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Estimating the Distribution of Net Worth and Disposable Income Using Johnson's S_U Distribution*

Pilsun Choi† Insik Min‡

Abstract While there has been increasing interest in the distribution of net worth and disposable income, these variables include negative values, making it impossible to use the distribution functions previously employed in estimating gross income distributions, as these distribution functions cannot accommodate negative values. In this study, we propose the Johnson's S_U distribution to estimate the distribution of net worth and disposable income. The S_U distribution function is defined over the entire real number space and is one of the most flexible parametric distribution functions for capturing a wide range of skewness and kurtosis. Therefore, it is highly suitable for estimating the various shapes of distributions for net worth and disposable income, which can include negative values. We derive the Lorenz curve for the S_U distribution in a closed-form expression. As an illustrative example, we apply both the univariate and bivariate S_U distributions to estimate the distributions and Lorenz curves of net worth and disposable income using the survey data of Korea.

Keywords Parametric distribution, Johnson's S_U, Lorenz curve, net worth, disposable income.

JEL Classification C13, C46, D31.

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†Department of International Trade, Konkuk University, 120 Neungdong-ro, Gwangjin-gu, Seoul, Republic of Korea 05029. E-mail: pchoi@konkuk.ac.kr.

‡Corresponding author. Department of Economics, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul, Republic of Korea 02447. E-mail: imin@khu.ac.kr.

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1. INTRODUCTION

Measuring the distribution and inequality of income and wealth in a society is always a significant concern, both in practical and policy contexts. Particularly, understanding and quantifying economic inequality is a central theme in socioeconomic disclosure. In this regard, the Lorenz curve stands as a major analytical tool, providing a graphical representation of income and wealth distributions. While several continuous parametric distribution functions have been proposed as methods for assessing the Lorenz curve and the Gini coefficient, most of them are focused on gross income and, as a result, are defined only in the positive or non-negative range. Examples include, among others, the log-normal, Weibull, Gamma, Singh-Maddala (Singh and Maddala, 1976), Dagum (Dagum, 1977), GB2 (McDonald, 1984), dPLN (Reed, 2003), and κG (Clementi *et al.*, 2007) distributions.

Recently, there has been an increasing interest in the distribution of net worth and disposable income, in addition to gross income. In analyses and policies related to wealth or income inequality, net worth is more meaningful than total assets, and disposable income is more relevant than gross income. Since net worth and disposable income data inevitably include negative values, most parametric distribution functions traditionally used for modeling the distribution and inequality of income (or assets) have the limitation of accommodating only positive values (Jäntti *et al.*, 2015).

In this study, we propose the Johnson's S_U distribution to estimate the distribution of net worth and disposable income. The S_U distribution function is defined over the entire real number space and is one of the most flexible parametric distribution functions for capturing a wide range of skewness and kurtosis. Therefore, it is highly suitable for estimating the various shapes of distributions for net worth or disposable income, which can include negative values. We derive the Lorenz curve for the S_U distribution function in a closed-form expression. As an illustrative example, we apply the S_U distribution to estimate the distribution and Lorenz curve of net worth and income using the dataset of Survey of Household Finances and Living Conditions (SHFLC) in Korea.

2. JOHNSON'S S_U DISTRIBUTION

The S_U distribution first appeared in the pathbreaking article of Johnson (1949a). The S_U variable X is generated by the transformation to normality in the following manner.

$$\sinh^{-1}\left(\frac{X-m}{s}\right) = \lambda + \theta Z, \quad -\infty < X < \infty, \quad s > 0, \quad \theta > 0,$$

where Z is a standard normal variable. The symbol S_U is for 'unbounded system' implying that the range of X is unbounded. The probability density function (PDF) of S_U is

$$f(x) = \frac{1}{\theta\sqrt{(x-m)^2+s^2}}\,\phi\left(\theta^{-1}\left[\sinh^{-1}\left(\frac{x-m}{s}\right)-\lambda\right]\right),$$

where $\phi(\cdot)$ is the PDF of a standard normal variable. The cumulative distribution function (CDF) of X is

$$F(x) = \Phi\left(\theta^{-1}\left[\sinh^{-1}\left(\frac{x-m}{s}\right)-\lambda\right]\right),$$

where $\Phi(\cdot)$ is the CDF of a standard normal variable. Johnson (1949a) provides the first four moments of X as follows.

