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Risks with semi-infinite support: characterizations and applications*

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Abstract There are many situations that we need to model as risky prospects whose values are censored below at 0, hence defined on semi-infinite support. As useful tools for such models, we derive analytic characterizations of risks such as certainty equivalent and risk premium, for gamma and lognormal distributions and utility functions that have constant risk aversion. As a main application, we consider an extension of the 'linear contract, exponential utility, normally distributed risks' (LEN) moral hazard model to gamma distributed risks. We also discuss other potential applications, ranging from loan contracts to comparison of income distributions.

Keywords certainty equivalent, gamma and lognormal distributions, CARA, CRRA, moral hazard

JEL Classification D81, C46, D82

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1. INTRODUCTION

This paper offers analytic characterizations of risks that are distributed over a semi-infinite interval $[0,\infty)$ or $(0,\infty)$. The familiar normal distribution has the whole real line $(-\infty,\infty)$ as its support, which may be inappropriate if the realized values are censored below at 0, e.g. if the random variable describes nominal interest rates, nominal incomes or stock prices. We aim to offer useful results for modeling such situations in this paper.

The *certainty equivalent* of risky prospects is a central concept in the economics of uncertainty. In short, the certainty equivalent is the sure amount of prize that is equivalent in utility to a risky prospect ("lottery"), hence is defined for a particular risk and a particular utility function. It is used in formulating the notion of risk aversion and in defining the risk premium. (Arrow, 1971; Pratt, 1964) For a risk averse person, the certainty equivalent of a risky prize is less than the expected value of the prize and the difference is the risk premium.

The certainly equivalent greatly simplifies analysis of risky situations as it is typically much easier to handle than the expected utility (integrals and probability measures). A particularly simple expression of the certainty equivalent can play a useful role in applications, e.g. in describing optimal contracts for the "LEN" model of moral hazard (Bolton and Dewatripont, 2005, Section 4.2). In fact, we will later discuss an extension of the LEN model.

Since our aim is geared towards potential theoretical applications, we seek results about specific classes of distributions and utility functions. We consider two probability distributions supported on a semi-infinite interval: the gamma distribution and the lognormal distribution. Both are widely used in economics and finance (Kleiber and Kotz, 2003) and flexible enough to accommodate diverse modeling situations. However, concrete characterizations of the certainty equivalent of such risks are not readily available in the literature, partly because the usage of these distributions has been mostly limited to empirical work.

The *gamma distribution* is a two-parameter continuous probability distribution. It is one of the most flexible distributions, and includes as its special cases the exponential distribution and the chi-square (χ^2) distribution. It has another attractive feature that the normal distribution lacks: the gamma distribution can accommodate both (almost) symmetric and skewed distributions by varying parameter values, which gives us useful flexibility in modeling.

The *lognormal distribution* is another two-parameter continuous probability distribution supported on a semi-infinite interval. Perhaps not as flexible as the gamma distribution, it has the advantage of a close connection with the normal distribution: the log of a lognormally distributed random variable is normally distributed. The logarithmic scale is convenient for modeling growth phenomena and the lognormal distribution is often employed in applications involving income distributions or stock market indices, for example.

In Sections 2 and 3, we derive the certainty equivalent and the risk premium of lotteries following either of the two distributions, for utility functions that are constant in risk aversion—in absolute and relative measures of Arrow and Pratt. These utility functions are tractable and easily parameterized, as they involve familiar forms of exponential or power functions (and the log function as a limiting case), hence are often used in theoretical as well as empirical work. In order to demonstrate usefulness of our results, we discuss applications. In Section 4, we extend the LEN model to allow for gamma distributed risks and compare the form of optimal contracts. Section 5 sketches other potential applications ranging from loan contracts with random interest rates to comparisons of income distributions.

Throughout the paper, we assume that the preferences can be represented by von Neumann-Morgenstern utility functions and that the expected utility hypothesis applies.

2. CHARACTERIZATION OF GAMMA DISTRIBUTED RISKS

Consider a lottery with random prize w that follows the gamma distribution[1] whose probability density function is given by

$$f(w) = \frac{\beta^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\beta w} \tag{1}$$

where $\alpha > 0$, $\beta > 0$ and $\Gamma(\cdot)$ is the gamma function.[2] This distribution is supported on the semi-infinite interval $[0, \infty)$ and has two positive parameters α and β, commonly referred to as 'shape' (or 'skewness') and 'rate' (or 'scale') respectively. Its mean is $E[w] = \alpha/\beta \equiv \mu$ and the variance is $Var[w] = \alpha/\beta^2 \equiv \sigma^2$. With some abuse of notation, we will write $w \sim \Gamma(\alpha, \beta)$.

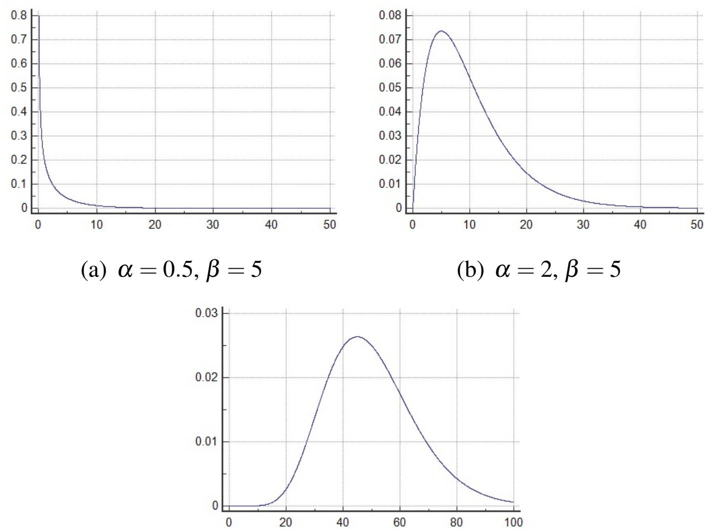
The gamma distribution is qualitatively different for $0 < \alpha \leq 1$ and for $\alpha > 1$. If $0 < \alpha \leq 1$, the gamma density $f(w)$ is strictly decreasing with the mode (the highest density) occurring at $w = 0$ (see Figure 1(a)); a special case is $\alpha =$

[1] We follow DeGroot's (1970, p.39) notation. Some authors use the parameter $1/\beta$ in place of β. See Hogg et al (2005, Section 3.3) for introductory treatment. Kleiber and Kotz (2003) Section 5.2 provides a survey related to applications in economics.

[2] The gamma function is defined as $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$. Some notable properties are: $\Gamma(1) = 1$, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ and if n is an integer, then $\Gamma(n) = (n-1)!$.

