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# Concavity, Partial Concavity and Quasiconcavity: Characterizations by Modularity and Homogeneity*

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**Abstract** We study the interrelationships among concavity and weaker concavity notions of multi-variable functions, with or without differentiability. We show that modularity and homogeneity play central roles. Our characterizations offer illuminations on the nature of and the linkages among these function properties, and can be used in establishing economically meaningful results without appealing to derivatives.

**Keywords** Quasiconcavity, modularity, marginal rate of substitution, marginal utility, marginal productivity.

**JEL Classification** C60, D11, D24.

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## 1. INTRODUCTION

This paper's main objects of study are two weaker concavity notions of multi-variable functions: partial concavity and quasiconcavity. These two notions have familiar applications: for example, they each correspond to diminishing marginal productivities and diminishing marginal rates of technological substitution of a production function. Each notion *separately* has well-known characterizations when the function is *differentiable*. This paper explores the interrelationships among concavity, partial concavity and quasiconcavity, with or without differentiability.

Quasiconcavity of a multi-variable function refers to the shape of its contour. Formally, a function is (strictly) quasiconcave if its upper contour sets are (strictly) convex. In more familiar terms, when a utility function is strictly quasiconcave, its indifference curves show diminishing marginal rates of substitution (MRS); similarly, if a production function is strictly quasiconcave, then its isoquants show diminishing marginal rates of technological substitution (MRTS). On the other hand, partial concavity of a utility function means diminishing marginal utilities (MU) and partial concavity of a production function means diminishing marginal productivities (MP).

It is sometimes mistakenly thought that diminishing MRS or MRTS is closely related to diminishing MU or MP, but we can easily show that the two are independent notions. MRS or MRTS is measured along a single contour and computed as a "total" derivative, while MU or MP is measured across several contours and computed as a "partial" derivative.

The basic results on these concavity notions are already available but they are usually expressed for the special case of differentiable functions by using derivative conditions. Moreover, the exact relationships among them (other than the independence noted above) are not given prominence.

It is our goal to provide a systematic presentation of these concavity notions. Figure 1 shows the overview of our results. The "inner" relations (hollow arrows, $\Longrightarrow$) are trivial, but the "non-relation" (crossed-out two-headed arrow $\Longleftrightarrow$) may not be apparent, so we offer illustrations. Our major results provide the "outer" relations (filled arrows ➡), with or without differentiability.

It is of interest that our non-differentiable conditions involve modularity and homogeneity, which are themselves important and interesting in economics literature. In Section 2, we present basic results with an emphasis on differentiable functions (hollow arrows). Section 3 contains our main results involving modularity and homogeneity (filled arrows). In Section 4, we summarize and discuss the results.

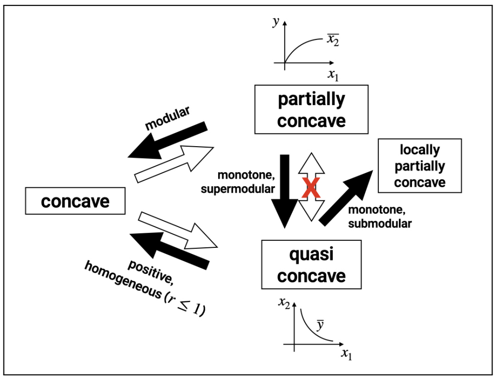



Figure 1: OVERVIEW OF RESULTS. The figure diagrammatically shows the logical relationships among concavity, partial concavity and quasiconcavity.

## 2. DEFINITIONS AND BASIC RESULTS

This section lays out some background by giving definitions and stating basic results represented by the "hollow" arrows in Figure 1. Most of it is standard material. We offer proofs and examples if they are short, intuitive and useful for later results. For more information, the reader can consult, for example, Simon and Blume (1994) chapter 21.

We will emphasize two-variable functions, but definitions can be easily stated for $n$-variable functions. So consider real-valued functions on $\mathbb{R}^n$, the finite-dimensional Euclidean space.[1] For any pair of vectors $\mathbf{x}^A = (x_1^A, \ldots, x_n^A), \mathbf{x}^B = (x_1^B, \ldots, x_n^B) \in \mathbb{R}^n$, let a convex combination be denoted by $\mathbf{x}^t = t\mathbf{x}^A + (1-t)\mathbf{x}^B$ for $0 \leq t \leq 1$. A function is concave if its value at a convex combination of two points is not lower than the convex combination of the two values. Formally, a function $F : \mathbb{R}^n \to \mathbb{R}$ is *concave* if $F(\mathbf{x}^t) \geq tF(\mathbf{x}^A) + (1-t)F(\mathbf{x}^B)$ and is *strictly concave* if the inequality is strict for $0 < t < 1$. Most results on concavity extend to strict concavity, but some don't and we will note such distinctions when needed.

We can also define *convexity* of $F$ as concavity of $-F$. Then similar results also hold for convexity (with inequalities appropriately reversed). For convenience in exposition, we will focus on concavity.

[1]We can extend to more general vector spaces, but we will remain under the finite Euclidean space setting for concreteness.

## 2.1. CONCAVITY AND PARTIAL CONCAVITY

By fixing values of $(n-1)$ variables, we can restrict $F$ to a one-variable function:

$$F(\bar{x}_1, \ldots, x_i, \ldots, \bar{x}_n) \equiv \phi_i(x_i),$$

for some $(\bar{x}_1, \ldots, \bar{x}_n) \in \mathbb{R}^{n-1}$. Borrowing from convention in game theory, we can also write

$$F(\bar{x}_1, \ldots, x_i, \ldots, \bar{x}_n) \equiv F(x_i, \bar{x}_{-i}).$$

We say that $F$ is *partially concave* if $\phi_i(x_i) = F(x_i, \bar{x}_{-i})$ is concave for arbitrary $\bar{x}_{-i} \in \mathbb{R}^{n-1}$, for each $i = 1, 2, \ldots, n$. It is *strictly partially concave* if the inequality is strict for $0 < t < 1$. Note that partial concavity means concavity in each and every variable when viewed as a one-variable function.

A (strictly) concave function is (strictly) partially concave: since $t\bar{x}_{-i} + (1-t)\bar{x}_{-i} = \bar{x}_{-i}$ we have $\phi_i(x_i^t) = \phi_i(tx_i^A + (1-t)x_i^B) = F(tx_i^A + (1-t)x_i^B, \bar{x}_{-i}) = F(t(x_i^A, \bar{x}_{-i}) + (1-t)(x_i^B, \bar{x}_{-i})) \geq tF(x_i^A, \bar{x}_{-i}) + (1-t)F(x_i^B, \bar{x}_{-i}) = t\phi_i(x_i^A) + (1-t)\phi_i(x_i^B)$. The converse is not true as shown by the following examples.

**Example 1** (partial concavity does not imply concavity)**.** We offer two examples. In (a), $F$ is partially concave (but not strictly) and it is not concave. In (b), $G$ is strictly partially concave and is not concave.

(a) Let $F(x_1, x_2) = x_1 + x_2 - 2x_1x_2$. Since $F$ is linear in each $x_i$, it is partially concave (but not strictly). Now consider $\mathbf{x}^A = (1,0)$ and $\mathbf{x}^B = (0,1)$, then $F(1,0) = F(0,1) = 1$, but $F(\mathbf{x}^{0.5}) = F(0.5, 0.5) = 0.5$. So $F(\mathbf{x}^{0.5}) < 0.5F(\mathbf{x}^A) + 0.5F(\mathbf{x}^B)$. Hence $F$ is not concave.

(b) Let $G(x_1, x_2) = \log_2(x_1 + x_2^2)$. That $G$ is strictly partially concave can be seen most easily by recognizing that $G(x_i, \bar{x}_j)$ is a log function "shifted" horizontally. Now consider $\mathbf{x}^A = (12, 2)$ and $\mathbf{x}^B = (0, 4)$ and $\mathbf{x}^{0.5} = (6, 3)$. $G(12, 2) = G(0, 4) = 4$ but $G(6, 3) = \log_2(15) \approx 3.9$. So $G(\mathbf{x}^{0.5}) < 0.5G(\mathbf{x}^A) + 0.5G(\mathbf{x}^B)$. Hence $G$ is not concave. □

### 2.1.1. Concavity and partial concavity for differentiable functions

Suppose $F$ is twice continuously differentiable. It is well known that concavity and partial concavity can be characterized using partial derivatives. $F$ is partially concave if and only if $\phi_i''(x_i) = F_{ii}(\mathbf{x}) \leq 0$ for each $i = 1, \ldots, n$, where $F_{ii} = \frac{\partial^2 F}{\partial x_i^2}$. A sufficient (but not necessary) condition for *strict* partial concavity is

$F_{ii}(\mathbf{x}) < 0$.[2] On the other hand, $F$ is concave if and only if its Hessian is negative semi-definite. A sufficient (but not necessary) condition for *strict* concavity is its Hessian being negative definite. One can check that in Example 1 both $F$ and $G$ have indefinite Hessians.