$$\begin{aligned}
\mu_1 &\equiv \mu = m + s\omega^{1/2}\sinh(\lambda) \\
\mu_2 &\equiv \sigma^2 = \frac{1}{2}s^2(\omega-1)(\omega\cosh(2\lambda)+1) \\
\mu_3 &= \frac{1}{4}s^3\omega^{1/2}(\omega-1)^2\left[\omega(\omega+2)\sinh(3\lambda)+3\sinh(\lambda)\right] \\
\mu_4 &= \frac{1}{8}s^4(\omega-1)^2\Big[\omega^2(\omega^4+2\omega^3+3\omega^2-3)\cosh(4\lambda) \\
&\quad +4\omega^2(\omega+2)\cosh(2\lambda)+3(2\omega+1)\Big],
\end{aligned}$$

where $\omega = e^{\theta^2}$. The coefficients of skewness and kurtosis of X are respectively μ_3/σ^3 and μ_4/σ^4.

Johnson (1949a) shows that the S_U distribution is an extremely flexible distribution function capable of capturing the widest range of combinations of skewness and excess kurtosis. Due to this flexibility, the S_U distribution finds applications in various fields. Notably, it has performed well in modeling univariate and multivariate financial returns and estimating Value-at-Risk (Choi and Nam, 2008).

The Lorenz curve is a graphical representation of the distribution of wealth. For the S_U variable X and $p = F(x)$, the Lorenz curve is as follows (see Appendix).

$$L(p) = \frac{mp + \frac{1}{2}se^{\theta^2/2}\left(e^{\lambda}\Phi\left(\Phi^{-1}(p)-\theta\right) - e^{-\lambda}\Phi\left(\Phi^{-1}(p)+\theta\right)\right)}{m + se^{\theta^2/2}\sinh(\lambda)}.$$

The S_U distribution can be easily extended to multivariate dimensions (Johnson, 1949b). When an $N \times 1$ random vector $\mathbf{Z}$ follows a multivariate standard normal distribution, the joint PDF of $\mathbf{Z}$ is expressed as:

$$\phi_{\mathbf{R}}(\mathbf{z}) = (2\pi)^{-N/2}|\mathbf{R}|^{-1/2}\exp\left(-\frac{1}{2}\mathbf{z}'\mathbf{R}^{-1}\mathbf{z}\right),$$

where $\mathbf{R}$ is the correlation coefficient matrix with an off-diagonal element r_{ij}, and $|\mathbf{R}|$ is the determinant of $\mathbf{R}$. A multivariate S_U random vector $\mathbf{X}$ can be obtained by the inverse hyperbolic sine transformation of each variable X_i to a normal variable, i.e., $\sinh^{-1}\left(\frac{X_i - m_i}{s_i}\right) = \lambda_i + \theta_i Z_i$ where $s_i > 0$ and $\theta_i > 0$. Hence, the joint PDF of $\mathbf{X}$ is:

$$f(\mathbf{x}) = (2\pi)^{-N/2}|\mathbf{R}|^{-1/2}J \cdot \exp\left(-\frac{1}{2}\mathbf{z}'\mathbf{R}^{-1}\mathbf{z}\right), \tag{1}$$

where $z_i = \theta_i^{-1}\left[\sinh^{-1}\left(\frac{x_i - m_i}{s_i}\right) - \lambda_i\right]$, and $J = \prod_{i=1}^{N}\theta_i^{-1}\left[(x_i - m_i)^2 + s_i^2\right]^{-1/2}$.