1 when it is the exponential distribution. In contrast, if $\alpha > 1$, the density is unimodal with an interior mode (Figure 1(b)).

As α increases, the gamma density becomes almost symmetric and approximates the shape of the normal density, but supported on the semi-infinite interval (Figure 1(c)). Large values of α may be useful for some modeling situations if the distribution is almost symmetric but restricted to positive values.



(a) $\alpha = 0.5, \beta = 5$ (b) $\alpha = 2, \beta = 5$

(c) $\alpha = 10, \beta = 5$

Figure 1: Gamma density functions

2.1. CARA UTILITY

Consider the utility function parametrized by θ:

$$u(w;\theta) = -\exp(-\theta w) \tag{2}$$

so that $-u''/u' = \theta$ is the constant absolute risk aversion coefficient.

For the lottery having the gamma density (1), the expected utility can be computed as[3]

$$\begin{aligned} E[u(w)] &= \int -e^{-\theta w} \frac{\beta^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\beta w} dw \\ &= -\int \frac{\beta^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-(\beta+\theta)w} dw \\ &= -\frac{\beta^\alpha}{(\beta+\theta)^\alpha} \underbrace{\int \frac{(\beta+\theta)^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-(\beta+\theta)w} dw}_{=1} \\ &= -\left(\frac{\beta}{\beta+\theta}\right)^\alpha \end{aligned}$$

where in the penultimate line, the integrand has been transformed into a gamma distribution with parameters α and $(\beta+\theta)$. The expected utility turns out to have a simple expression involving the risk aversion parameter θ and the distribution parameters α and β.

We now derive the expression for the certainty equivalent of the lottery. Denoting it as w_C, we must have $u(w_C) = E[u(w)]$ so

$$-\exp(-\theta w_C) = -\left(\frac{\beta}{\beta+\theta}\right)^\alpha \quad \Longrightarrow \quad w_C = \frac{\alpha}{\theta} \ln\left(\frac{\beta+\theta}{\beta}\right)$$

Hence, we obtained the expression for the certainty equivalent of a gamma-distributed risk for the CARA utility function. Noting $E[w] = \alpha/\beta$, the risk premium is also easily computed.

$$RP = \frac{\alpha}{\beta}\left[1 - \frac{\beta}{\theta}\ln\left(\frac{\beta+\theta}{\beta}\right)\right]$$

The following proposition summarizes the results obtained so far.

Proposition 1. *If the random prize w follows $\Gamma(\alpha,\beta)$ and if preferences are represented by the CARA utility function (2), then the certainty equivalent w_C of the prize is given by*

$$w_C = \frac{\alpha}{\theta}\ln\left(1+\frac{\theta}{\beta}\right) \tag{3}$$

and the risk premium is given by

$$RP = \frac{\alpha}{\beta}\left[1 - \frac{\beta}{\theta}\ln\left(1+\frac{\theta}{\beta}\right)\right] \tag{4}$$

[3]We omit ranges of integration throughout the paper for ease of notations.

Some corollaries are immediate and they will be useful for applications later. The first one offers alternative formulas involving the mean μ and the variance σ^2 of the distribution.

Corollary 1. *If we let $\mu = \alpha/\beta$ and $\sigma^2 = \alpha/\beta^2$ be the mean and the variance of w, then we have*

$$w_C = \frac{\mu^2}{\theta\sigma^2} \ln\left(1 + \frac{\theta\sigma^2}{\mu}\right) \tag{5}$$

$$RP = \mu\left[1 - \frac{\mu}{\theta\sigma^2} \ln\left(1 + \frac{\theta\sigma^2}{\mu}\right)\right] \tag{6}$$

Proof: *From $\mu = \alpha/\beta$ and $\sigma^2 = \alpha/\beta^2$, we have $\alpha = \mu^2/\sigma^2$ and $\beta = \mu/\sigma^2$. Substituting these into (3) and (4) yield (5) and (6), respectively.* ∎

The next corollary considers scalar multiples of a gamma distributed risk. This will come in handy when we consider linear contracts later.

Corollary 2. *Suppose $w \sim \Gamma(\alpha, \beta)$ and consider a new random variable $v \equiv kw$ for some $k \in \mathbb{R}$. Then we have $v \sim \Gamma(\alpha, \beta/k)$. Hence if $w \sim \Gamma(\alpha, \beta)$, then the certainty equivalent of $v = kw$ is*

$$v_C = \frac{\alpha}{\theta} \ln\left(1 + \frac{k\theta}{\beta}\right) = \frac{\mu^2}{\theta\sigma^2} \ln\left(1 + \frac{k\theta\sigma^2}{\mu}\right) \tag{7}$$

Proof. That v also follows a gamma distribution is obvious. Let $v \sim \Gamma(\alpha', \beta')$. Since $E[kw] = kE[w]$ and $Var[kw] = k^2 Var[w]$, we have $\alpha' = (E[kw])^2/Var[kw] = (E[w])^2/Var[w] = \alpha$ and $\beta' = E[kw]/Var[kw] = E[w]/(kVar[w]) = \beta/k$. Therefore we can use the formula (3) with β replaced by β/k. ∎

Corollary 3 gives approximate formulas that apply for limited parameter values.

Corollary 3. *Suppose $w \sim \Gamma(\alpha, \beta)$. If θ/β is sufficiently small or equivalently if $\theta\sigma^2/\mu$ is sufficiently small, then*

$$w_C \approx \frac{\alpha}{\beta}\left(1 - \frac{\theta}{2\beta}\right) = \mu - \frac{1}{2}\theta\sigma^2 \tag{8}$$

Proof. By first-order Taylor expansion, we have $\ln(1+x) \approx x - \frac{1}{2}x^2$ when x is sufficiently small. Use this approximation in (3) and (5) to obtain (8). ∎

The latter formula in (8) in terms of μ and σ^2 is the same as the one for a normally distributed risk. We see that there is continuity between a normally distributed risk and a gamma distributed risk when θ (the extent of risk aversion) or σ^2 (the size of risk) is relatively small.[4]

The next proposition offers consistency check and comparative statics.