It is instructive to look into the Hessian conditions. Consider a two-variable function $F(x_1, x_2)$. Then $F$ is negative semi-definite when (i) $F_{11} \leq 0$, $F_{22} \leq 0$ and (ii) $F_{11}F_{22} - F_{12}F_{21} = F_{11}F_{22} - F_{12}^2 \geq 0$. The condition (i) is in fact partial concavity of $F$. It says that as $x_i$ increases, its marginal effect $F_i$ (weakly) decreases. The condition (ii) says that $F_{12}(= F_{21})$ must not overshadow $F_{11}$ and $F_{22}$. The cross-partial derivative $F_{ij}$ measures how one variable ($x_j$) affects another variable's marginal effect ($F_i$). If $|F_{12}|$ is too great, as two variables increase jointly, one variable's own effect $F_{ii}$ can be dominated by another variable's cross effect $F_{ij}$, which can lead to non-concavity even when it is partially concave.

In economic terms, let $F(K, L)$ be a (twice continuously differentiable) production function where $K$ denotes capital input and $L$ labor input. Partial concavity corresponds to diminishing marginal productivities for both factors: $F_{KK} \leq 0$, $F_{LL} \leq 0$. On the other hand, $F_{LK} = F_{KL}$ is interpreted as how one factor affects another factor's marginal productivity. If $F_{KL} > 0$, then the cross effect is complementary ("synergy" between two factors) and if $F_{KL} < 0$, then it is substitutary. Regardless of the sign, if the cross effect $|F_{KL}|$ is too great, then $F(K, L)$ may not be concave even if it is partially concave.

For $n \geq 3$, the Hessian conditions have additional inequalities involving several cross effects as well as interactions between own and cross effects. But the conditions still boil down to small cross effects.[3].

## 2.2. CONCAVITY AND QUASICONCAVITY (AND MONOTONICITY)

Quasiconcavity is a foundational notion in economic theory. Many standard texts invoke quasiconcavity when explaining consumer theory, producer theory,

[2] That the condition is not necessary can be seen by the simple example $f(x) = -x^4$ which is strictly concave but $f''(0) = 0$.

[3] For $n = 3$, the additional condition can be expressed as

$$F_{11}F_{22}F_{33}\left[1 - \left(\frac{F_{23}^2}{F_{22}F_{33}} + \frac{F_{13}^2}{F_{11}F_{33}} + \frac{F_{12}^2}{F_{11}F_{22}}\right) + \frac{2F_{12}F_{23}F_{31}}{F_{11}F_{22}F_{33}}\right] \leq 0.$$

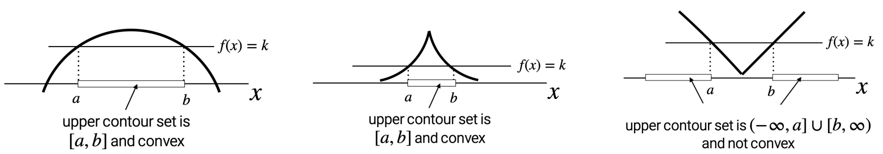
Since $F_{11}F_{22}F_{33} \leq 0$, we need the expressions in $[\cdots]$ to be non-negative, which is possible when $|F_{ij}| \ll |F_{ii}|$

general equilibrium theory, etc.[4] However, without some motivation, its meaning may not be apparent at first encounter.

There are two equivalent definitions. Given any two vectors $\mathbf{x}^A, \mathbf{x}^B \in \mathbb{R}^n$ and its convex combination $\mathbf{x}^t$, we say that a function $F(\mathbf{x})$ is *quasiconcave* if $F(\mathbf{x}^t) \geq \min\{F(\mathbf{x}^A), F(\mathbf{x}^B)\}$ or equivalently if the upper contour set $\{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) \geq k\}$ is convex for any $k \in \mathbb{R}$. It is *strictly quasiconcave* if the inequality is strict for $0 < t < 1$ in the first definition, or the upper contour set is strictly convex in the second definition.

A concave function is quasiconcave. To see this, without loss of generality consider $\mathbf{x}^A, \mathbf{x}^B$ such that $F(\mathbf{x}^A) \geq F(\mathbf{x}^B)$. Then by concavity, $F(\mathbf{x}^t) \geq tF(\mathbf{x}^A) + (1-t)F(\mathbf{x}^B) \geq tF(\mathbf{x}^B) + (1-t)F(\mathbf{x}^B) = F(\mathbf{x}^B) = \min\{F(\mathbf{x}^A), F(\mathbf{x}^B)\}$. The converse is not true as shown by the following examples.

**Example 2** (quasiconcavity and concavity for one-variable functions)**.** Figure 2 shows graphs of one-variable functions. Functions in (a) and (b) are both strictly quasiconcave,[5] but in (a) it is strictly concave and in (b) it is not concave. And (c) shows an example of a function that is neither quasiconcave nor concave. □




(a) strictly concave, quasiconcave. (b) not concave, quasiconcave. (c) not concave, not quasiconcave.

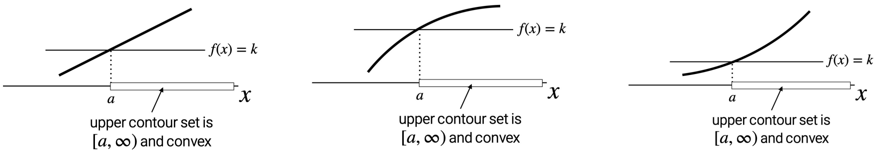
Figure 2: GRAPHS OF FUNCTIONS FOR EXAMPLE 2. These graphs illustrate the distinction between concavity and quasiconcavity.

The next set of examples shows that monotonicity (with or without concavity) implies quasiconcavity for one-variable functions. We also note that this observation doesn't extend to multi-variable functions.

**Example 3** (monotonicity, quasiconcavity and concavity)**.** Figure 3 shows graphs of monotone increasing one-variable functions. All three are (strictly) quasiconcave. The function in (c) is not concave (is in fact strictly convex). It should

[4] According to Guerraggio and Molho (2004), the term originates in the seminal monograph in convex analysis, Fenchel (1953), but the notion can be traced back to von Neumann (1928) in his early work on game theory and to de Finetti (1949) in his early work on expected utility theory.

[5] There is no distinction between quasiconcavity and strict quasiconcavity for one-variable functions.




(a) concave (linear.) (b) strictly concave. (c) not concave.

Figure 3: GRAPHS OF MONOTONE & QUASICONCAVE FUNCTIONS FOR EXAMPLE 3. These graphs show that one-variable monotone functions are quasiconcave, whether concave or not.

be obvious that monotone decreasing one-variable functions are also (strictly) quasiconcave.

In contrast, $F(x_1,x_2) = x_1^2 + x_2^2$ is monotone, but is not quasiconcave (hence not concave): $F(2,0) = F(0,2) = 4$ and $F(1,1) = 2 \ngeq \min\{F(2,0), F(0,2)\} = 4$. This shows that monotonicity and quasiconcavity are independent notions for multi-variable functions. □

The next is a multi-variable example showing that (strict) quasiconcavity does not imply concavity.

**Example 4** (quasiconcavity does not imply concavity)**.**

- Consider $F(x_1,x_2) = x_1x_2$. For $x_1 > 0$, $x_2 > 0$, $F$ is strictly quasiconcave, but is not concave.

- That $F$ is strictly quasiconcave can be seen most easily by drawing its upper contour set. The boundary of the upper contour set, or the contour satisfies $F(x_1,x_2) = x_1x_2 = k$ for some $k$. Hence, it is the graph of the function $x_2 = \frac{k}{x_1}$, which is the familiar Cobb-Douglas indifference/isoquant curve.

- To show strict quasiconcavity formally, let $U = \{(x_1,x_2) \in \mathbb{R}_{++} | x_1x_2 \geq k\}$ for some $k$. For $\mathbf{x}^A = (a,b), \mathbf{x}^B = (c,d) \in U$, $\mathbf{x}^t = (ta+(1-t)c, tb+(1-t)d)$ and $F(\mathbf{x}^t) = (ta+(1-t)c)(tb+(1-t)d) = t^2ab + t(1-t)ad + t(1-t)bc + (1-t)^2cd = t^2ab + t(1-t)(ad+bc) + (1-t)^2cd$. From $ab \geq k$ and $cd \geq k$, we have $a \geq \frac{k}{b} \implies ad \geq \frac{d}{b}k$ and $c \geq \frac{k}{d} \implies bc \geq \frac{d}{b}k$, so $ad+bc \geq (\frac{d}{b}+\frac{b}{d})k \geq 2\sqrt{\frac{d}{b}\cdot\frac{b}{d}}k = 2k$. Therefore we have $F(\mathbf{x}^t) = t^2ab + t(1-t)(ad+bc) + (1-t)^2cd \geq t^2k + 2t(1-t)k + (1-t)^2k = k$ implying $\mathbf{x}^t \in U$ as well.