In the literature, there is an approach that uses copula functions to construct the joint distribution of household income and wealth (Jäntti *et al.*, 2015, among others). The multivariate S_U model discussed in this study can also be understood using the concept of copulas. That is, the multivariate S_U distribution can be considered as a distribution that combines each marginal S_U variable using a Gaussian copula function. Since S_U variables are transformed from normal variables, it is quite simple to transform them back to normal variables and combine them using a Gaussian copula. It should be noted that $\mathbf{R}$ in (1) is the correlation matrix of $\mathbf{Z}$, which is the transformed variable from $\mathbf{X}$. Due to the nonlinear transformation, the correlation of $\mathbf{Z}$ is not the same as the correlation of $\mathbf{X}$. Rather, the Pearson's correlation coefficient ρ_{ij} between X_i and X_j is:

$$\begin{aligned}\rho_{ij} = \frac{e^{\frac{\theta_i^2+\theta_j^2}{2}}}{\sigma_i\sigma_j}\Big[&\frac{1}{2}e^{r_{ij}\theta_i\theta_j}\cosh(\lambda_i+\lambda_j) \\ &-\frac{1}{2}e^{-r_{ij}\theta_i\theta_j}\cosh(\lambda_i-\lambda_j) - \sinh(\lambda_i)\sinh(\lambda_j)\Big],\end{aligned} \tag{2}$$

where $\sigma_k = \left[\frac{1}{2}\left(e^{\theta_k^2}-1\right)\left(e^{\theta_k^2}\cosh(2\lambda_k)+1\right)\right]^{1/2}$, $k = i, j$. If $i = j$, then ρ_{ij} becomes 1. Inversely, when $\mathbf{X}$ follows a multivariate S_U distribution with correlation matrix Σ, whose off-diagonal element is ρ_{ij}, the correlation r_{ij} between Z_i and Z_j is:

$$r_{ij} = \frac{1}{\theta_i\theta_j}\ln\left(\frac{B_{ij}+\sqrt{B_{ij}^2+\cosh(\lambda_i+\lambda_j)\cosh(\lambda_i-\lambda_j)}}{\cosh(\lambda_i+\lambda_j)}\right),$$

where $B_{ij} = \rho_{ij}\sigma_i\sigma_j\exp\left(-\frac{1}{2}(\theta_i^2+\theta_j^2)\right)+\sinh(\lambda_i)\sinh(\lambda_j)$.

Consider a bivariate S_U distribution with r, the correlation coefficient between Z_1 and Z_2. From (1), the joint PDF is:

$$f(x_1,x_2) = \frac{1}{2\pi\sqrt{1-r^2}}J\exp\left(-\frac{1}{2(1-r^2)}\left(z_1^2-2rz_1z_2+z_2^2\right)\right),$$

where $z_1 = \theta_1^{-1}\left[\sinh^{-1}\left(\frac{x_1-m_1}{s_1}\right)-\lambda_1\right]$, $z_2 = \theta_2^{-1}\left[\sinh^{-1}\left(\frac{x_2-m_2}{s_2}\right)-\lambda_2\right]$, and $J = (\theta_1\theta_2)^{-1}\left([(x_1-m_1)^2+s_1^2][(x_2-m_2)^2+s_2^2]\right)^{-1/2}$. The conditional distribution of X_1 given $X_2 = x_2$ is of the same S_U system as X_1, but with λ_1 and θ_1 replaced, respectively, by $\lambda_1^* = \lambda_1 + r\theta_1\theta_2^{-1}\left(\sinh^{-1}\left(\frac{x_2-m_2}{s_2}\right)-\lambda_2\right)$, and $\theta_1^* = \theta_1\sqrt{1-r^2}$ (Kotz *et al.*, 2000):

$$X_1 \mid X_2 = x_2 \sim S_U(m_1, s_1, \lambda_1^*, \theta_1^*). \tag{3}$$

3. FITTING EXAMPLE

In the 2023 SHFLC dataset, we focus on the two variables that are of particular importance for household finances: net worth and disposable income. Both are measured in units of one million Korean won, with a total of 18,904 household observations. Table 1 presents summary statistics for the two variables.[1] It is noticeable that both variables exhibit extreme positive skewness and excess kurtosis, indicating that the S_U distribution can be an appropriate distribution. The last row in the table provides the empirical Gini coefficients, revealing that