Proposition 2. *For the gamma distributed risk and the CARA utility, we have the expected inequality*

$$E[w] > w_C \quad [\text{or equivalently, } RP > 0] \tag{9}$$

As we vary the parameters α, β, or θ, we have the following comparative static results:

$$\frac{\partial w_C}{\partial \alpha} > 0, \quad \frac{\partial RP}{\partial \alpha} > 0 \tag{10}$$

$$\frac{\partial w_C}{\partial \beta} < 0, \quad \frac{\partial RP}{\partial \beta} < 0 \tag{11}$$

$$\frac{\partial w_C}{\partial \theta} < 0, \quad \frac{\partial RP}{\partial \theta} > 0 \tag{12}$$

As we vary the mean μ or the variance σ^2, we have the following comparative static results:

$$\frac{\partial w_C}{\partial \mu} > 0, \quad \frac{\partial RP}{\partial \mu} > 0 \tag{13}$$

$$\frac{\partial w_C}{\partial \sigma^2} < 0, \quad \frac{\partial RP}{\partial \sigma^2} > 0 \tag{14}$$

Proof: See Appendix. ∎

Changes in either α or β affect both the mean and the variance of the random variable, and intuitions for (10) or (11) are not straightforward. They say that an increase in the shape parameter α leads to higher certainty equivalent and risk premium; an increase in the scale parameter β leads to lower certainty equivalent and risk premium.

(12) through (14) are more intuitive. As the degree of risk aversion increases, the certainty equivalent falls and the risk premium rises; as the mean (the expected prize) increases, both the certainty equivalent and the risk premium rises; and finally as the variance (the size of risk) increases, the certainty equivalent falls and the risk premium rises.

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[4] In fact, this first-order approximation is a general property; see Pratt (1964).

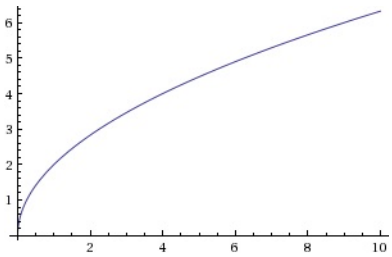
2.2. CRRA UTILITY

Now consider a different class of utility functions (again parametrized by θ):

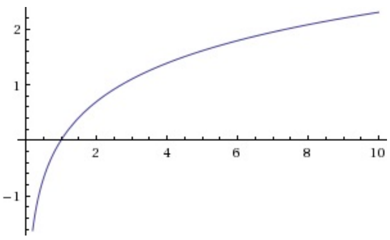
$$u(w;\theta) = \begin{cases} w^{1-\theta}, & 0 < \theta < 1 \\ \ln(w), & \theta = 1 \\ -w^{1-\theta}, & \theta > 1 \end{cases} \tag{15}$$

so that $-u''/u' = \theta/w$. Hence, the absolute risk aversion is decreasing in w (DARA) while the relative risk aversion is constant at θ (CRRA). The parameter θ now represents the constant *relative* (or *proportional*) risk aversion coefficient.

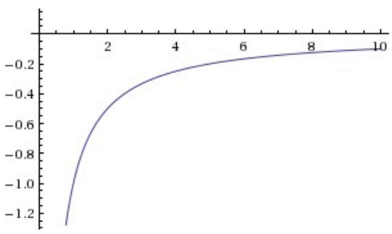
The parametrization of CRRA utility is a bit trickier than that of CARA utility, as shown by three sub-cases in (15).[5] As Figure 2 shows, all three sub-cases are similar in overall shape, but the technical difference lies in the boundedness of utility values: utilities are bounded below (and unbounded above) for $0 < \theta < 1$ and bounded above (and unbounded below) for $\theta > 1$, while they are unbounded for $\theta = 1$.



(a) $\theta = 0.5$



(b) $\theta = 1$



(c) $\theta = 2$

Figure 2: Illustrative graphs of CRRA utility function for different θ values

We will consider the case $0 < \theta < 1$ here. Mathematically the $\theta > 1$ case

[5]Some authors combine the $\theta \neq 1$ cases to $u(w) = w^{1-\theta}/(1-\theta)$, while Bikhchandani et al (2013, p.92) omits the $\theta > 1$ case. We follow Pratt's (1964) exposition.

is almost symmetric. The $\theta = 1$ case ($u = \ln(w)$) is singular and can be treated separately if needed.[6]

For $u(w) = w^{1-\theta}$, $0 < \theta < 1$, we have

$$
\begin{aligned}
E[u(w)] &= \int w^{1-\theta} \frac{\beta^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\beta w} dw \\
&= \int \frac{\beta^\alpha}{\Gamma(\alpha)} w^{(\alpha+1-\theta)-1} e^{-\beta w} dw \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1-\theta)}{\beta^{\alpha-\theta+1}} \underbrace{\int \frac{\beta^{\alpha-\beta+1}}{\Gamma(\alpha-\theta+1)} w^{(\alpha+1-\theta)-1} e^{-\beta w} dw}_{=1} \\
&= \frac{1}{\beta^{1-\theta}} \cdot \frac{\Gamma(\alpha+1-\theta)}{\Gamma(\alpha)}
\end{aligned}
$$

While we can now easily derive the formula for the certainty equivalent from the above expression, one difficulty is that the gamma function $\Gamma(\cdot)$ does not have an easy explicit expression for non-integer values. By the condition $0 < \theta < 1$, the two arguments of gamma function above, '$\alpha + 1 - \theta$' and 'α' would not be integers simultaneously in general.

Fortunately, for sufficiently large values of α, we also have a simpler asymptotic expression (Tricomi and Erdélyi, 1951) given by[7]

$$
\frac{\Gamma(\alpha-\theta+1)}{\Gamma(\alpha)} \approx \alpha^{1-\theta} \left[1 - \frac{\theta(1-\theta)}{2\alpha}\right] \quad \text{for large } \alpha
$$

in which case we can write

$$
E[u(w)] \approx \frac{1}{\beta^{1-\theta}} \cdot \alpha^{1-\theta} \left[1 - \frac{\theta(1-\theta)}{2\alpha}\right] = \left(\frac{\alpha}{\beta}\right)^{1-\theta} \left[1 - \frac{\theta(1-\theta)}{2\alpha}\right] \quad \text{for large } \alpha
$$

Therefore, for large values of α, we have the approximate identity

$$
\begin{aligned}
[w_C]^{1-\theta} &\approx \left(\frac{\alpha}{\beta}\right)^{1-\theta} \left[1 - \frac{\theta(1-\theta)}{2\alpha}\right] \quad \text{for large } \alpha \\
\Longrightarrow w_C &\approx \frac{\alpha}{\beta} \left[1 - \frac{\theta(1-\theta)}{2\alpha}\right]^{1/(1-\theta)} \quad \text{for large } \alpha
\end{aligned}
$$

These are summarized in the following proposition.

[6] A characterization of the $E[\ln(x)]$ when x follows a gamma distribution can be given in terms of the so-called digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$. See Johnson et al (1995, Section 17.5).

[7] Tricomi and Erdélyi (1951) give more refined approximate formulas applicable for smaller values of α.