- To see that $F$ is not concave, we can check that its Hessian $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is indefinite because $\det(H) = -1 < 0$, or by considering $F(1,1) = 1$, $F(3,3) = 9$, $F(2,2) = 4 < 0.5F(1,1) + 0.5F(3,3)$. □

### 2.2.1. Quasiconcavity for differentiable functions

If $F(x_1, \ldots, x_n)$ is twice continuously differentiable, quasiconcavity can be characterized using the so-called bordered Hessian

$$\overline{H} = \begin{pmatrix} 0 & F_1 & \cdots & F_n \\ F_1 & F_{11} & \cdots & F_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ F_n & F_{n1} & \cdots & F_{nn} \end{pmatrix}.$$

The characterization consists of alternating signs of $\overline{H}$'s leading principal minors.[6] A sufficient (but not necessary) condition for *strict* quasiconcavity involves monotonicity in addition to the sign-alternating leading principal minors (Arrow and Enthoven*et al.*, 1961; de la Fuente, 2000, Theorem 3.11).

We can examine these characterizations in more depth for two-variable function $F(x_1, x_2)$. A sufficient set of two conditions are: (i) $-F_1^2 < 0 \iff F_1 \neq 0$ and (ii) $-F_1^2 F_{22} + 2F_1 F_2 F_{12} - F_2^2 F_{11} > 0$.

These conditions have easy economic and geometric interpretations. Quasiconcavity expresses diminishing "absolute slope" of the contour. In economics, it is diminishing marginal rates of substitution (MRS) for utility functions or diminishing marginal rates of technological substitution (MRTS) for production functions. For convenience, we shall refer to both MRS and MRTS as simply MRS hereafter.

If $F$ is twice continuously differentiable, both MRS and diminishing MRS can be characterized by partial derivatives as follows. For a contour $F(x_1, x_2) = k$, its slope is $\frac{dx_2}{dx_1}\big|_{F(\mathbf{x})=k} = -\frac{F_1(\mathbf{x})}{F_2(\mathbf{x})}$. Then the slope's absolute value is $MRS =$

[6] Let

$$\overline{H}_r = \begin{pmatrix} 0 & F_1 & \cdots & F_r \\ F_1 & F_{11} & \cdots & F_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ F_r & F_{r1} & \cdots & F_{rr} \end{pmatrix}$$

for $r = 1, \ldots, n$. Then $F$ is quasiconcave if $(-1)^r |\overline{H}_r| > 0$ for all $r$ and only if $(-1)^r |\overline{H}_r| \geq 0$ for all $r$.

$\frac{F_1(\mathbf{x})}{F_2(\mathbf{x})}$.[7] Then quasiconcavity (diminishing MRS) says $\frac{d}{dx_1}(MRS) \leq 0$. This total derivative can be expressed in terms of partial derivatives as follows:

$$\frac{d}{dx_1}\left(\frac{F_1}{F_2}\right) = \frac{\frac{dF_1}{dx_1} \cdot F_2 - F_1 \frac{dF_2}{dx_1}}{F_2^2} = \frac{F_{11}F_2^2 - 2F_1F_2F_{12} + F_{22}F_1^2}{F_2^3}.$$

From this we obtain an important characterization of quasiconcavity for differentiable two-variable functions. While the result is not new, it sometimes goes unnoticed so is worth writing down. (Proofs are omitted as they are obvious by inspection of the above formula.)

**Lemma 1.** *Suppose $F(x_1,x_2)$ is twice continuously differentiable and strictly increasing ($F_1 > 0$, $F_2 > 0$). Then $F$ is quasiconcave (shows diminishing MRS) if and only if*

$$F_{11}F_2^2 - 2F_1F_2F_{12} + F_{22}F_1^2 \leq 0. \tag{1}$$

*and is strictly quasiconcave if and only if (1) holds with strict inequality.*

Note that the condition (1) is consistent with the bordered Hessian sign condition (ii) from the above.[8] Moreover, the following is immediate.

**Corollary 1.** *Suppose $F$ is twice continuously differentiable, strictly increasing ($F_i > 0$) and strictly partially concave ($F_{ii} < 0$). Then $F_{12} \geq 0$ is a sufficient condition for strict quasiconcavity of $F$.*

This corollary establishes a first relationship between (strict) partial concavity and (strict) quasiconcavity. Note that it uses monotonicity as well as differentiability of the function. Hence if $F$ is differentiable and monotone, then partial concavity (diminishing marginal effects) and a non-negative cross effect (independence or complementarity) imply quasiconcavity (diminishing MRS).

The conditions for $n \geq 3$ cases include further sign restrictions.[9] It is difficult to state an easy set of restrictions on $F_{ij}$ to satisfy these. Not surprisingly, one strong restriction is that $|F_{ij}|$ be small, which would guarantee the function's (full) concavity as noted previously in Section 2.1.

[7] We are making an unnecessary but economically sensible assumption of $F_i > 0$. As per the condition (i) it is sufficient for the following that $F_i \neq 0$.

[8] As noted in the previous footnote, we can show that the same condition (1) applies even when $F_1 < 0, F_2 < 0$.

[9] For $n = 3$, the last condition is $-F_1^2(F_{22}F_{33} - F_{23}^2) - F_2^2(F_{11}F_{33} - F_{13}^2) - F_3^2(F_{11}F_{22} - F_{12}^2) + 2F_1F_2(F_{33}F_{12} - F_{13}F_{23}) + 2F_1F_3(F_{22}F_{13} - F_{12}F_{23}) + 2F_2F_3(F_{11}F_{23} - F_{12}F_{13}) < 0$.

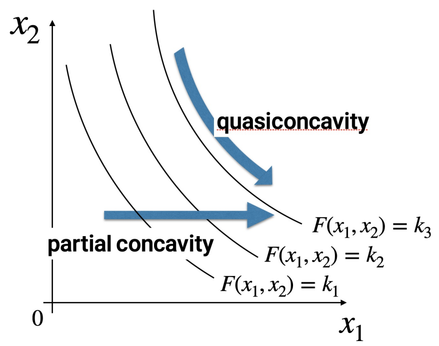



Figure 4: PARTIAL CONCAVITY VS QUASICONCAVITY. The figure illustrates that partial concavity and quasiconcavity are checked along different directions on contours.

## 2.3. PARTIAL CONCAVITY AND QUASICONCAVITY DO NOT IMPLY EACH OTHER

Before turning to our main results, let us elaborate on the non-relation between partial concavity and quasiconcavity. In economic applications, this observation alerts us that diminishing MRS and diminishing MU or MP are distinct and independent properties.

Both partial concavity and quasiconcavity are implied by concavity of a multi-variable function. But the two are weaker than concavity in different ways. While this is well established, concern has been raised that even some textbooks do not correctly deliver this message (Dittmer, 2005; Kim, 2008).

The distinction between two notions can most easily be seen graphically (Figure 4). Partial concavity involves partial derivatives of $F(\mathbf{x})$ at different values of the fixed variable, so is checked either horizontally or vertically. In contrast, quasiconcavity concerns total derivatives between $x_2$ and $x_1$ at a fixed function value and is checked across a single contour line. To put it differently, concavity concerns all three variables in $y = F(x_1, x_2)$, partial concavity concerns $y$ and $x_i$ only while keeping $x_j$ fixed as $F(x_i, \overline{x_j})$, and quasiconcavity concerns $x_1$ and $x_2$ while keeping $y$ fixed as $F(x_1, x_2) = k$. Partial concavity and quasiconcavity weaken concavity in different directions.

We sometimes mistakenly think that diminishing MU and diminishing MRS are closely related. A loose reasoning is as follows. MRS measures relative "value" of $x_1$ against $x_2$. If we increase $x_1$ and decrease $x_2$, then $x_1$ becomes more plentiful and $x_2$ becomes more scarce. Diminishing MU suggests $x_1$ becomes less valuable while $x_2$ becomes more valuable, and this is consistent with diminishing relative value (MRS) of $x_1$. And this reasoning seems to be supported by the formula $MRS = \frac{F_1}{F_2} = \frac{MU_1}{MU_2}$.

The main source of confusion can be identified by the differentiable condition (1) given in Lemma 1 above. We noted in Corollary 1 that $F_{12} \geq 0$ is a sufficient condition for a (strictly) partially concave function to be (strictly) quasiconcave. Interestingly, we also noted the role of the cross partial derivative $F_{12}$ in contrasting concavity and partial concavity. If $|F_{12}|$ is too great, a partially concave function may not be concave. In contrast, if $F_{12} < 0$ *and* $|F_{12}|$ is too great, a partially concave function may not be quasiconcave. When $F_{12} < 0$, the above loose reasoning fails because the increase in $x_1$ not only lowers $MU_1$ but $MU_2$ as well, while the decrease in $x_2$ raises both $MU_1$ and $MU_2$, leading to an ambiguous final outcome. The following example may be useful for removing any remaining doubts.

**Example 5** (quasiconcavity and partial concavity are independent)**.** (a) offers a strictly quasiconcave but not partially concave (convex in fact) function; (b) offers partially concave but not quasiconcave functions.

(a) $F(x_1,x_2) = x_1^2 x_2^2$. For $x_1 > 0, x_2 > 0$, $F$ is strictly quasiconcave with a contour $x_2 = \frac{\sqrt{k}}{x_1}$ for $F(x_1,x_2) = k$. But $F_{11} = 2x_2^2 > 0$, $F_{22} = 2x_1^2 > 0$ for $x_1 > 0, x_2 > 0$. So $F$ is not concave in either variable.