[1]In our study, all statistics were calculated using the weights provided in the SHFLC, and the empirical Gini coefficients were also calculated using the weighted method as described by Lerman and Yitzhaki (1989). Additionally, the maximum likelihood estimation (MLE) performed below maximizes the weighted log-likelihood function.

	Net Worth	Disposable Income
Mean	435.4	54.8
Standard Deviation	726.3	46.5
Skewness	10.3	5.1
Kurtosis	264.7	137.1
Gini ($\times$100)	60.9	40.6

Table 1: SUMMARY STATISTICS. Except for the Gini coefficient, all statistics are in million won. It is noticeable that both net worth and disposable income exhibit extreme positive skewness and excess kurtosis.

	Net Worth	Disposable Income
$\hat{m}$	-21.0	-3.5
$\hat{s}$	34.4	15.2
$\hat{\lambda}$	2.6	1.8
$\hat{\theta}$	1.2	0.7
Gini ($\times$100)	64.8	41.3

Table 2: ESTIMATION RESULTS OF S_U MODEL. Parameter estimation was performed using maximum likelihood estimation with the L-BFGS-B algorithm in the Python SciPy optimize module.

the level of inequality measured by asset is considerably higher than income inequality.

Maximum likelihood estimation was performed using the Python SciPy 'optimize' module with limited-memory BFGS (L-BFGS-B) algorithm. When comparing the Gini coefficient estimated by the S_U model [2] in Table 2 with the empirical Gini coefficient in Table 1, the former is higher than the latter for both variables.

As shown in the histograms in the left panels of Figures 1 and 2, it is evident that both net worth and disposable income exhibit distributions that are extremely positively skewed. The estimated S_U distributions appear to represent the empirical distributions quite well. The right panels of Figures 1 and 2 display the empirical and estimated Lorenz curves. For disposable income, the Lorenz curves of the S_U model and the empirical one are nearly identical, consistent with the close similarity of their Gini coefficients, as observed in Tables 1 and 2.

[2]The Gini coefficients of the S_U model were calculated by numerical integration of the estimated Lorenz curve.

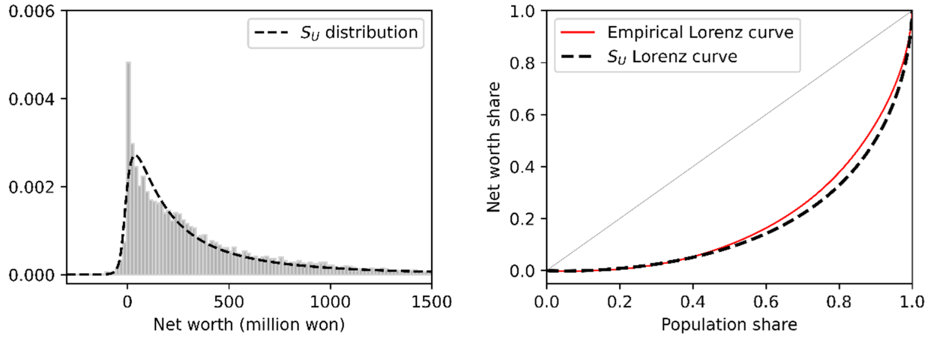

Figure 1: ESTIMATED DISTRIBUTION AND LORENZ CURVE USING S_U MODEL FOR NET WORTH. The estimated S_U distribution appear to represent the empirical distribution quite well.