Proposition 3. *If the random prize w follows $\Gamma(\alpha,\beta)$ and preferences are represented by the CRRA utility function $u(w;\theta) = w^{1-\theta}$, where $0 < \theta < 1$, then the certainty equivalent w_C of the prize is given by the expression*

$$w_C = \frac{1}{\beta}\left[\frac{\Gamma(\alpha+1-\theta)}{\Gamma(\alpha)}\right]^{1/(1-\theta)} \tag{16}$$

(which can be written in terms of μ and σ^2 as well but the expression is unwieldy and omitted). In addition, for sufficiently large α, we have the approximate expression

$$w_C \approx \frac{\alpha}{\beta}\left[1-\frac{\theta(1-\theta)}{2\alpha}\right]^{1/(1-\theta)} = \mu\left[1-\frac{\theta(1-\theta)\sigma^2}{2\mu^2}\right]^{1/(1-\theta)} \tag{17}$$

The exact and approximate risk premium expressions are as follows

$$RP = \frac{\alpha}{\beta} - \frac{1}{\beta}\left[\frac{\Gamma(\alpha+1-\theta)}{\Gamma(\alpha)}\right]^{1/(1-\theta)} \tag{18}$$

$$RP \approx \frac{\alpha}{\beta}\left[1-\left[1-\frac{\theta(1-\theta)}{2\alpha}\right]^{1/(1-\theta)}\right] = \mu\left[1-\left[1-\frac{\theta(1-\theta)\sigma^2}{2\mu^2}\right]^{1/(1-\theta)}\right] \tag{19}$$

, for large α

We could also state results on consistency check and comparative statics, analogous to Proposition 2. The inequality $w_C < E[w]$ should hold in principle for the exact expression (16), as the utility function shows risk aversion but writing out the proof is challenging because of the gamma functions.[8] The inequality is easy to show for the approximate expression (17).[9] Most of the comparative

[8] A non-rigorous but heuristic argument for $\alpha < 1$ and very small θ is as follows: Rewrite the equation (16) as

$$w_C = \frac{\alpha}{\beta}\left[\frac{\Gamma(\alpha+1-\theta)}{\alpha^{1-\theta}\Gamma(\alpha)}\right]^{\frac{1}{1-\theta}} = E[w]\cdot\left[\frac{\Gamma(\alpha+1-\theta)}{\alpha^{1-\theta}\Gamma(\alpha)}\right]^{\frac{1}{1-\theta}}$$

Since $1/(1-\theta) > 1$, we need to show the expression within the brackets $[\cdot]$ is less than 1. The gamma function satisfies the recurrence relation $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. If θ is very small $1-\theta \approx 1$, and we could say $\Gamma(\alpha+1-\theta) \approx \alpha\Gamma(\alpha)$. Then the expression inside the brackets becomes α^{θ}, which is less than 1 for $\alpha < 1$.

[9] This follows easily because $w_C = E[w]\cdot\left[1-\frac{\theta(1-\theta)}{2\alpha}\right]^{1/(1-\theta)}$ and $1-\frac{\theta(1-\theta)}{2\alpha} < 1$.

static results are as expected, although some are a bit involved and are omitted for brevity. A few remarks are in order in view of potential applications.

Remark 1. The formula (17) holds for large α only. As noted before, if α is large, the gamma distribution is almost symmetric. On the other hand, both mean and variance are determined by α and β together, so a large α alone doesn't necessarily constrain sizes of mean or variance. ■

Remark 2. Even the simple formula (17) may seem too complicated for easy manipulations. But the certainty equivalent can be safely transformed monotonically as it is a "sure" amount. The von Neumann-Morgenstern expected utility is cardinal (expressing an attitude towards risk), but when $u'(\cdot) > 0$, the agent maximizing $E[u(w)]$ is equivalent to maximizing w_C, which is again equivalent to maximizing $(w_C)^k$ for an arbitrary $k > 0$. If $0 < \theta < 1$, then we can safely focus on

$$1 - \frac{\theta(1-\theta)}{2\alpha} \quad \text{or} \quad 1 - \frac{\theta(1-\theta)\sigma^2}{2\mu^2}$$

which are relatively simple to handle. For example, the above expressions show that w_C is non-monotonic with regards to the degree of relative risk aversion θ, while it is increasing in α, μ and decreasing in σ^2 (and independent of β). ■

3. CHARACTERIZATION OF LOGNORMALLY DISTRIBUTED RISKS

We now consider a different continuous distribution with the semi-infinite support $(0,\infty)$.[10] The lognormal distribution is the distribution of a random variable whose logarithm is normally distributed. In other words, w is lognormally distributed if and only if $y = \ln(w)$ is normally distributed. If w is lognormally distributed, then its probability density function is

$$f(w) = \frac{1}{w\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln w - \mu)^2}{2\sigma^2}\right) \tag{20}$$

with the mean $\exp(\mu + \frac{1}{2}\sigma^2)$ and the variance $(\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)$. We write $w \sim LN(\mu, \sigma^2)$ when $\ln(w) \sim N(\mu, \sigma^2)$. Note that μ and σ^2 are the mean and the variance of $\ln(w)$ (*not* of w) when w is lognormally distributed.

Perhaps not as commonly known as the normal distribution, it has been appreciated and used since early stages of formal statistics by statisticians such as

[10]The point 0 is not included in the support for the lognormal distribution.

Francis Galton.[11] The lognormal distribution is positively skewed, as it is essentially a logarithm of a symmetric distribution. Appeals to logarithm makes it particularly useful in dealing with growth phenomena, e.g. so-called Gibrat's law, etc, hence is used widely in a variety of fields.

While conceptually easy (at least in relation to the normal distribution), the formulaic expressions themselves are rather unwieldy for most purposes. As simple inspections reveal, the CARA utility function is not well suited to lognormally distributed risks, but a pleasant surprise is that the the CRRA utility functions are quite suitable for the purpose.

3.1. SPECIAL CASE (CRRA, $\theta = 1$): $u = \ln(w)$

When the coefficient of relative risk aversion is $\theta = 1$, the CRRA utility reduces to $u = \ln(w)$. In this case, computing the expected utility of a lognormally distributed lottery is very simple because if w is lognormally distributed, then $\ln(w)$ is normally distributed. Hence, if $f(w)$ is given as (20), then $y = \ln(w)$ has the normal density. In other words,

$$
\begin{aligned}
E[u(w)] &= \int \ln(w) \underbrace{\frac{1}{w\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln w - \mu)^2}{2\sigma^2}\right)}_{\text{lognormal density}} dw \\
&= \int y \underbrace{\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{\sigma^2}\right)}_{\text{normal density}} dy \\
&= \mu
\end{aligned}
$$

so that *any* lognormally distributed risk has a *constant* expected utility for the utility function $u = \ln(w)$. From $\ln(w_C) = \mu$, the certainty equivalent is

$$w_C = e^{\mu} < e^{\mu + \frac{1}{2}\sigma^2} = E[w]$$

and the risk premium is

$$RP = e^{\mu}[e^{\frac{1}{2}\sigma^2} - 1]$$

[11] See Chapter 1 of Aitchison and Brown (1957) for a brief history of the distribution. Also see Kleiber and Kotz (2003, Chapter 4) for definition and survey of applications in economics.

