from (p_1, p_2) to $(p_1 + dp_1, p_2 + dp_2) \in C$, where $dp_2 = -\alpha\, dp_1$ and $dp_1 < 0$ (i.e., move away from p along the line $\phi_2 = 1 - \alpha^2$ in the north-west direction). In this case, $n_1(p) = 1 + \alpha - p_1$ and $n_2(p) = 1$. Thus,

$$\begin{aligned} d\pi(p_1, p_2) &= \frac{\partial \pi(p_1, p_2)}{\partial p_1} dp_1 + \frac{\partial \pi(p_1, p_2)}{\partial p_2} dp_2 \\ &= (c + 1 - 2p_1)\, dp_1 \\ &> 0, \end{aligned}$$

where the inequality follows from $dp_1 < 0$ and $c + 1 - 2p_1 < c + 1 - 2\alpha < 0$. Thus, p is not profit-maximizing. □

Finally, we consider the case where c is large enough, that is, $c > 1$. Our next lemma shows that $(1,1)$ solves (5).

Lemma 7. *Assume that* $c > 1$. *Then* $(p_1^*, p_2^*) = (1,1)$ *is a solution to* (5).

Proof. It is enough to show that for each $(p_1, p_2) \in C$, if $\phi_1(p) > 0$ or $\phi_2(p) > 0$, then p is not a solution to (5). Assume that $\phi_1(p) > 0$; the case $\phi_2(p) > 0$ is similar. We distinguish two cases.

Case 1: $\phi_2(p) > 0$.

Then it is possible to increase p_1 slightly without leaving C (and without affecting p_2). Further, using $n_1(p) = \frac{1-p_1+\alpha(1-p_2)}{1-\alpha^2}$ and $n_2(p) = \frac{1-p_2+\alpha(1-p_1)}{1-\alpha^2}$,

$$\begin{aligned}\frac{\partial \pi(p)}{\partial p_1} &= n_1(p) - \frac{1}{1-\alpha^2}(p_1 - c) - \frac{\alpha}{1-\alpha^2}(p_2 - c) \\ &= \frac{1}{1-\alpha^2}[2(1-p_1) + 2\alpha(1-p_2) + (1+\alpha)(c-1)] \\ &> \frac{c-1}{1-\alpha} \\ &> 0,\end{aligned}$$

where the first inequality follows from $\phi_1(p) > 0$. Thus, increasing p_1 slightly will increase π, so that p is not profit-maximizing.

Case 2: $\phi_2(p) = 0$.

First, note that $\phi_2(p) = 0$ is possible only if $p_2 < 1$. Consider a small change from (p_1, p_2) to $(p_1 + dp_1, p_2 + dp_2) \in C$, where $dp_2 = -\alpha\, dp_1$ and $dp_1 > 0$ (i.e., move away from p along the line $\varphi_2 = 0$ in the south-east direction). In this case, $n_1(p) = 1 - p_1$ and $n_2(p) = 0$. Thus,

$$\begin{aligned}d\pi(p_1, p_2) &= \frac{\partial \pi(p_1, p_2)}{\partial p_1} dp_1 + \frac{\partial \pi(p_1, p_2)}{\partial p_2} dp_2 \\ &= (c + 1 - 2p_1) dp_1 \\ &> 0,\end{aligned}$$

where the inequality follows from $dp_1 > 0$ and $c + 1 - 2p_1 > c - 1 > 0$. Thus, p is not profit-maximizing. □

Now we characterize the set of equilibria and show that all equilibria are outcome-equivalent.

Lemma 8. *Assume that $c > 1$. Then each $(p_1^*, p_2^*) \in \mathbb{R}_+^2$ is a solution to* (5) *if and only if $p_1^*, p_2^* \geq 1$. Further, all solutions are outcome-equivalent.*

Proof. Let $(p_1^*, p_2^*) \in \mathbb{R}_+^2$. To prove the "if" part, it is enough to note that any (p_1^*, p_2^*) with $p_1^*, p_2^* \geq 1$ yields $n_1(p_1^*, p_2^*) = 0 = n_2(p_1^*, p_2^*)$ and $\pi(p_1^*, p_2^*) = 0 = \pi(1,1)$, so that (p_1^*, p_2^*) is an equilibrium, with its outcome equivalent to that of $(1,1)$.