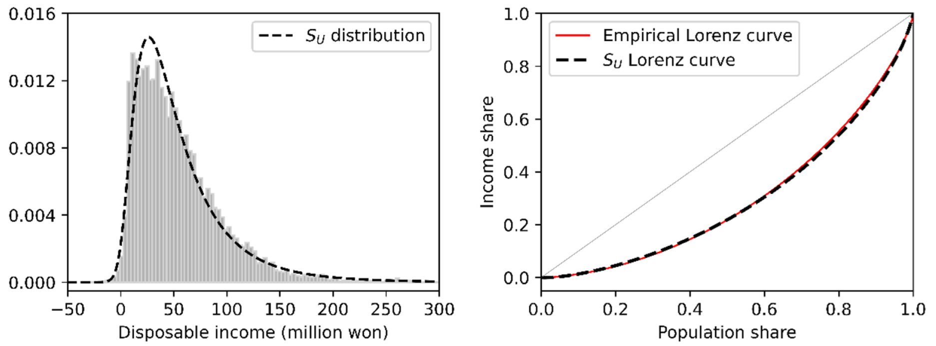

Figure 2: ESTIMATED DISTRIBUTION AND LORENZ CURVE USING S_U MODEL FOR DISPOSABLE INCOME. The Lorenz curves of the S_U model and the empirical one are nearly identical.

To assess the usefulness and flexibility of the S_U distribution, we compare its goodness of fit with two other distribution functions: the log-normal and the Generalized Beta of the Second Kind (GB2). The former is the most traditional and basic distribution function for estimating income distribution, while the latter is considered one of the best-performing distributions in terms of goodness of fit in the income distribution literature (McDonald *et al.*, 2013). The log-normal distribution has two parameters, while the GB2 distribution, like the S_U distribution, has four parameters.

One issue here is that the log-normal and GB2 distribution functions cannot

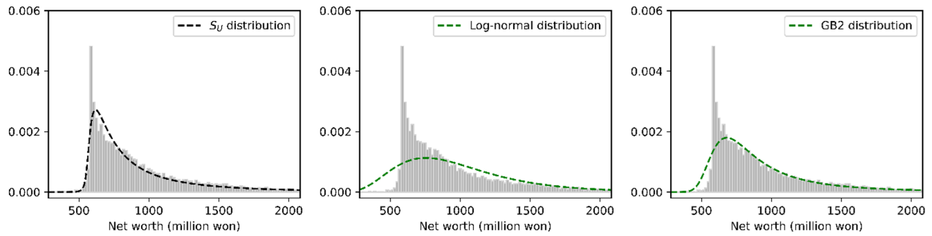

Figure 3: ESTIMATED PDFS FOR LOCATION-SHIFTED NET WORTH. The S_U distribution (left) performs significantly better than the log-normal (center) and appears slightly better than the GB2 (right).

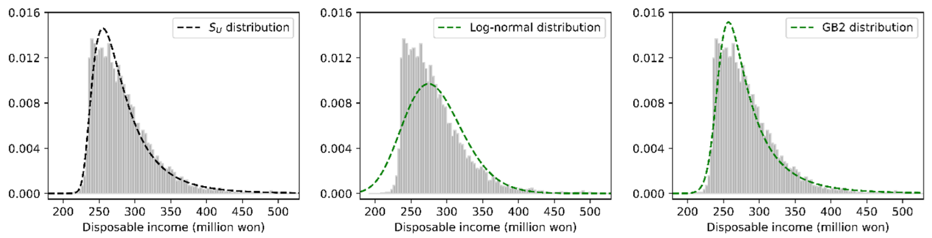

Figure 4: ESTIMATED PDFS FOR LOCATION-SHIFTED DISPOSABLE INCOME. It is evident that the log-normal distribution (center) has a poorer fit compared to the S_U (left) and GB2 (right).

accommodate zero or negative values. As a result, it is inherently impossible to use these distributions to estimate the distributions of net worth and disposable income, which are the focus of this study. That said, selecting a new variable that only takes positive values (such as gross income or assets) to compare the goodness of fit of the three distributions would not align with the objective of this study—namely, introducing a flexible distribution function capable of covering the entire real number range for estimating economic well-being variables that include zero and negative values.