3.2. CRRA UTILITY FUNCTION: $u = \pm w^{1-\theta}$

We now turn to the more general case. Consider $0 < \theta < 1$. The expected utility of a lottery following a lognormal distribution is given below.

$$
\begin{aligned}
E[u(w)] &= \int w^{1-\theta} \frac{1}{w\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln w - \mu)^2}{2\sigma^2}\right) dw \\
&= \int e^{(1-\theta)\ln w} \frac{1}{w\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln w - \mu)^2}{2\sigma^2}\right) dw \\
&= \int \frac{1}{w\sigma\sqrt{2\pi}} \exp\left((1-\theta)\ln w - \frac{(\ln w - \mu)^2}{2\sigma^2}\right) dw \\
&= \int \frac{1}{w\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln w)^2 - 2(\mu - (1-\theta)\sigma^2)\ln w + \mu^2}{2\sigma^2}\right) dw \\
&= \int \frac{1}{w\sigma\sqrt{2\pi}} \\
&\exp\left(-\frac{(\ln w - [\mu - (1-\theta)\sigma^2])^2}{2\sigma^2} + [\mu + \frac{1}{2}\sigma^2(1-\theta)](1-\theta)\right) dw \\
&= \int \frac{1}{w\sigma\sqrt{2\pi}} \\
&\exp\left(-\frac{(\ln w - [\mu - (1-\theta)\sigma^2])^2}{2\sigma^2}\right) \exp([\mu + \frac{1}{2}\sigma^2(1-\theta)](1-\theta)) dw \\
&= \exp([\mu + \frac{1}{2}\sigma^2(1-\theta)](1-\theta)) \\
&\underbrace{\int \frac{1}{w\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln w - [\mu - (1-\theta)\sigma^2])^2}{2\sigma^2}\right) dw}_{=1} \\
&= \exp([\mu + \frac{1}{2}\sigma^2(1-\theta)](1-\theta))
\end{aligned}
$$

where in the penultimate line, we produced another lognormal density function which integrates out to 1. From $u(w_C) = E[u(w)]$, we can easily derive the certainty equivalent.

$$\exp((1-\theta)\ln w_C) = \exp([\mu + \frac{1}{2}\sigma^2(1-\theta)](1-\theta))$$

$$\ln w_C = \mu + \frac{1}{2}\sigma^2(1-\theta)$$

$$\Longrightarrow w_C = \exp(\mu + \frac{1}{2}\sigma^2(1-\theta)) < E[w] = \exp(\mu + \frac{1}{2}\sigma^2)$$

where the last inequality follows from $1-\theta < 1$. The risk premium is then

$$RP = \exp(\mu + \frac{1}{2}\sigma^2) - \exp(\mu + \frac{1}{2}\sigma^2(1-\theta)) = \exp(\mu + \frac{1}{2}\sigma^2)\left[1 - \exp(-\frac{1}{2}\sigma^2\theta)\right]$$

The case for $\theta > 1$ can proceed almost identically. The expression for $E[u(w)]$ would have a 'minus' sign before and the resulting formula for w_C would be the same as above. The inequality $w_C < E[w]$ would still hold because $1-\theta < 0$ in this case. Finally, note that for $\theta = 1$, the certainty equivalent is e^{μ}, consistent with the result we derived in Section 3.1.

Proposition 4. *If the lottery has random prize w with the lognormal density function (20) and if preferences are represented by the CRRA utility function (15), then the certainty equivalent w_C of the lottery is given by*

$$w_C = \exp(\mu + \frac{1}{2}\sigma^2(1-\theta)) \tag{21}$$

and the risk premium RP is given by

$$RP = \exp(\mu + \frac{1}{2}\sigma^2)\left[1 - \exp(-\frac{1}{2}\sigma^2\theta)\right] \tag{22}$$

The formulas have the expected behavior with respect to changes in θ: as $\theta \to 0$ (less risk averse), w_C approaches the expected prize $E[w]$; as $\theta \to 1$, w_C approaches e^{μ}. On the other hand, as $\theta \to \infty$, the risk premium RP increases and approaches $\exp(\mu + \frac{1}{2}\sigma^2)$.

The formula (21) may seem a bit difficult to manipulate but Remark 2 from Section 2.2 again applies here. If we are mostly interested in a strictly increasing utility function of w_C, then we can freely take a monotone transformation and focus on

$$\mu + \frac{1}{2}\sigma^2(1-\theta)$$

which is relatively simple to handle.

For reference, Table 1 summarizes the key formulas we have obtained in Sections 2 and 3.

Table 1: Formulas for Certainty Equivalent

	gamma distribution $w \sim \Gamma(\alpha,\beta)$	lognormal distribution $w \sim LN(\mu,\sigma^2)$
$u = -e^{-\theta w}$ (CARA)	$\frac{\alpha}{\theta}\ln(1+\frac{\theta}{\beta}) = \frac{\mu^2}{\theta\sigma^2}\ln(1+\frac{\theta\sigma^2}{\mu})$ $\approx \frac{\alpha}{\beta}\left(1-\frac{\theta}{2\beta}\right) = \mu - \frac{1}{2}\theta\sigma^2$ for small $\frac{\theta}{\beta}$ and $\frac{\theta\sigma^2}{\mu}$	
$v = -e^{-\theta k w}$ (CARA)	$\frac{\alpha}{\theta}\ln(1+\frac{k\theta}{\beta})$ $= \frac{\mu^2}{\theta\sigma^2}\ln(1+\frac{k\theta\sigma^2}{\mu})$	
$u = \pm w^{1-\theta}$ (CRRA)	$\frac{1}{\beta}\left[\frac{\Gamma(\alpha+1-\theta)}{\Gamma(\alpha)}\right]^{1/(1-\theta)}$ $\approx \frac{\alpha}{\beta}\left[1-\frac{\theta(1-\theta)}{2\alpha}\right]^{\frac{1}{1-\theta}}$ $= \mu\left[1-\frac{\theta(1-\theta)\sigma^2}{2\mu^2}\right]^{\frac{1}{1-\theta}}$ for large α	$\exp(\mu+\frac{1}{2}\sigma^2(1-\theta))$

Notes:

1. (α,β) are parameters for the gamma distribution, with $\mu = \alpha/\beta$ and $\sigma^2 = \alpha/\beta^2$.
2. For the lognormal distribution, μ and σ^2 are *not* the mean and the variance. They are the mean and the variance for the log of the distribution. The distribution itself has the mean $\exp(\mu+\frac{1}{2}\sigma^2)$ and the variance $(\exp(\sigma^2)-1)\exp(2\mu+\sigma^2)$.