(b) $G(x_1,x_2) = \alpha x_1 + \beta x_2 - x_1 x_2$ for sufficiently large constants $\alpha, \beta$. For $x_1 < \beta$ and $x_2 < \alpha$, $G$ is partially concave; in fact it is linear increasing in each variable. That $G$ is not quasiconcave (in fact strictly quasiconvex) can be checked by using (1). Since $G_1 = \alpha - x_2$, $G_2 = \beta - x_1$, $G_{11} = G_{22} = 0$ and $G_{12} = -1$

$$G_{11}G_2^2 - 2G_1G_2G_{12} + G_{22}G_1^2 = 2(\alpha - x_2)(\beta - x_1) > 0, \text{ for } x_1 < \beta, x_2 < \alpha,$$

hence MRS is increasing. We can also show that $\log G(x_1,x_2)$ is strictly partially concave (see Kim, 2008) and being a monotone increasing transform of $G(\mathbf{x})$ it also is not quasiconcave.

□

## 3. MAIN RESULTS ON PARTIAL CONCAVITY AND QUASICONCAVITY

Equipped with the background provided in the previous section, we now explore the remaining parts of Figure 1, the “filled” arrows that tell us sufficient conditions for linking partial concavity and quasiconcavity. We will put emphasis on two-variable functions $y = F(x_1, x_2)$ for ease of exposition. Specifically, the direct linkages between partial concavity and quasiconcavity are limited to two-variable functions. But the indirect linkages (via full concavity) can be established for general multi-variable functions.

As noted previously, partial concavity and quasiconcavity are concerned with different "directions" of the function. Hence, any linkage must somehow fill the missing direction. To go from partial concavity to quasiconcavity, we need to put some structure on the interaction between $x_1$ and $x_2$, while to go from quasiconcavity to partial concavity, we need to specify how dependent variable $y$ responds to changes in $(x_1, x_2)$. So our goal is to identify and fill such missing links. We will show that important roles are played by modularity and homogeneity, as well as separability and monotonicity.

### 3.1. MODULARITY, SUPERMODULARITY AND SUBMODULARITY

Since modularity is less familiar than other notions used in this paper, we will explicitly define it for our setting of $\mathbb{R}^n$.[10] First we define $\mathsf{sup}$ (or the join $\vee$) and $\mathsf{inf}$ (or the meet $\wedge$) of a pair of vectors $\mathbf{x}^A = (x_1^A, \ldots, x_n^A)$ and $\mathbf{x}^B = (x_1^B, \ldots, x_n^B)$.

$$\mathsf{sup}\{\mathbf{x}^A, \mathbf{x}^B\} \equiv \mathbf{x}^A \vee \mathbf{x}^B \equiv (\max\{x_1^A, x_1^B\}, \ldots, \max\{x_n^A, x_n^B\}),$$
$$\mathsf{inf}\{\mathbf{x}^A, \mathbf{x}^B\} \equiv \mathbf{x}^A \wedge \mathbf{x}^B \equiv (\min\{x_1^A, x_1^B\}, \ldots, \min\{x_n^A, x_n^B\}).$$

In words, $\mathsf{sup}$ operation takes the higher number for every coordinate while $\mathsf{inf}$ operation takes the lower number for every coordinate. First note that if two vectors are naturally ordered, for example $\mathbf{x}^A \gg \mathbf{x}^B \iff x_i^A > x_i^B$ for every $i = 1, 2, \ldots, n$, then $\mathbf{x}^A \vee \mathbf{x}^B = \mathbf{x}^A$ and $\mathbf{x}^A \wedge \mathbf{x}^B = \mathbf{x}^B$. The join and the meet have the real "bite" only when the two vectors are not comparable by coordinate-wise ordering.

Figure 5 illustrates the notions for $\mathbb{R}^2$. Given any two incomparable vectors in $\mathbb{R}^2$, $\mathsf{sup}$ is at the northeast vertex and $\mathsf{inf}$ is at the southwest vertex of the rectangle formed by the four vectors.

A function $F : \mathbb{R}^n \to \mathbb{R}$ is *supermodular* if

$$\begin{aligned} &F(\mathbf{x}^A \vee \mathbf{x}^B) + F(\mathbf{x}^A \wedge \mathbf{x}^B) \geq F(\mathbf{x}^A) + F(\mathbf{x}^B) \\ \iff &F(\mathbf{x}^A \vee \mathbf{x}^B) - F(\mathbf{x}^B) \geq F(\mathbf{x}^A) - F(\mathbf{x}^A \wedge \mathbf{x}^B), \end{aligned} \tag{SPM}$$

and $F$ is *submodular* if the inequalities are reversed in (SPM). Finally, $F$ is *modular* if it is both supermodular and submodular.

[10] More generally, modularity is defined on lattices. See Topkis (1998) for details.

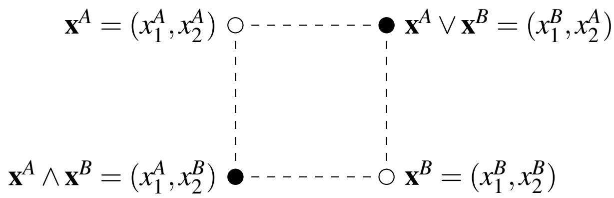



Figure 5: $\mathbf{x}^A \vee \mathbf{x}^B$ AND $\mathbf{x}^A \wedge \mathbf{x}^B$ IN $\mathbb{R}^2$. The figure shows the positions of sup and inf.

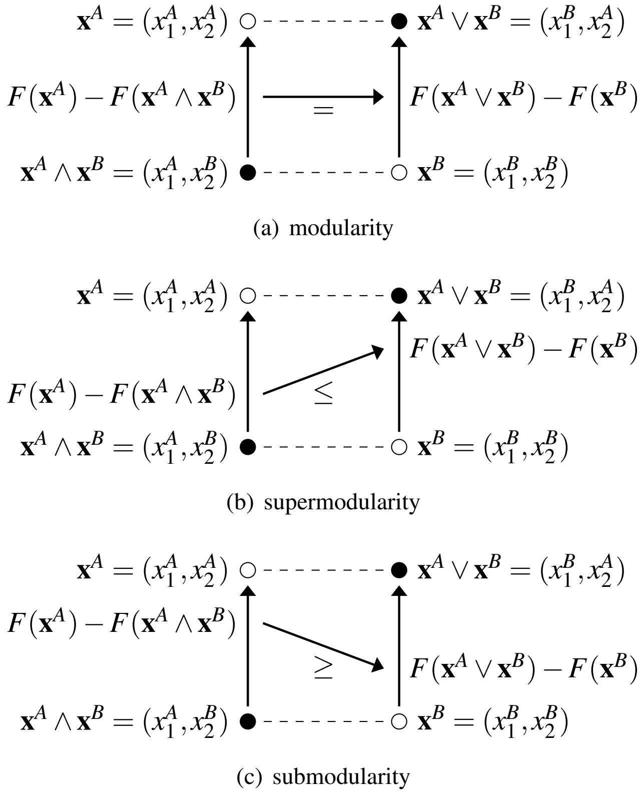



Figure 6: MODULARITY OF FUNCTION IN $\mathbb{R}^2$. The figures compare the notions of modularity, supermodularity and submodularity.

Figure 6 illustrates modularity notions in $\mathbb{R}^2$. As we move horizontally ($x_1^A \to x_1^B$), the vertical effect, *i.e.* the marginal effect of the same $\Delta x_2 = x_2^A - x_2^B$, on $F(\cdot)$ stays the same, increases or decreases according to whether $F$ is modular, supermodular or submodular. One can show that if $F$ is differentiable, then supermodularity reduces to $F_{12} \geq 0$, submodularity to $F_{12} \leq 0$ and modularity to $F_{12} = 0$.

Supermodularity, in particular, has appeared under disguises widely in economic theory and game theory. For example, it is equivalent to, or closely related to, such notions as convex games (Shapley, 1971) in cooperative game theory, Spence-Mirrlees single crossing condition (Milgrom and Shannon, 1994) in information economics, strategic complements (Bulow *et al.*, 1985) in industrial organization theory, and ambiguity aversion (Gilboa and Schmeidler, 1989) in maximin expected utility. The monotone comparative statics literature crucially relies on supermodularity: see Milgrom and Shannon (1994). Amir (2005) is a good introduction.

## 3.2. PARTIAL CONCAVITY AND QUASICONCAVITY

We can start from the results in Section 2 for differentiable functions. To recapitulate briefly, if a differentiable, increasing and (strictly) partially concave $F(x_1, x_2)$ has $F_{12} \geq 0$, it is (strictly) quasiconcave (Corollary 1). We now want to generalize this to possibly non-differentiable functions. The fact that supermodularity reduces to $F_{12} \geq 0$ for differentiable functions suggests a natural generalization.

### 3.2.1. Supermodularity, monotonicity and partial concavity imply quasiconcavity

Our first main result says that supermodularity combined with monotonicity allows us to go from partial concavity to quasiconcavity. This is a generalization of Corollary 1.