Next, to prove the "only if" part, suppose, by contradiction, that (p_1^*, p_2^*) solves (5) but $p_1^* < 1$; the case $p_2^* < 1$ is similar. Then $n_1(p_1^*, p_2^*) > 0$, so that $n_1(p_1^*, p_2^*)(p_1^* - c) < 0$. If $p_2^* \leq c$, then $\pi(p_1^*, p_2^*) < 0 = \pi(1,1)$, a contradiction. Assume, henceforth, that $p_2^* > c$. If $\phi_2(p_1^*, p_2^*) \leq 0$, then $n_2(p_1^*, p_2^*) = 0$, so that $\pi(p_1^*, p_2^*) < 0 = \pi(1,1)$, a contradiction. If $\phi_2(p_1^*, p_2^*) > 0$, then as we saw in the proof of Lemma 7 (Case 1), increasing p_1^* slightly will increase the platform's profit slightly, a contradiction. □

A.5. PROOF OF THEOREM 3

Part (1). Let $c \in [0,1)$ and $\alpha \in (0,1)$. Choose $K \in \mathbb{N}$ with $K \geq 2$. We show that $SW(K,\alpha,c) > SW(K+1,\alpha,c)$. First, if $\alpha \leq Kc$, then $SW(K,\alpha,c) = \left[\frac{K(1-c)}{K-\alpha}\right]^2 > \left[\frac{(K+1)(1-c)}{K+1-\alpha}\right]^2 = SW(K+1,\alpha,c)$. Next, if $Kc < \alpha \leq (K+1)c$, then $SW(K,\alpha,c) = 1 + 2\left(\frac{\alpha}{K} - c\right) > 1 = SW(K+1,(K+1)c,c) \geq SW(K+1,\alpha,c)$. Finally, if $\alpha > (K+1)c$, then $SW(K,\alpha,c) = 1 + 2\left(\frac{\alpha}{K} - c\right) > 1 + 2\left(\frac{\alpha}{K+1} - c\right) = SW(K+1,\alpha,c)$.

Part (2). For each $\alpha \in (0,1)$, let $g(\alpha) \equiv \frac{3-2\alpha}{4(1-\alpha)^2} - \frac{4}{(2-\alpha)^2}$. It is simple to see that $g(\alpha) = 0$ has a unique solution in $(0,1)$; let us denote it by $\overline{\alpha}$ (numerically, $\overline{\alpha} \approx 0.418545\cdots$).[8] Then for each $\alpha \in (0,1)$, $\alpha > \overline{\alpha}$ implies $g(\alpha) > 0$; and $\alpha < \overline{\alpha}$ implies $g(\alpha) < 0$. Let us distinguish four cases.

Case 1: $\frac{1}{2} \leq c < 1$.

[8]To see this, since $\lim_{\alpha \to 0^+} g(\alpha) = -\frac{1}{4}$ and $\lim_{\alpha \to 1^-} g(\alpha) = \infty$, the intermediate value theorem implies that $g(\alpha) = 0$ has a sol in $(0,1)$. For uniqueness, it is enough to show that for each $\alpha \in (0,1)$, $g'(\alpha) > 0$. Let $\alpha \in (0,1)$. Note that $g'(\alpha) = \frac{1-\frac{\alpha}{2}}{(1-\alpha)^3} - \frac{1}{(1-\frac{\alpha}{2})^3}$ and $g''(\alpha) = \frac{\frac{5}{2}-\alpha}{(1-\alpha)^4} - \frac{\frac{3}{2}}{(1-\frac{\alpha}{K})^4}$. Since $\frac{5}{2} - \alpha > \frac{3}{2}$ and $1 - \alpha < 1 - \frac{\alpha}{2}$, $g''(\alpha) > 0$. Together with $\lim_{\alpha \to 0^+} g'(\alpha) = 0$, this means that for each $\alpha \in (0,1)$, $g'(\alpha) > 0$.