4. AN APPLICATION: OPTIMAL CONTRACTING UNDER MORAL HAZARD

4.1. THE LEN MODEL

The so-called LEN model is a simple and popular model in the moral hazard literature.[12] The acronym LEN stands for 'linear contract', 'exponential utility,' and 'normally distributed risks'. In particular, the 'exponential utility' refers to CARA utility function $u(w;\theta) = -e^{-\theta w}$. It is well-known (Bolton and Dewatripont, 2005, Section 4.2) that the certainty equivalent of a risk that follows $N(\mu, \sigma^2)$ for the CARA utility is

$$w_C = \mu - \frac{1}{2}\theta\sigma^2 \tag{23}$$

(identical to the approximate formula (8) for a gamma distributed risk in Corollary 3).

The 'linear contract' refers to the compensation scheme in the form of $w = t + sq$, where t is the "fixed payment" and s is the "incentive power" proportional to the random outcome q. A linear contract is not only practical and prevalent in the real life; it can be theoretically optimal under some assumptions.[13]

Bolton and Dewatripont (2005, Section 4.2) outline the following simple LEN model. Performance of the agent is assumed to be $q = a + \varepsilon$, where a is the agent's effort (hidden action) with quadratic disutility $\frac{1}{2}ca^2$ and $\varepsilon \sim N(0, \sigma^2)$. If the agent has the CARA utility function and the principal uses the linear contract of the form $w = t + sq$, then the certainty equivalent of the compensation takes a simple form based on (23), which leads to a particularly simple (and intuitive) formula for the optimal incentive power s^* (with the subscript N for 'normal'):

$$s_N^* = \frac{1}{1 + \theta c\sigma^2} < 1 \tag{24}$$

It is optimal to use a low s_N^* (low-powered incentive) when θ (agent's degree of risk aversion) and c (agent's marginal disutility of effort) and σ^2 (noise associated with the performance measure) are high.

[12] For the standard textbook treatment of this model in a moral hazard setting, see Bolton and Dewatripont (2005, Section 4.2). Although they do not use the term 'LEN', Holmström and Milgrom (1987, 1991) are considered to be pioneers of this model. Also see Kirkegaard (2015).

[13] See Carroll (2015), Barlo and Özdoğan (2014), Holmström and Milgrom (1987).

4.2. EXTENSION: THE LEG MODEL

As we have argued in Introduction, one potential problem with the LEN model is the fact that the noise ε follows a normal distribution, hence could take an arbitrarily large negative value. This may be problematic if, for example, the agent's effort cannot produce a negative output; the worst an agent can do for the principal may be to produce nothing.

So now suppose that ε follows the gamma distribution with parameters (α,β). If the agent chooses effort level a, then the outcome is almost surely positive with mean $(a+\alpha/\beta)$ and variance α/β^2. Also suppose that the principal uses a linear contract.[14] We might as well call this a "LEG" (linear-exponential-gamma) model.

It is possible to show that the optimal incentive power s_G^* in this model is greater than s_N^* and may even be greater than 1.[15] The following proposition presents an approximate formula of s_G^* which parallels that of s_N^* for the case of sufficiently small values of $\theta\beta$. (The proof in Appendix derives the exact formula for s_G^* as well.)

Proposition 5. *Consider the LEG model, where the agent receives linear compensation $t+sq$, has CARA utility function and the agent's effort is subject to noise $\varepsilon \sim \Gamma(\alpha,\beta)$. Assume that the disutility of effort is $\frac{1}{2}ca^2$. If $\theta\beta$ is sufficiently small, then the optimal incentive power s_G^* is:*

$$s_G^* \approx \frac{\beta^2}{\beta^2+\theta(\alpha c-\beta)} = \frac{1}{1+\theta(c-1/\mu)\sigma^2} \tag{25}$$

Proof: See Appendix. ■

Comparing (25) with (24), we first notice the similarity of their dependence on θ, c, and σ^2. As in the LEN model, the principal will optimally impose lower-powered incentives on the agent when the agent is more risk averse, the effort is more costly and the performance measure is subject to higher noise.

s_G^* differs from s_N^* in the appearance of the term $(c-1/\mu)$ in place of c. Since $\mu>0$, we have $s_G^*>s_N^*$. This difference disappears as $\mu\to\infty$. Furthermore, if $c\mu<1$, then we have $s_G^*>1$. For these observations, we may offer following rough intuitions.

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[14] It is not clear whether the optimal contract in this setting is indeed linear. A recent paper by Carroll and Meng (2016) offers a foundation for linear contracts for a situation where a lower bound on the shocks is perceived by the principal.

[15] The incentive share greater than 1 is unconventional, but it can be accommodated in principle if the fixed payment t is sufficiently small or even negative.

First, the agent is assured of producing positive output with any (minimal) effort, so inducing effort requires higher incentive power. Secondly, note that the mean and the variance of a gamma distribution may not be independent of each other, since $\mu = \alpha/\beta$ and $\sigma^2 = \alpha/\beta^2$. If μ increases, then σ^2 may increase as well. So a noise with high μ may also be associated with high variance. Furthermore, if μ (expected output from no effort) is very high, effort is relatively insignificant. Hence it makes sense to lower the incentive power when μ is high. Finally, if μ is very low (but still positive) and c (marginal disutility of effort) is also very low, then it may be necessary to grant the whole output and more to the agent to induce the effort (the principal can profit by imposing a negative t, i.e. fixed participation fee on the agent).

5. FURTHER APPLICATIONS

To stress the potential for applications, we discuss some sketchy ideas for further applications. We leave fuller explorations of these ideas to future work.

LOAN CONTRACTS WITH UNCERTAIN INTEREST RATES

An interesting possibility of application is loan contracts with uncertain interest rates, as many loan products offered by banks leave interest rates open to future fluctuations at the time of issuance. Using a normal distribution for interest rates is clearly inappropriate in many real life situations. The so-called Cox-Ingersoll-Ross model shows that the interest rate has the gamma distribution as its asymptotic distribution (Cox et al, 1985).