**Proposition 1.** *If $F(x_1, x_2)$ is supermodular, monotone* (*increasing or decreasing*)*, and partially concave, then it is quasiconcave.*

*Proof.*[11] Pick any two vectors $\mathbf{x}^A, \mathbf{x}^B$ from $U = \{\mathbf{x} \in \mathbb{R}^2 | F(\mathbf{x}) \geq k\}$ for some $k \in \mathbb{R}$. Without loss of generality, we may assume $x_1^A \leq x_2^B$. If $x_2^A \leq x_2^B$ as well so

[11] I am grateful that an anonymous referee's suggestions led to significant corrections and simplifications of the proof.

that $\mathbf{x}^B \geq \mathbf{x}^A$, then $F(\mathbf{x}^B) \geq F(\mathbf{x}^t) \geq F(\mathbf{x}^A) = \min\{F(\mathbf{x}^A), F(\mathbf{x}^B)\}$ if $F$ is monotone increasing and $F(\mathbf{x}^A) \geq F(\mathbf{x}^t) \geq F(\mathbf{x}^B) = \min\{F(\mathbf{x}^A), F(\mathbf{x}^B)\}$ if $F$ is monotone decreasing and we are done for $x_2^A \leq x_2^B$ case.

Now consider the remaining possibility $x_2^A > x_2^B$. Then we have $\mathbf{x}^A \vee \mathbf{x}^B = (x_1^B, x_2^A)$ and $\mathbf{x}^A \wedge \mathbf{x}^B = (x_1^A, x_2^B)$. By supermodularity, we have $F(x_1^B, x_2^A) - F(x_1^B, x_2^B) \geq F(x_1^A, x_2^A) - F(x_1^A, x_2^B) \iff F(x_1^A, x_2^B) + F(x_1^B, x_2^A) \geq F(x_1^A, x_2^A) + F(x_1^B, x_2^B)$. (Consult Figure 6(b).) Therefore

$$\begin{aligned}
F(x_1^t, x_2^t) &\geq tF(x_1^A, x_2^t) + (1-t)F(x_1^B, x_2^t) && \text{[concave in } x_1] \\
&\geq t\left(tF(x_1^A, x_2^A) + (1-t)F(x_1^A, x_2^B)\right) + (1-t)\left(tF(x_1^B, x_2^A) + (1-t)F(x_1^B, x_2^B)\right) && \text{[concave in } x_2] \\
&= t^2F(x_1^A, x_2^A) + t(1-t)\left(F(x_1^A, x_2^B) + F(x_1^B, x_2^A)\right) + (1-t)^2F(x_1^B, x_2^B) \\
&\geq t^2F(x_1^A, x_2^A) + t(1-t)\left(F(x_1^A, x_2^A) + F(x_1^B, x_2^B)\right) + (1-t)^2F(x_1^B, x_2^B) && \text{[supermodular]} \\
&= tF(x_1^A, x_2^A) + (1-t)F(x_1^B, x_2^B) \\
&\geq tk + (1-t)k = k, && [\mathbf{x}^A, \mathbf{x}^B \in U]
\end{aligned}$$

where the crucial step relies on supermodularity to ensure $F(x_1^A, x_2^B) + F(x_1^B, x_2^B) \geq F(\mathbf{x}^A) + F(\mathbf{x}^B)$. □

**Remark 1**. As can been from the proof, for linearly ordered vectors, monotonicity alone ensures quasiconcavity, as in one-variable functions (Example 3 in Section 2.2), while for unordered, incomparable vectors, partial concavity and supermodularity imply quasiconcavity. □

The following example is modified from Example 5(b) in Section 2 and shows the monotonicity condition has a bite.

**Example 6** (without monotonicity, the proposition fails)**.**

- Consider $F(x_1, x_2) = -4x_1 - x_2 + x_1x_2$. (This is the negative of $G(x_1, x_2)$ from Example 5(b) with $\alpha = 4$, $\beta = 1$.) From $F_1 = x_2 - 4$ and $F_2 = x_1 - 1$, we observe that $F$ is *not* monotone for $x_1 > 1$ and $x_2 < 4$.

- We can see from $F_{12} = 1$ that $F$ is supermodular. To check directly, consider four vectors $(0,0)$, $(a,0)$, $(0,b)$, $(a,b)$ with $a > 0$, $b > 0$. Then we have $F(a,b) + F(0,0) = -4a - b + ab > F(a,0) + F(0,b) = -4a - b$. Also $F$ is partially concave, as it is linear with respect to each variable.

- But $F$ is not quasiconcave for $x_1 > 1$ and $x_2 < 4$. For example, $F(\frac{1}{2}(11,0) + \frac{1}{2}(21,2)) = F(16,1) = -49 < F(11,0) = -44 = F(21,2)$. (We can actually verify that $F$ is strictly quasiconvex.) So $F$ fails to be quasiconcave because it is not monotone. □

**Remark 2**. For $n \geq 3$ cases, supermodularity (complementarity) alone would not be sufficient. We would need the size of cross effects to be sufficiently small. If not, interactions among 3 or more variables would complicate the outcomes. We will later show that one strong sufficient condition is modularity. □

### 3.2.2. Submodularity, monotonicity and quasiconcavity imply partial non-convexity

Now we take to the converse. While we are unable to get a direct link from quasiconcavity to partial concavity, we do have a weaker but interesting characterization based on submodularity.

**Proposition 2.** *Suppose $F(x_1,x_2)$ is submodular and monotone. If $F(x_1,x_2)$ is quasiconcave, then it is not strictly partially convex. That is, $F(\mathbf{x})$ is locally (over some part of the domain) concave with respect to at least one variable.*

*Proof.* We shall prove the contrapositive: Given submodularity and monotonicity, if $F$ is strictly partially convex, then it is not quasiconcave.

Pick two vectors $\mathbf{x}^A, \mathbf{x}^B \in \mathbb{R}^2$ such that $F(\mathbf{x}^A) = F(\mathbf{x}^B) = k$. We will exhibit a convex combination $\mathbf{x}^t = t\mathbf{x}^A + (1-t)\mathbf{x}^B$ such that $F(\mathbf{x}^t) < k$. (See Figure 7 for visual guidance.)

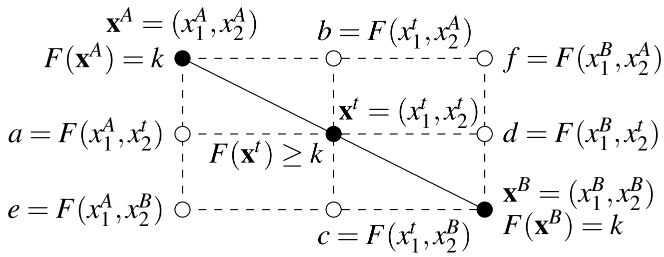



Figure 7: VISUAL GUIDE FOR PROOF OF PROPOSITION 2. The figure shows how to form a convex combination of vectors so that quasiconcavity fails in the proof of Proposition 2.

Let $x_1^A < x_1^t < x_1^B$, then we must have $x_2^A > x_2^t > x_2^B$ by monotonicity. Let $t = 1/2$. Then by strict partial convexity, we have

$$F(\mathbf{x}^t) - c < b - F(\mathbf{x}^t) \iff F(\mathbf{x}^t) < \frac{1}{2}(b+c),$$
$$F(\mathbf{x}^t) - a < d - F(\mathbf{x}^t) \iff F(\mathbf{x}^t) < \frac{1}{2}(a+d).$$

On the other hand, by submodularity, we have $k+F(\mathbf{x}^t) > a+b$ and $F(\mathbf{x}^t)+k > c+d$, which imply

$$k+F(\mathbf{x}^t) > \frac{1}{2}(a+b+c+d).$$

Combine the three inequalities to derive $k+F(\mathbf{x}^t)+\frac{1}{2}(a+b+c+d) > 2F(\mathbf{x}^t)+\frac{1}{2}(a+b+c+d) \implies k > F(\mathbf{x}^t)$. □

Although the relationship is not quite symmetric, it is interesting that supermodularity is used in one direction and submodularity is used in the other direction. Moreover, since this is a "negative" result, it can be extended to $n \geq 3$ case without any modification in the proof.[12]

**Corollary 2.** *Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is submodular and monotone. If $F(\mathbf{x})$ is quasiconcave with respect to $(x_i, x_j)$ where $i \neq j \in \{1,\ldots,n\}$, then it is locally concave with respect to $x_i$ or $x_j$.*

As an application, this implies that when all the factors are productive and substitutary, if the production function exhibits diminishing MRTS in two (out of several) factors, then at least one of the two factors must exhibit diminishing MP over some domain.

### 3.3. CONCAVITY AND PARTIAL CONCAVITY

In order to fully explore the interrelationships among concavity notions, we now consider going from weaker concavity to "full" concavity. This will also provide indirect linkages between partial concavity and quasiconcavity. It turns out that what we need to go from partial concavity to concavity is modularity. This is rather satisfying in that supermodularity (and submodularity to a lesser extent) played an important role in linking partial concavity and quasiconcavity.