Then

$$\begin{aligned} SW(1,\frac{c+1}{2},c) &= 2-c \\ &> 2(1-c) \\ &> [2(1-c)]^2 \\ &= \sup_{\alpha\in(0,1)} SW(2,\alpha,c), \end{aligned}$$

where the last equality holds because given $c \geq \frac{1}{2}$, for each $\alpha \in (0,1)$, $\alpha < 1 \leq 2c$, so that for each $\alpha \in (0,1)$, $SW(2,\alpha,c) = \left(\frac{2(1-c)}{2-\alpha}\right)^2$. Thus, with c fixed in $[\frac{1}{2},1)$, $SW(2,\alpha,c)$ and $SW(1,\alpha,c)$ cross only in $(0,\frac{c+1}{2})$; further, for each $\alpha \in (0,\frac{c+1}{2})$, $SW(2,\alpha,c) = \frac{4(1-c)^2}{(2-\alpha)^2}$ and $SW(1,\alpha,c) = \frac{(1-c)^2(3-2\alpha)}{4(1-\alpha)^2}$. As we saw above, $g(\alpha) = 0$ has a unique solution $\overline{\alpha}$ in $(0,\frac{c+1}{2})$ and that is where $SW(1,\alpha,c)$ crosses $SW(2,\alpha,c)$ from below. Thus, $\hat{\alpha}(c) = \overline{\alpha}$.

Case 2: $\frac{1}{3} \leq c < \frac{1}{2}$.

Then

$$\begin{aligned} SW(1,\frac{c+1}{2},c) &= 2-c \\ &> 2(1-c) \\ &= \sup_{\alpha\in(0,1)} SW(2,\alpha,c), \end{aligned}$$

where the last equality follows from the fact that for $\alpha > 2c$, $SW(2,\alpha,c) = 1+\alpha-2c$. Thus, with c fixed in $[\frac{1}{3},\frac{1}{2})$, $SW(2,\alpha,c)$ and $SW(1,\alpha,c)$ cross only in $(0,\frac{c+1}{2})$; further, for each $\alpha \in (0,\frac{c+1}{2})$, $SW(2,\alpha,c) = \frac{4(1-c)^2}{(2-\alpha)^2}$ and $SW(1,\alpha,c) = \frac{(1-c)^2(3-2\alpha)}{4(1-\alpha)^2}$. Proceeding as in Case 1, we may take $\hat{\alpha}(c) = \overline{\alpha}$.

Case 3: $\frac{\overline{\alpha}}{2} \leq c < \frac{1}{3}$.

Then

$$\begin{aligned} SW(1,\frac{c+1}{2},c) &= 2-c \\ &> 2(1-c) \\ &= \sup_{\alpha\in(0,1)} SW(2,\alpha,c), \end{aligned}$$

where the last equality follows from the fact that for $\alpha > 2c$, $SW(2,\alpha,c) = 1+\alpha-2c$. Thus, with c fixed in $[\frac{\overline{\alpha}}{2},\frac{1}{3})$, $SW(2,\alpha,c)$ and $SW(1,\alpha,c)$ cross only in $(0, \frac{c+1}{2})$. Since for each $\alpha \in (0,2c]$, $SW(2,\alpha,c) = \frac{4(1-c)^2}{(2-\alpha)^2}$ and $SW(1,\alpha,c) = \frac{(1-c)^2(3-2\alpha)}{4(1-\alpha)^2}$, proceeding as in Case 2, it follows that in $(0,2c]$, $SW(1,\alpha,c)$ crosses $SW(2,\alpha,c)$ from below at $\alpha = \overline{\alpha}$. On the other hand, the two do not cross in $(2c, \frac{c+1}{2})$.[9] Thus, $\hat{\alpha}(c) = \overline{\alpha}$.

Case 4: $c < \frac{\overline{\alpha}}{2}$.

For each $\alpha \in (0,1)$, let $h(\alpha) \equiv SW(1,\alpha,c) - SW(2,\alpha,c)$. We proceed in four steps.

Step 1: *For each $\alpha \in (0,2c)$, $h(\alpha) < 0$.*

Let $\alpha \in (0,2c)$. Then $\alpha < \overline{\alpha}$, so that by our observation about $\overline{\alpha}$ above, $g(\alpha) < 0$. Since $h(\alpha) = (1-c)^2 g(\alpha)$, $h(\alpha) < 0$.