A potential borrower has a project that requires investment of 1 unit of money but does not have any wealth to draw on. Ex ante, the return R to the project is uniformly distributed on $[0,\overline{R}]$. Assume that the borrower learns the realized value of R before making a loan application but the bank cannot observe it, so expects it to be $\frac{1}{2}\overline{R}$. The bank offers a loan contract with gamma-distributed interest rate with the mean $\frac{1}{2}\overline{R}$. In other words, the repayment is going to be D, which follows the gamma distribution with the mean $\frac{1}{2}\overline{R}$ and some variance σ^2.

If the potential borrower has the CARA utility function $u(w) = -\exp(-\theta w)$, with $w = R - D$, then using the formula from Table 1, the certainty equivalent of the loan will be

$$w_C = R - \frac{\overline{R}^2}{4\theta\sigma^2}\ln(1 + \frac{2\theta\sigma^2}{\overline{R}})$$

She will accept the loan contract only if

$$R \geq \frac{\overline{R}^2}{4\theta\sigma^2} \ln(1 + \frac{2\theta\sigma^2}{\overline{R}}) \tag{26}$$

We could further simplify (26). Since $\ln(1+x) \approx x - \frac{1}{2}x^2$ for $x \approx 0$, we have, when $2\theta\sigma^2/\overline{R}$ is sufficiently small,

$$R \geq \frac{1}{2}\overline{R} - \frac{1}{2}\theta\sigma^2 \tag{27}$$

the right-hand side of which has resemblance to the expression of the certainty equivalent in the LEN model. If $\theta = 0$ (risk neutral) or $\sigma^2 = 0$ (no risk), then (27) is an obvious requirement that the borrower accept the loan only if her return is greater than the expected interest rate. If $\theta > 0$ and $\sigma^2 > 0$, this requirement is *weakened*: if the borrower is more risk averse and the interest rate is more variable, the borrower will get the loan for the (sure) return that falls below the expected interest rate. This somewhat paradoxical result may be explained by the fact that the return is fixed while the repayment is random. A risk averse borrower assesses a random interest rate as being *lower* than its mean rate.

COMPARISON BETWEEN DIFFERENT INCOME DISTRIBUTIONS

One conspicuous area where both the gamma distribution and the lognormal distribution are employed is the study of income (or wealth) distribution. Kleiber and Kotz (2003, Section 4.9) report on a number of empirical studies that attempted to fit income or wealth distribution data to lognormal distributions with varying degrees of fitness. It is evident that many authors deemed it worthwhile to examine the lognormal distribution, although it seems to be outperformed by the gamma distribution in many studies. Kleiber and Kotz (2003, Section 5.2.6) also report on the use of the gamma distribution in studying income or wealth distribution.

Suppose a society's income distribution follows $w \sim \Gamma(\alpha, \beta)$. Also suppose that the representative voter or the benevolent policy maker has the CARA utility function. In this context, the degree of 'risk aversion' θ would be interpreted as inequality aversion or preference for equal distribution. Then from Table 1, we know that the utility of the distribution is equivalent to the utility of

$$\frac{\mu^2}{\theta\sigma^2} \ln(1 + \frac{\theta\sigma^2}{\mu}) \approx \mu - \frac{1}{2}\theta\sigma^2$$

where $\mu = \alpha/\beta$ is the mean income and $\sigma^2 = \alpha/\beta^2$ is the variance.

The first-order approximation applies if $\theta\sigma^2/\mu$ is sufficiently small. As long as the mean income is sufficiently high, the inequality is not too great and the inequality aversion is not too strong, the policy maker compares alternative policies that entail different income distributions by looking at the means and the variances. The marginal rate of substitution between μ and σ^2 would be then approximately constant at $\frac{1}{2}\theta$. A policy that increases the variance of income by 1 unit needs to ensure that the mean of income also increase by at least $\frac{1}{2}\theta$ units.

APPLICATIONS OF RESULTS FOR CRRA UTILITY FUNCTION

All the applications mentioned so far use CARA utility and gamma distribution. For completeness, we give a brief account of potential applications of CRRA utility and gamma or lognormal distribution.

One interesting topic is the cost of macroeconomic risks. Gollier (2001, Section 2.8) gives a sketch of the argument using ideas of Lucas (1987). Assuming the CRRA utility (with the degree of relative risk aversion unspecified) for the population, we compute the certainty equivalent of GDP per capita from times series data. The risk premium implied by this computation is interpreted as the macroeconomic cost of the fluctuation in per capita income. For reasonable degrees of risk aversion, it is found that the cost is negligible.

We can employ our results from Table 1 to conduct further hypothetical analyses. If we focus on the GDP per capita itself rather than its growth rate, then we can use the gamma distribution. When α is sufficiently high (the distribution is almost symmetric), we can carry out similar computations as in Gollier (2001, Section 2.8).

Or we can model the GDP as following a lognormal distribution $LN(\mu,\sigma^2)$ where μ and σ^2 would designate the mean and the variance of *growth rates* rather than the GDP itself. The certainty equivalent in this case is a simple formula which is proportional to $\mu + \frac{1}{2}\sigma^2(1-\theta)$. We can compare alternative scenarios focusing on three parameters μ, σ^2 and θ.

6. CONCLUSION

In this paper, we argued for the importance of allowing semi-infinite support (with a lower bound) in modeling risks. Then we characterized the expected utility, the certainty equivalent and the risk premium for risky prospects that follow either a gamma or a lognormal distribution for constant risk aversion utility functions. These tools may be applicable to several different areas; in particular they complement those associated with the LEN model in moral hazard literature.

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APPENDIX

Proof of Proposition 2

To prove (9), note that $E[w] = \alpha/\beta$, so we have the inequality (9) if and only if

$$\theta > \beta \ln\left(\frac{\beta+\theta}{\beta}\right)$$

Let $g(\theta) \equiv \theta - \beta\ln(\beta+\theta) + \beta\ln(\beta)$. Then we need to show $g(\theta) > 0$ for $\theta > 0$. First, it is easily checked that $g(0) = 0$. Moreover, $g'(\theta) = 1 - \beta/(\beta+\theta) > 0$ for all $\theta > 0$. Hence $g(\cdot)$ is a strictly increasing function for $\theta > 0$. (10) is trivial, since both w_C and RP are proportional to α.