Since we are generalizing from the differentiable case, it is natural to start from the Hessian conditions. For two-variable functions, the relevant condition is $F_{11}F_{22} > F_{12}^2$ or that the cross partial derivative is small relative to the own second-order partial derivatives. The derivative condition applies to an arbitrarily small neighborhood. But for a non-differentiable condition, any pair of vectors with a finite distance must be applicable. So it seems that the condition must allow for any sizes of $F_{ii}$'s, hence we are led to consider the limiting condition $F_{12} = 0$. Modularity suggests itself as an obvious analog of the differentiable condition $F_{12} = 0$. In fact, modularity has a more concrete characterization of additive separability.

[12] I am grateful to an anonymous referee for alerting me to this extension.

### 3.3.1. Modularity is equivalent to additive separability

A multi-variable function $F(x_1,\ldots,x_n)$ is *additively separable* if it can be decomposed into the sum of one-variable functions $F(x_1,\ldots,x_n) = f_1(x_1)+\cdots+f_n(x_n)$. Such a function obviously satisfies $F_{ij}=0$ for $i \neq j$ if differentiable. The following lemma establishes that modularity is equivalent to additive separability and will be used in proving Proposition 3 in the next subsection. (For simplicity, we restrict to $\mathbb{R}^2$ but our arguments generalize to $\mathbb{R}^n$.)

**Lemma 2.** *Consider a set of four vectors* $\{(a,b),(c,d),(a,d),(c,b)\}$ *that form a rectangle.* (*See Figure 8.*) *Then* $F(x,y)$ *is additively separable if and only if*

$$F(a,b)+F(c,d) = F(a,d)+F(c,b) \tag{2}$$

*holds for the rectangular vectors.*

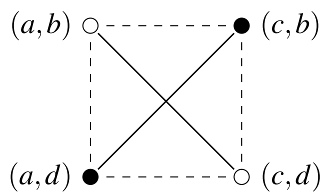



Figure 8: VISUAL GUIDE FOR LEMMA 2. The figure shows the positions of four vectors for characterizing additive separability in the proof of Lemma 2.

*Proof*: ($\Longrightarrow$) Suppose $F(x_1,x_2) = f_1(x_1)+f_2(x_2)$. Then $F(a,b)+F(c,d) = (f_1(a)+f_2(b))+(f_1(c)+f_2(d)) = (f_1(a)+f_2(d))+(f_1(c)+f_2(b)) = F(a,d)+F(c,b)$.

($\Longleftarrow$) Suppose that we have $F(a,b)+F(c,d) = F(a,d)+F(c,b)$ for an arbitrary set of four rectangular vectors. Consider $\{\underbrace{(0,0)}_{(a,b)},\underbrace{(x_1,x_2)}_{(c,d)},\underbrace{(x_1,0)}_{(a,d)},\underbrace{(0,x_2)}_{(c,b)}\}$. Since this set forms a rectangle we have

$$F(0,0)+F(x_1,x_2) = F(x_1,0)+F(0,x_2)$$
$$\Longrightarrow F(x_1,x_2) = F(x_1,0)+F(0,x_2)-F(0,0).$$

Define $f_1(x_1) = F(x_1,0)$ and $f_2(x_2) = F(0,x_2)-F(0,0)$. Then $F(x_1,x_2) = f(x_1)+f(x_2)$. □

To see that the lemma establishes equivalence between additive separability and modularity, let $\mathbf{x}^A = (a,b)$ and $\mathbf{x}^B = (c,d)$ in Figure 8. Then $\mathbf{x}^A \vee \mathbf{x}^B = (c,b)$

and $\mathbf{x}^A \wedge \mathbf{x}^B = (a,d)$. So the condition (3) becomes $F(\mathbf{x}^A \vee \mathbf{x}^B) + F(\mathbf{x}^A \wedge \mathbf{x}^B) = F(c,b) + F(a,d) = F(a,b) + F(c,d) = F(\mathbf{x}^A) + F(\mathbf{x}^B)$.

Now we turn to our next main result.

### 3.3.2. Modularity and partial concavity imply concavity

**Proposition 3.** *If $F(\mathbf{x})$ is modular (hence additively separable) and partially concave, then it is concave (hence quasiconcave).*

*Proof.* For simplicity, we will spell it out for $F(x_1,x_2)$. Consider two arbitrary vectors $\mathbf{x}^A = (x_1^A, x_2^A)$ and $\mathbf{x}^B = (x_1^B, x_2^B)$. We wish to show that $F(\mathbf{x}^t) \geq tF(\mathbf{x}^A) + (1-t)F(\mathbf{x}^B)$.

If $x_1^A = x_1^B$, then by partial concavity, $F(\cdot, x_2)$ is concave in $x_2$ and we are done. Similarly, if $x_2^A = x_2^B$, then $F(x_1,\cdot)$ is concave in $x_1$ so we are done. Therefore, let $x_1^A \neq x_1^B$ and $x_2^A \neq x_2^B$. Furthermore, without loss of generality, let $x_1^A < x_1^B$. Then there are two possible cases: (a) $x_2^A > x_2^B$, (b) $x_2^A < x_2^B$. In the first case, the two vectors are on the (downward) "diagonal" vertices of the rectangle, while in the second case, they are comparable ($\mathbf{x}^B \gg \mathbf{x}^A$) and on the (upward) "anti-diagonal" vertices. (See Figure 9.) In either case, we have $F(x_1^A, x_2^A) + F(x_1^B, x_2^B) = F(x_1^A, x_2^B) + F(x_1^B, x_2^A)$ by Lemma 2.

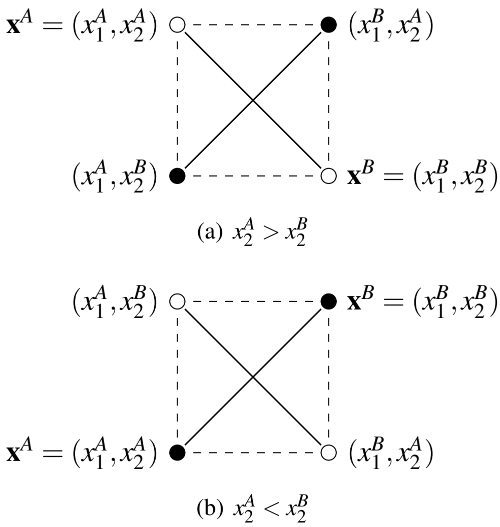



Figure 9: VISUAL GUIDE FOR PROOF OF PROPOSITION 3. The figures show two possible cases of the positions of four vectors in the proof of Proposition 3.

Therefore, for any two vectors $\mathbf{x}^A$ and $\mathbf{x}^B$ such that $x_1^A < x_1^B$ and $x_2^A \neq x_2^B$, we have

$$\begin{aligned}
F(x_1^t, x_2^t) &\geq tF(x_1^A, x_2^t) + (1-t)F(x_1^B, x_2^t) && \text{[concave in } x_1\text{]} \\
&\geq t\left(tF(x_1^A, x_2^A) + (1-t)F(x_1^A, x_2^B)\right) + (1-t)\left(tF(x_1^B, x_2^A) + (1-t)F(x_1^B, x_2^B)\right) && \text{[concave in } x_2\text{]} \\
&= t^2F(x_1^A, x_2^A) + t(1-t)\left(F(x_1^A, x_2^B) + F(x_1^B, x_2^A)\right) + (1-t)^2F(x_1^B, x_2^B) \\
&= t^2F(x_1^A, x_2^A) + t(1-t)\left(F(x_1^A, x_2^A) + F(x_1^B, x_2^B)\right) + (1-t)^2F(x_1^B, x_2^B) && \text{[add. separable]} \\
&= tF(x_1^A, x_2^A) + (1-t)F(x_1^B, x_2^B).
\end{aligned}$$

We can obviously extend this reasoning to $n \geq 3$. □

**Corollary 3.** *If $F(\mathbf{x})$ is modular and not concave, then it is not partially concave. That is, it is strictly convex with respect to at least one variable over some part of the domain.*

Modularity is obviously a strong assumption. It basically assumes away interaction between $x_i$ and $x_j$. We present some familiar examples below.

**Example 7** (modular and partially concave functions in economics)**.**

(a) The log form Cobb-Douglas utility $u(x_1, x_2) = \log(x_1^\alpha x_2^\beta) = \alpha \log(x_1) + \beta \log(x_2)$: This function shows strict partial concavity (diminishing marginal utilities) and modularity ($U_{12} = 0$), so is strictly concave and also strictly quasiconcave (diminishing MRS, indifference curves are convex to the origin).

(b) A CES production function $f(K, L) = \sqrt{K} + \sqrt{L}$: This production function shows strict partial concavity (diminishing marginal productivities) and modularity ($f_{KL} = 0$), so is strictly concave and also strictly quasiconcave (diminishing MRTS, isoquants are convex to the origin).

(c) A Cobb-Douglas form $f(x_1, x_2) = x_1^{1/4} x_2^{1/4}$, where $x_1 > 0$, $x_2 > 0$ is concave (hence partially concave) but not modular (not additively separable). So the converse of Proposition 3 is false: although concavity of $F$ implies partial concavity, it does not imply modularity. □

## 3.4. CONCAVITY AND QUASICONCAVITY

We now turn our attention to quasiconcavity and see under what conditions we can derive concavity from it. Quasiconcavity concerns the curvature of a level curve for any given function value, and is invariant to monotone transforms of function values. To derive concavity, we need to have some control over how different level curves are related. Homogeneity of the function provides such control.