Considering this, we decided to use the original net worth and disposable income dataset but apply a location-shift by adding a constant value to all data points, ensuring that all values become positive. We then compare the goodness of fit of the three distribution functions based on the transformed data. For the S_U model, such a location-shift does not practically affect its goodness of fit, making it possible to conduct a valid comparison with the log-normal and GB2 distributions. For each variable, we added a constant just enough to shift the minimum value slightly above zero.

Figures 3 and 4 show the estimated PDFs of net worth and disposable in-

	Net Worth			Disposable Income		
	S_U	Log-normal	GB2	S_U	Log-normal	GB2
χ^2	522.1	9,450.8	3,175.7	854.0	5,301.9	1,061.2
MAD	1.4	7.2	3.4	4.0	7.6	3.7

Table 3: GOODNESS-OF-FIT. χ^2 measures the differences between observed and expected frequencies, while MAD (multiplied by 100) measures the differences between the empirical and model-based distribution functions.

come, after applying location-shifts to ensure all values are positive. In the figures, the leftmost panel presents the estimation results based on the S_U distribution, which remain virtually identical to those in Figures 1 and 2, except for the shift in location. When comparing the empirical histogram with the model-based estimated distributions in the figures, it is evident that the log-normal distribution has a poorer fit compared to the S_U and GB2 distributions.

To conduct a more formal comparison of goodness of fit, we use two measures. The first is derived from Pearson's chi-square goodness-of-fit test, which is defined as the sum of the squared differences between observed and expected frequencies (i.e., counts of observations), each divided by the expected frequency:

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

where O_i is the observed count for bin i, and E_i is the expected count for bin i, as asserted by the model. The number of bins k was determined according to Sturges's rule.

The other goodness-of-fit measure is based on the Kolmogorov–Smirnov (KS) test, which is a nonparametric test for the equality of one-dimensional probability distributions. The KS-type goodness-of-fit measure is defined as the mean absolute difference (MAD) between the empirical distribution function $F_n(x)$ and the model-based distribution function $F(x)$ across all n observations:

$$\text{MAD} = \frac{1}{n} \sum_{i=1}^{n} |F_n(x_i) - F(x_i)| .$$

Table 3 presents the goodness-of-fit results for the three distribution functions. Consistent with the findings from Figures 3 and 4, the log-normal distribution shows a significantly poorer fit compared to the S_U and GB2. When

Parameters	Net Worth		Disposable Income
$\hat{m}$	-19.8		-3.0
$\hat{s}$	40.6		19.5
$\hat{\lambda}$	2.5		1.6
$\hat{\theta}$	1.2		0.7
$\hat{r}$		0.52	

Table 4: ESTIMATION RESULTS OF BIVARIATE S_U MODEL. The correlation parameter $\hat{r}$ takes a positive value exceeding 0.5, indicating a strong correlation between net worth and disposable income.

comparing the performance of S_U and GB2 distribution functions, the S_U outperforms GB2 in both goodness-of-fit measures for net worth. For disposable income, however, S_U exhibits a better fit than GB2 based on the χ^2, whereas GB2 performs better than S_U in terms of the MAD. The goodness of fit for the S_U distribution in this context can be considered at least comparable to the performance of GB2, which is regarded as one of the best-fitting distributions for variables that take only positive values.[3]

Next, we estimated the joint distribution of net worth and disposable income using the bivariate S_U distribution function presented in (2). In fact, it is possible to perform the estimation in a two-step manner, where individual marginal distributions are estimated first and then the correlation parameters among them are estimated. Such a two-step estimation may be used when dealing with a large number of variables. However, in our case with only two variables, we estimated all parameters in one step.