Step 2: *On $[2c, \frac{c+1}{2}]$, $h(\alpha) = 0$ has a unique solution, which we denote by $\hat{\alpha}(c)$.*

For each $\alpha \in [2c, \frac{c+1}{2}]$, $h(\alpha) = \frac{(1-c)^2(3-2\alpha)}{4(1-\alpha)^2} - (1+\alpha-2c)$. First, $h(\frac{c+1}{2}) = \frac{c+1}{2} > 0$. Second, since $h(2c) = \frac{(1-c)^2(3-4c)}{4(1-2c)^2} - 1 = (1-c)^2 g(2c)$ and for $c < \frac{\overline{\alpha}}{2}$, $g(2c) < 0$ (by our argument above), $h(2c) < 0$. Now the intermediate value theorem implies that $h(\alpha) = 0$ has a solution in $[2c, \frac{c+1}{2}]$. Next, concerning uniqueness, it is enough to show that $h' > 0$ on $[2c, \frac{c+1}{2}]$. Note that $h'(\alpha) = \frac{(1-c)^2(2-\alpha)}{2(1-\alpha)^3} - 1$ and $h''(\alpha) = \frac{(1-c)^2(5-2\alpha)}{2(1-\alpha)^4}$. Since $h''(\alpha) > 0$, $h'(\alpha)$ is minimized at $\alpha = 2c$. Since $h'(2c) = \frac{7c}{(1-2c)^3}\left[(c-\frac{9}{14})^2 + \frac{3}{196}\right] \geq 0$, with equality only if $c = 0$, the claim follows.

Step 3: *For each $\alpha \in (\frac{c+1}{2}, 1)$, $h(\alpha) > 0$.*

For each $\alpha \in [\frac{c+1}{2}, 1)$, $h(\alpha) = \alpha$. Since $h(\frac{c+1}{2}) = \frac{c+1}{2} > 0$ and for each $\alpha \in (\frac{c+1}{2}, 1)$, $h'(\alpha) = 1 > 0$, the claim follows.

Step 4: *$\hat{\alpha}(c)$ is an increasing function of c on $[0, \frac{\overline{c}}{2})$ and for each $c \in [0, \frac{\overline{\alpha}}{2})$, $\hat{\alpha}(c) < \overline{\alpha}$.*

The above three steps uniquely defines $\hat{\alpha}(c) \in [2c, \frac{c+1}{2}]$ for each $c \in [0, \frac{\overline{\alpha}}{2})$. Note that $h(\hat{\alpha}(c)) = 0$ can be written as $\left(1 - \frac{1-\hat{\alpha}(c)}{1-c}\right)^2 = \frac{1}{2} - \frac{1}{4(1-\hat{\alpha}(c))}$. To prove monotonicity of $\hat{\alpha}(c)$, suppose, by contradiction, that there exist $c, c' \in [0, \frac{\overline{\alpha}}{2})$ with $c < c'$ and $\hat{\alpha}(c) \geq \hat{\alpha}(c')$. Since $\hat{\alpha}(c) > c$ and $\hat{\alpha}(c') > c'$, $\left(1 - \frac{1-\hat{\alpha}(c)}{1-c}\right)^2 > \left(1 - \frac{1-\hat{\alpha}(c')}{1-c'}\right)^2$ but $\frac{1}{2} - \frac{1}{4(1-\hat{\alpha}(c))} \leq \frac{1}{2} - \frac{1}{4(1-\hat{\alpha}(c'))}$, a contradiction. Thus, $\hat{\alpha}(c)$ is

[9] For each $\alpha \in (2c, \frac{c+1}{2})$, $SW(2,\alpha,c) = 1+\alpha-2c$ and $SW(1,\alpha,c) = \frac{(1-c)^2(3-2\alpha)}{4(1-\alpha)^2}$. Now it is enough to note that $SW(2,2c,c) < SW(1,2c,c)$; and for each $\alpha \in (2c, \frac{c+1}{2})$, $\frac{\partial}{\partial\alpha}SW(2,\alpha,c) = 1$ and $\frac{\partial}{\partial\alpha}SW(1,\alpha,c) = \frac{(1-c)^2(2-\alpha)}{2(1-\alpha)^3}$

increasing in c.

Finally, the claim that for each $c \in [0, \frac{\overline{\alpha}}{2})$, $\hat{\alpha}(c) < \overline{\alpha}$ follows from $\hat{\alpha}\left(\frac{\overline{\alpha}}{2}\right) = \overline{\alpha}$ and monotonicity of $\hat{\alpha}(c)$ on $[0, \frac{\overline{\alpha}}{2})$.