A multi-variable function $F(\mathbf{x})$ is *homogeneous of degree r* if $F(\alpha\mathbf{x}) = \alpha^r F(\mathbf{x})$ for $\alpha > 0$. If $r = 1$, we have $F(\alpha\mathbf{x}) = \alpha F(\mathbf{x})$ and the function is essentially linear. If $0 < r \leq 1$, then $F(\alpha\mathbf{x}) = \alpha^r F(\mathbf{x}) \leq \alpha F(\mathbf{x})$ for $\alpha > 1$ so the function value increases less than linearly when the variables increase jointly in the direction of a ray from the origin. For production functions, $r = 1$ is equivalent to constant returns to scale and $0 < r < 1$ implies decreasing returns to scale. On the other hand, concavity means that the function value increases less than linearly in *any* direction.

### 3.4.1. Positivity, homogeneity and quasiconcavity imply concavity

Our final result says that quasiconcavity (diminishing MRTS) and homogeneity of $0 < r \leq 1$ (constant or decreasing returns to scale), combined with positivity of a function ensures that it is concave. Positivity of function $F(\mathbf{x})$ means $F(\mathbf{x}) > 0$ for all $\mathbf{x}$. In economic models, positivity is mostly a natural or harmless assumption. For example, production quantities can be assumed to be positive and utility scales can be chosen so that they are positive.

**Proposition 4.** *Suppose $F(\mathbf{x})$ is positive, homogeneous of degree $0 < r \leq 1$ and (strictly) quasiconcave, then $F$ is (strictly) concave.*

**Remark**. Theorem 21.15 in Simon and Blume (1994) proves concavity under the assumptions of positivity, homogeneity of degree 1, and quasiconcavity. Silberberg and Suen (2000)(p.140) offer two claims without proofs: that quasiconcavity and homogeneity of degree 1 imply concavity and that strict quasiconcavity and homogeneity of degree $0 < r < 1$ imply strict concavity. Our proposition clarifies and improves upon the theorem and the claims. □

*Proof.* If $F(\mathbf{x}) = k > 0$, then by homogeneity of degree $r$,

$$F(\frac{\mathbf{x}}{k^{1/r}}) = \frac{1}{k}F(\mathbf{x}) = 1.$$

(For the above to make sense, we need $k > 0$, which holds by positivity of $F$.)

Now consider two vectors $\mathbf{x}^1, \mathbf{x}^2$ such that $F(\mathbf{x}^1) = k_1 > 0$ and $F(\mathbf{x}^2) = k_2 > 0$. We want to show $F(\mathbf{x}^t) = F(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \geq tF(\mathbf{x}^1) + (1-t)F(\mathbf{x}^2) = tk_1 + (1-t)k_2$. First define

$$\theta \equiv \frac{tk_1^{1/r}}{tk_1^{1/r} + (1-t)k_2^{1/r}}.$$

Then by quasiconcavity

$$\begin{aligned} F\left(\theta\frac{\mathbf{x}^1}{k_1^{1/r}}+(1-\theta)\frac{\mathbf{x}^2}{k_2^{1/r}}\right) &\geq \min\left\{F\left(\frac{\mathbf{x}^1}{k_1^{1/r}}\right),F\left(\frac{\mathbf{x}^2}{k_2^{1/r}}\right)\right\} \\ &= \min\left\{\frac{1}{k_1}F(\mathbf{x}^1),\frac{1}{k_2}F(\mathbf{x}^2)\right\}=1, \end{aligned} \quad (3)$$

where the inequality would be strict for $0<t<1$ if $F$ is strictly quasiconcave. On the other hand, we have

$$\begin{aligned} \theta\frac{\mathbf{x}^1}{k_1^{1/r}}+(1-\theta)\frac{\mathbf{x}^2}{k_2^{1/r}} &= \frac{tk_1^{1/r}}{tk_1^{1/r}+(1-t)k_2^{1/r}}\cdot\frac{\mathbf{x}^1}{k_1^{1/r}}+\frac{(1-t)k_2^{1/r}}{tk_1^{1/r}+(1-t)k_2^{1/r}}\cdot\frac{\mathbf{x}^2}{k_2^{1/r}} \\ &= \frac{t\mathbf{x}^1+(1-t)\mathbf{x}^2}{tk_1^{1/r}+(1-t)k_2^{1/r}}. \end{aligned}$$

Plugging this back into (3),

$$\begin{aligned} &F\left(\frac{t\mathbf{x}^1+(1-t)\mathbf{x}^2}{tk_1^{1/r}+(1-t)k_2^{1/r}}\right)=\frac{1}{(tk_1^{1/r}+(1-t)k_2^{1/r})^r}F(\mathbf{x}^t)\geq 1 \\ \Longrightarrow\ &F(\mathbf{x}^t)\geq(tk_1^{1/r}+(1-t)k_2^{1/r})^r. \end{aligned}$$

where the inequality would be strict for $0<t<1$ if $F$ is strictly quasiconcave. Note also that the inequality is preserved by positivity of $k_1, k_2$. Now the proof is concluded by the following lemma. □

**Lemma 3.** *$(tk_1^{1/r}+(1-t)k_2^{1/r})^r \geq tk_1+(1-t)k_2$ for $0<r\leq 1$, $k_1>0$, $k_2>0$ and $0\leq t\leq 1$.*

*Proof of Lemma.* Since $0<r\leq 1$, $g(x)=x^r$ is concave, which implies

$$\begin{aligned} (tk_1^{1/r}+(1-t)k_2^{1/r})^r &= g(tk_1^{1/r}+(1-t)k_2^{1/r}) \\ &\geq tg(k_1^{1/r})+(1-t)g(k_2^{1/r}) \\ &= tk_1+(1-t)k_2, \end{aligned}$$

where the inequality would be strict if $0<r<1$ and $0<t<1$. □

**Remark**: The lemma is a special case of the weighted power mean inequality (e.g., Qi *et al.*, 2000) . □

The proof involves some difficult-to-grasp constructions. So the following example tries to offer some (imperfect) intuitions.

**Example 8.**

- Suppose $r = 1/2$. Consider $\mathbf{x}^1$ and $\mathbf{x}^2$ such that $F(\mathbf{x}^1) = \sqrt{2}$ and $F(\mathbf{x}^2) = \sqrt{3}$. If $F$ is to be (strictly) concave, we should have $F(\frac{1}{2}\mathbf{x}^1 + \frac{1}{2}\mathbf{x}^2) > \frac{\sqrt{2}+\sqrt{3}}{2}$ where we chose $t = 1/2$. See Figure 10 for visual guide.

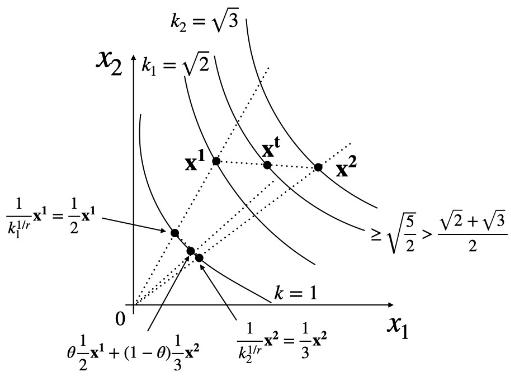



Figure 10: VISUAL GUIDE FOR EXAMPLE 8. The figure illustrates how various constructions are related in the proof of Proposition 4.

- The first step in the proof is to "shrink" vectors $\mathbf{x}^1$ and $\mathbf{x}^2$ so that they are on the contour for $F(\mathbf{x}) = 1$. This is done by choosing $\frac{1}{k_1^{1/r}}\mathbf{x}^1 = \frac{1}{2}\mathbf{x}^1$ and $\frac{1}{k_2^{1/r}}\mathbf{x}^2 = \frac{1}{3}\mathbf{x}^2$. Now take a convex combination of these two shrunk vectors on the same contour, with $\theta = \frac{tk_1^{1/r}}{tk_1^{1/r}+(1-t)k_2^{1/r}} = \frac{\frac{1}{2}\times 2}{\frac{1}{2}\times 2+\frac{1}{2}\times 3} = \frac{2}{5}$. By quasiconcavity this combination is in the upper contour set for $F(\mathbf{x}) = 1$, or $F(\theta\frac{1}{2}\mathbf{x}^1 + (1-\theta)\frac{1}{3}\mathbf{x}^2) \geq 1$.

- This combination is in fact a shrunk version of $\mathbf{x}^t$ where the shrinking factor is $tk_1^{1/r} + (1-t)k_2^{1/r} = \frac{1}{2} \times 2 + \frac{1}{2} \times 3 = \frac{5}{2}$. Since $F$ is homogeneous of degree $1/2$, this means that $F(\mathbf{x}^t) \geq \sqrt{\frac{5}{2}}(\approx 1.58)$, strictly greater than $\frac{\sqrt{2}+\sqrt{3}}{2}(\approx 1.57)$.