Table 4 presents the estimated parameters of the bivariate S_U distribution. Comparing with the individual univariate model in Table 1, we find that the parameter estimates, particularly the shape parameters (λ and θ), are very close to each other. Additionally, the correlation parameter r takes a positive value exceeding 0.5, and the Pearson's correlation coefficient between net worth and disposable income, calculated using equation (2), is 0.38, which is slightly lower than the empirical correlation coefficient of 0.43.

Finally, based on the bivariate estimation, we illustratively derived the con-

[3]Our objective is to quantitatively assess how well the S_U distribution approximates the empirical data compared to other distributions, notably the log-normal and GB2. Accordingly, we do not present p-values, as our primary goal is to evaluate the models' relative performance through goodness-of-fit statistics.

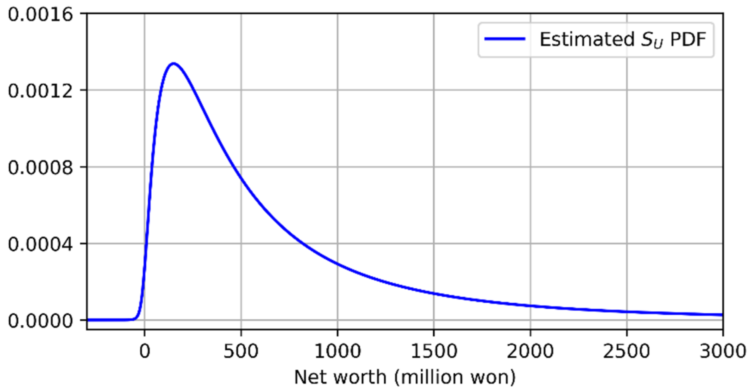

Figure 5: NET WORTH DISTRIBUTION FOR HOUSEHOLDS WITH A DISPOSABLE INCOME OF 100 MILLION WON. The ability to explicitly derive this kind of conditional distribution is another advantage of the S_U distribution.

ditional distribution. Specifically, we chose to estimate the net worth distribution for households with a disposable income of 100 million won. We used the equation (3), and the result is shown in Figure 5. The ability to explicitly derive such conditional distributions is another advantage of the S_U distribution.

4. CONCLUSION

Parametric distribution models for wealth and income offer the advantage of capturing all features of the distribution with a small number of parameters. They also have the advantage of estimating distributions and inequality/poverty indices even when survey microdata is unavailable and only grouped data is provided. Various parametric distributions have been used to estimate income distributions, but they cannot be applied to variables like net worth and disposable income that include negative values.

When it comes to estimating the distribution of these variables, the S_U distribution can be considered one of the best candidates, because it is defined over the entire real number space and capable of capturing extreme skewness and kurtosis significantly well. Another advantage of the S_U distribution is that, since the S_U distribution is essentially a transformation of the normal distribution, it can be easily extended to multivariate dimensions using multivariate normal distribution. Due to the fact that it is derived by transforming the normal distribution, the S_U distribution has several advantages. The joint PDF has a simple form, making maximum likelihood estimation relatively straightforward, even in one-step estimation. Furthermore, generating multivariate S_U random numbers is also straightforward, making it advantageous for simulation analyses in a multivariate dimension. In our example, we considered two variables: net worth and disposable income. However, when dealing with more than two variables—for instance, when estimating the joint distribution of wealth, income, and consumption—the multivariate S_U model is likely to be an attractive option.

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APPENDIX: LORENZ CURVE OF S_U DISTRIBUTION

This appendix derives the Lorenz curve for the S_U distribution, providing a closed-form expression and relevant mathematical formulations.