For (11),

$$\frac{\partial w_C}{\partial \beta} = \frac{\alpha}{\theta}\left(\frac{1}{\beta+\theta} - \frac{1}{\beta}\right) < 0$$

$$\frac{\partial RP}{\partial \beta} = \frac{\partial E[w]}{\partial \beta} - \frac{\partial w_C}{\partial \beta} = -\frac{\alpha}{\beta^2} - \frac{\alpha}{\theta}\left(\frac{1}{\beta+\theta} - \frac{1}{\beta}\right) = -\frac{\alpha\theta}{\beta^2(\beta+\theta)} < 0$$

(12) involves some messy computations, so we simplify the presentation by writing "$A \sim B$" to mean "A has the same sign as B",

$$\begin{aligned}\frac{\partial w_C}{\partial \theta} &\sim \frac{\theta}{\beta+\theta} - \ln\left(\frac{\beta+\theta}{\beta}\right)\\ &= \frac{\theta/\beta}{1+\theta/\beta} - \ln\left(1+\frac{\theta}{\beta}\right), \quad \text{letting } k \equiv \frac{\theta}{\beta} > 0\\ &= \frac{k}{1+k} - \ln(1+k) \equiv h(k)\end{aligned}$$

Note that $h(0) = 0$ and $h'(k) = -k/(1+k)^2 < 0$ for $k > 0$. Therefore $h(k) < 0$ for $k > 0$, hence we know the sign of $\partial w_C/\partial\theta$ to be positive. Then since $E[w]$ does not involve θ, we see that

$$\frac{\partial RP}{\partial \theta} = -\frac{\partial w_C}{\partial \theta} > 0$$

To show (13), from (5)

$$\begin{aligned}\frac{\partial w_C}{\partial \mu} &= \frac{2\mu}{\theta\sigma^2}\ln\left(1+\frac{\theta\sigma^2}{\mu}\right) - \frac{1}{1+\frac{\theta\sigma^2}{\mu}}, \quad \text{letting } k \equiv \frac{\theta\sigma^2}{\mu} > 0\\ &= \frac{2}{k}\ln(1+k) - \frac{1}{1+k} = \frac{2(1+k)\ln(1+k) - k}{k(1+k)}\\ &\sim 2(1+k)\ln(1+k) - k \equiv i(k)\end{aligned}$$

To determine the sign of $i(k)$, note that $i(0) = 0$ and $i'(k) = 2\ln(1+k) + 1 > 0$ for $k > 0$, hence $i(k) > 0$ for $k > 0$. Therefore, we showed the sign of $\partial w_C/\partial\mu$ to be positive. As for the risk premium

$$\frac{\partial RP}{\partial \mu} = 1 - \frac{\partial w_C}{\partial \mu} = \frac{k(2+k) - 2(1+k)\ln(1+k)}{k(1+k)} \sim k(2+k) - 2(1+k)\ln(1+k) \equiv j(k)$$

Again $j(0) = 0$ and $j'(k) = 2[k - \ln(1+k)] > 0$ for $k > 0$, so $j(k) > 0$ for $k > 0$ and $\partial RP/\partial\mu > 0$.

Finally to show (14), again from (5)

$$\begin{aligned}
\frac{\partial w_C}{\partial \sigma^2} &= -\frac{\mu^2}{\theta}\cdot\frac{1}{\sigma^4}\ln\left(1+\frac{\theta\sigma^2}{\mu}\right)+\frac{\mu}{\sigma^2}\cdot\frac{1}{1+\frac{\theta\sigma^2}{\mu}} \\
&= -\theta\left(\frac{\mu}{\theta\sigma^2}\right)^2\ln\left(1+\frac{\theta\sigma^2}{\mu}\right)+\theta\cdot\frac{\mu}{\theta\sigma^2}\cdot\frac{1}{1+\frac{\theta\sigma^2}{\mu}}, \quad \text{letting } k\equiv\frac{\theta\sigma^2}{\mu}>0 \\
&= \theta\left[\frac{-(1+k)\ln(1+k)+k}{k^2(1+k)}\right]\sim -(1+k)\ln(1+k)+k\equiv l(k)
\end{aligned}$$

Since $l(0)=0$ and $l'(k)=-\ln(1+k)<0$, we see that $l(k)<0$ for $k>0$ and we showed the sign of $\partial w_C/\partial\sigma^2$ to be negative. Since $E[w]$ does not depend on σ^2, we have

$$\frac{\partial RP}{\partial\sigma^2}=-\frac{\partial w_C}{\partial\sigma^2}>0 \quad \blacksquare$$

Proof of Proposition 5

If the agent chooses a, then the output is $q=a+\varepsilon$ and the compensation (net of effort disutility) would be $w=t+sq-\frac{1}{2}ca^2=t+s(a+\varepsilon)-\frac{1}{2}ca^2=t+sa-\frac{1}{2}ca^2+s\varepsilon$, where the only random term is "$s\varepsilon$". The certainty equivalent of w is derived from the certainty equivalent of $s\varepsilon$ by (7) from Corollary 2:

$$w_C=t+sa-\frac{1}{2}ca^2+\frac{\alpha}{\theta}\ln(1+\frac{s\theta}{\beta})$$

The effort a that maximizes the expected utility of the agent is the same as that maximizes the certainty equivalent w_C: from $dw_C/da=0$ we have $a^*=s/c$.

The principal wants to maximize the expected outcome (net of compensation) $E[q-(t+sq)]$ subject to the familiar participation constraint (PC) and the incentive compatibility constraint (IC) for the agent. (IC) is satisfied by $a^*=s/c$. (PC) leads to $w=\overline{w}$ (reservation wage) or combined with (IC):

$$t+\frac{s^2}{c}-\frac{1}{2}c\frac{s^2}{c^2}+\frac{\alpha}{\theta}\ln(1+\frac{s\theta}{\beta})=\overline{w}$$

We can now rewrite the principal's objective by using (PC) and (IC) as follows:

$$\begin{aligned}
E[q-(t+sq)] = a+\frac{\alpha}{\beta}-\left(t+s(a+\frac{\alpha}{\beta})\right) &= a-(t+sa)+(1-s)\frac{\alpha}{\beta} \\
&= \frac{s}{c}-\left(\overline{w}+\frac{s^2}{2c}-\frac{\alpha}{\theta}\ln(1+\frac{s\theta}{\beta})\right)+(1-s)\frac{\alpha}{\beta}
\end{aligned}$$

FOC in $s>0$ yields

$$s_G^*=\frac{1}{2\beta\theta}\left(-[\alpha c\theta+\beta(\beta-\theta)]+\sqrt{[\alpha c\theta+\beta(\beta-\theta)]^2+4\beta^3\theta}\right)$$

If $\theta\beta^3$ is sufficiently small, the first-order approximation yields

$$s_G^*\approx\frac{1}{2\beta\theta}\times\frac{1}{2[\alpha c\theta+\beta(\beta-\theta)]}\times 4\beta^3\theta=\frac{\beta^2}{\beta^2+\theta(\alpha c-\beta)}=\frac{1}{1+\theta(c-1/\mu)\sigma^2} \quad \blacksquare$$