- For concreteness, let $F(x_1, x_2) = x_1^{1/4}x_2^{1/4}$. This is positive on $\mathbb{R}^2_{++}$, homogeneous of degree $1/2$ and strictly quasiconcave. Choose $\mathbf{x}^1 = (1,4)$ so $F(1,4) = \sqrt{2}$ and $\mathbf{x}^2 = (9,1)$ so $F(9,1) = \sqrt{3}$. Then for example, $F(\frac{1}{2}\mathbf{x}^1 + \frac{1}{2}\mathbf{x}^2) = F(5, \frac{5}{2}) = \frac{\sqrt{20}}{2}(\approx 2.24) > \frac{\sqrt{2}+\sqrt{3}}{2} = \frac{1}{2}F(\mathbf{x}^1) + \frac{1}{2}F(\mathbf{x}^2)$. $\square$

The proposition is rather useful precisely because the proof is not so straightforward. To illustrate this point, we offer two examples.

**Example 9** (an optimal growth model)**.**

- Stokey *et al.* (1989)(Section 2.1) outline a benchmark model of optimal growth by beginning with a production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$. The function $F(k,n)$ is assumed to be continuously differentiable, strictly increasing, homogeneous of degree 1 and strictly quasiconcave (with additional parametric conditions). They simplify by normalizing $n = 1$ and introducing a one-variable per capita production function $f(k) = F(k,1) + (1-\delta)k$. Then in Exercise 2.1, they assert that given the assumptions on $F$, the function $f$ is continuously differentiable, strictly increasing and strictly concave (with some parametric conditions). These properties are exploited in their ensuing analysis.

- Most of their assertions on $f(k)$ in Exercise 2.1 are immediate except for strict concavity of $f(k)$. Our Proposition 4 can be invoked to prove this assertion and more. In fact, by the supposed positivity (the range of the function is $\mathbb{R}_+$), homogeneity of degree 1 and strict quasiconcavity, we know $F$ is strictly concave and hence strictly partially concave (strictly concave in each variable). We don't need differentiability or monotonicity of $F$. We can allow for homogeneity of degree less than 1 as well. □

The next example shows that Proposition 4 can work even when derivative conditions are not sufficient.

**Example 10** (derivative conditions vs proposition 4)**.**

- Consider production function $F(K,L)$, which is twice continuously differentiable, increasing, homogeneous of degree one, and strictly quasiconcave. We wish to show that $F$ is strictly partially concave (both factors have diminishing marginal productivity). We first try derivative conditions.

- By Euler's theorem on homogeneous functions we have

$$F(K,L) = F_K(K,L)K + F_L(K,L)L. \tag{4}$$

  Partially differentiate (4) with respect to $K$ to obtain

$$F_K(K,L) = F_{KK}(K,L)K + F_K(K,L) + F_{LK}(K,L)L. \tag{5}$$

Rearranging (5) and appealing to Young's theorem on cross-partials, we have

$$F_{KK}(K,L)K + F_{KL}(K,L)L = 0. \tag{6}$$

(6) also confirms that $F_K$ is homogeneous of degree zero. Similarly, we can show

$$F_{LL}(K,L)L + F_{KL}(K,L)K = 0. \tag{7}$$

From (6) and (7), we have

$$F_{KK} = -\frac{L}{K}F_{KL}, \quad F_{LL} = -\frac{K}{L}F_{KL}, \tag{8}$$

which can be plugged into (1) of Lemma 1 in Section 2.2 to obtain

$$F_{KK}F_L^2 - 2F_KF_LF_{KL} + F_{LL}F_K^2 = -\left(\frac{L}{K}F_L^2 + 2F_KF_L + \frac{K}{L}F_K^2\right)F_{KL}. \tag{9}$$

By strict quasi-concavity of $F$, the expression (9) must be negative:

$$-\underbrace{\left(\frac{L}{K}F_L^2 + 2F_KF_L + \frac{K}{L}F_K^2\right)}_{>0}F_{KL} < 0.$$

- We already know that the expressions within the parentheses are positive from the assumptions. Therefore we must have $F_{KL} > 0$. Going back to (8), we conclude $F_{KK} < 0$ and $F_{LL} < 0$. Hence, we are able to establish that

  **Lemma 4.** *If $F(\cdot)$ is twice continuously differentiable, increasing, homogeneous of degree one, and strictly quasi-concave, then $F$ is strictly partially concave.*

- Moreover, by checking the condition $F_{KK}F_{LL} - F_{KL}^2 = (-\frac{L}{K}F_{KL}) \times (-\frac{K}{L} F_{KL}) - F_{KL}^2 = 0$, we see that the Hessian is negative semi-definite, so $F$ is concave.

- However, our Proposition 4 assures us that $F$ is strictly concave because it is positive, homogeneous of degree $0 < r \leq 1$ and strictly quasiconcave. Note the differences between Lemma 4 and Proposition 4: the lemma is limited to differentiable, increasing and homogeneous of degree one functions; the proposition drops differentiability and monotonicity, adds positivity, and extends to homogeneity of degree not exceeding one to produce a stronger conclusion of strictly concavity. □

## 4. DISCUSSION AND CONCLUDING REMARKS

Our main results consist of four propositions. In these concluding remarks, we summarize them and discuss limitations, potential applications and/or extensions of each.

First of all, we note that the monotonicity assumption underlying Propositions 1 and 2 and the positivity assumption in Proposition 4 seem harmless in typical economic models. But obviously care must be taken to account for these assumptions when we consider more general models. For example, when considering non-monotone preferences, we need to be careful about applying or interpreting concavity or quasiconcavity.

Proposition 1 shows (for two-variable functions) that supermodularity, monotonicity and partial concavity imply quasiconcavity. For a production function, it means that if two factors show complementarity, positive and strictly diminishing marginal productivities, then the MRTS between the factors is strictly diminishing, which then would imply the existence of a unique cost-minimizing choice of factors for a given quantity.

Proposition 1's supermodularity is a natural extension of the derivative condition $F_{12} \geq 0$ or complementarity between the variables. It is interesting to note that Proposition 1 derives an ordinal property (quasiconcavity) from a cardinal property (partial concavity) with the aid of complementarity. Given that most results in consumer theory rely only on ordinal preferences, we can actually extend Proposition 1 by considering monotone transforms of supermodular functions.

This amounts to the consideration of *quasi-supermodularity* (QSM).[13] A function $F$ is QSM if $F(\mathbf{x}^A) \geq F(\mathbf{x}^A \wedge \mathbf{x}^B) \implies F(\mathbf{x}^A \vee \mathbf{x}^B) \geq F(\mathbf{x}^B)$ and $F(\mathbf{x}^A) > F(\mathbf{x}^A \wedge \mathbf{x}^B) \implies F(\mathbf{x}^A \vee \mathbf{x}^B) > F(\mathbf{x}^B)$. For our purposes, it suffices to know that monotone transforms of a supermodular function are quasi-supermodular. A QSM utility function represents essentially the same preferences as a SPM utility function, hence we can utilize Proposition 1 in analyzing such cases as well.

Proposition 2 (for two-variable functions) and its corollary (for $n$-variable functions) show that submodularity (between two variables), monotonicity and quasiconcavity imply local concavity with respect to at least one variable. Proposition 2 uses submodularity, which corresponds to the derivative condition $F_{12} \leq 0$ or substitutarity between the variables. It is not a very strong characterization but it can yield some interesting insights. For example, when we have a production function with diminishing MRTS (quasiconcavity) and if the factors are

[13]QSM is an ordinal version of SPM in that it preserves the 'ordering' of vertical distance across a horizontal movement. See Milgrom and Shannon (1994) for more details.

substitutary, then the technology exhibits decreasing marginal productivity for at least one factor over some part of the domain.

Proposition 3 invokes modularity, equivalent to additive separability, to derive concavity (hence quasiconcavity as well) from partial concavity. So if there are no interactions between the variables, being concave in each variable is sufficient to ensure the function is concave (and also its contours have diminishing absolute slopes). This can be a convenient tool to quickly determine concavity or quasiconcavity of some popular basic functional forms. In fact, many textbook functional forms do have additive separability, *e.g.* (i) linear utility $u(x_1,x_2) = x_1 + x_2$, (ii) quasilinear utility $u(x_1,x_2) = v(x_1) + x_2$ (with $v(\cdot)$ a concave function), (iii) a form of CES utility $u(x_1,x_2) = x_1^r + x_2^r$ (with $r < 1$) as well as (iv) any intertemporal model involving discounted sum of instantaneous utilities such as $U(x_1,x_2,\ldots) = \sum_{t=0} \beta^t u_t(x_t)$.

Finally, Proposition 4 shows that positivity and homogeneity of degree not exceeding 1 allow us to derive concavity (and partial concavity) from quasiconcavity. In other words, if production technology shows constant or decreasing returns to scale (over the domain where it has positive production), assuming diminishing MRTS is sufficient to guarantee that the function is concave (and its factors show diminishing marginal productivities). As illustrated by two examples in the previous section, this can be a useful tool for deriving strong results.

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