Consider an S_U variable $X = m + s \cdot \sinh(\lambda + \theta Z)$ with the PDF $f(x)$ and the CDF $F(x)$. Let p be the proportion of people in the population with wealth lower than x, i.e., $p = F(x) = \Phi(\theta^{-1}[\sinh^{-1}((x-m)/s) - \lambda])$. Then the Lorenz curve $L(p)$ for X is defined as follows Lubrano (2017):

$$L(p) = \frac{1}{E[X]} \int_{-\infty}^{x} t f(t) dt = \frac{1}{m + s e^{\theta^2/2} \sinh(\lambda)} \int_{-\infty}^{x} t f(t) dt.$$

For a normal random variable $Y \sim N(\mu, \sigma)$ conditional on $a \leq Y \leq b$, its moment generating function is:

$$E[e^{tY} | a \leq Y \leq b] = \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2} \left[\Phi\left(\frac{b-\mu}{\sigma} - t\sigma\right) - \Phi\left(\frac{a-\mu}{\sigma} - t\sigma\right) \right]}{\Phi((b-\mu)/\sigma) - \Phi((a-\mu)/\sigma)}.$$

Consider an S_U random variable X conditional on $a \leq X \leq b$, The first moment about zero of X is:

$$\begin{aligned}
&E[X | a \leq X \leq b] \\
&= E[m + s\sinh(Y) | \sinh^{-1}((a-m)/s) \leq Y \leq \sinh^{-1}((b-m)/s)] \\
&= m + sE\left[\frac{1}{2}(e^Y - e^{-Y}) \Big| \sinh^{-1}((a-m)/s) \leq Y \leq \sinh^{-1}((b-m)/s) \right] \\
&= m + \frac{s}{2} \left\{ E[e^Y | \sinh^{-1}((a-m)/s) \leq Y \leq \sinh^{-1}((b-m)/s)] \right. \\
&\qquad \left. - E[e^{-Y} | \sinh^{-1}((a-m)/s) \leq Y \leq \sinh^{-1}((b-m)/s)] \right\} \\
&= m + \frac{s}{2} e^{\theta^2/2} \frac{e^{\lambda}[\Phi(\beta - \theta) - \Phi(\alpha - \theta)] - e^{-\lambda}[\Phi(\beta + \theta) - \Phi(\alpha + \theta)]}{\Phi(\beta) - \Phi(\alpha)}, \qquad \text{(A1)}
\end{aligned}$$

where $\alpha = \theta^{-1}[\sinh^{-1}((a-m)/s) - \lambda]$ and $\beta = \theta^{-1}[\sinh^{-1}((b-m)/s) - \lambda]$. When $a = -\infty$ and $b = x$, the (A1) simplifies to:

$$E[X|-\infty < X \leq x] = m + \frac{s}{2}e^{\theta^2/2} \times \frac{e^{\lambda}\Phi(\theta^{-1}[\sinh^{-1}((x-m)/s)-\lambda]-\theta) - e^{-\lambda}\Phi(\theta^{-1}[\sinh^{-1}((x-m)/s)-\lambda]+\theta)}{\Phi(\theta^{-1}[\sinh^{-1}((x-m)/s)-\lambda])}.$$

Since $E[X|-\infty \leq X \leq x] = \frac{1}{F(x)}\int_{-\infty}^{x} tf(t)dt$, we obtain:

$$\int_{-\infty}^{x} tf(t)dt = mp + \frac{s}{2}e^{\theta^2/2}\left[e^{\lambda}\Phi(\Phi^{-1}(p)-\theta) - e^{-\lambda}\Phi(\Phi^{-1}(p)+\theta)\right].$$

Therefore, the Lorenz curve $L(p)$ simplifies to:

$$L(p) = \frac{mp + \frac{s}{2}e^{\theta^2/2}\left[e^{\lambda}\Phi(\Phi^{-1}(p)-\theta) - e^{-\lambda}\Phi(\Phi^{-1}(p)+\theta)\right]}{m + se^{\theta^2/2}\sinh(\lambda)}.$$