*Journal of Economic Theory and Econometrics*, Vol. 28, No. 3, Sep. 2017, 1–21

# A New Approach to Computing Equilibrium in Incomplete Markets*

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**Abstract** The paper is a sequel to Won (2016) which attempts to characterize full-rank GEI equilibrium as pre-GEI equilibrium. The pre-GEI equilibrium approach for computing full-rank GEI equilibrium is useful because pre-GEI equilibrium always exist under standard conditions. The goal of the current paper is to develop algorithms which can be implemented to compute pre-GEI equilibrium. The algorithms are built on a prototype system of equations which consist of the first-order conditions for utility maximization and market clearing conditions. The prototype system can be directly encoded into an algorithm or can be transformed into implementable forms for algorithms such as homotopy path-following algorithms. Two examples are presented where the algorithms are implemented to compute pre-GEI equilibrium and their performance are comparatively discussed.

**Keywords** incomplete markets, pre-GEI equilibrium, real assets, homotopy path-following algorithms, quasi-solution.

**JEL Classification** D52, D53, C62, C63, C02

*This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2012-S1A5A2A-01018422).

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*Received March 13, 2017, Revised August 12, 2017, Accepted August 21, 2017*

## 1.INTRODUCTION

A general equilibrium model with incomplete markets (GEI model) provides a rich environment for studying economic issues such as financial innovation, unhedgeable risk and Pareto-improving economic policies which go beyond the classical complete-market framework. A main difficulty with incomplete markets is it does not allow for an explicit solution of equilibrium outcomes even with simple preferences such as logarithmic or exponential utility functions. Moreover, the GEI model may fail to have equilibrium under standard conditions which guarantee the existence of Arrow-Debreu equilibrium.[1] When an explicit solution of equilibrium outcomes is unavailable, it is hard to characterize conditions under which the existential failure occurs to the GEI model. A computational approach can provide a good solution for such a conundrum with the GEI model.

The paper is a sequel to Won (2016) which attempts to characterize full-rank GEI equilibrium as pre-GEI equilibrium. The pre-GEI equilibrium approach for computing full-rank GEI equilibrium is useful because pre-GEI equilibrium always exists under standard conditions and coincides with GEI equilibrium when the payoff matrix has full rank. The goal of the current paper is to develop algorithms which can be implemented to compute pre-GEI equilibrium. The algorithms are built on a prototype system of equations which consist of the first-order conditions for utility maximization and market clearing conditions. The prototype system can be directly encoded into an algorithm or can be transformed into implementable forms for algorithms such as homotopy path-following algorithms. Two examples are presented where the algorithms are implemented to compute pre-GEI equilibrium and their performance are comparatively discussed.

The traditional algorithms based on successive approximation (for instance, the Newton-Raphson method) often display unstable performance around singularity during the computational procedure. It is worth recalling that the existential failure of the GEI model occurs due to the presence of ‘bad prices’ which make the payoff matrix singular. When algorithms fail to compute GEI equilibrium, it is hard to judge whether the computational failure stems from the algorithmic failure or the existential failure.[2] Moreover, it is illustrated later

[1] Standard conditions include the convexity and continuity of preferences, the convexity and closedness of consumption sets, and the survival requirement for the initial endowments.

[2] The existential failure is a measure-zero phenomenon in incomplete markets with real assets. As illustrated in Ku and Polemarchakis (1990), however, it is not an exceptional case any more in the presence of options.

in Example 5.2 that the homotopy algorithms developed in Schmedders (1998, 1999) can converge to a quasi-solution in the case that GEI equilibrium does not exist. The dilemma can arise when algorithms attempt to compute GEI equilibrium directly through a system of equations with singularity in the payoff matrix. The pre-GEI-equilibrium approach to computing full-rank GEI equilibrium which is free from the singularity problem has an advantage in distinguishing the existential failure from algorithmic failure and avoiding the quasi-solution problem. Algorithms which are designed to compute pre-GEI equilibrium produces successfully a computational outcome in Example 5.2 which vindicates the existential failure.

A simple example gives an intuition into the possible pitfall of homotopy systems which attempt to compute a solution for a system of equations possibly lacking a solution. Let's consider the following two equations

$$e^{x+y} - x = 0,$$
$$e^{x+ay} - 2x = 0.$$

Let $F(x,y,a)$ denote the system $(e^{x+y} - x, e^{x+ay} - 2x)$. The system $F(x,y,a) = 0$ has a solution if and only if

$$1 - \ln 2 \leq a < 1.$$

A linear homotopy algorithm is implemented to solve $F(x,y,1/10) = 0$ by taking the start system $F(x,y,9/10) = 0$ with the solution $(x,y) = (0.0009775, -6.93147)$. Although the system $F(x,y,1/10) = 0$ has no solution for $a = 1/10 < 1 - \ln 2$, the numerical procedure produces a great 'quasi-solution' $(x,y) = (0, -320.611)$. By plugging it into $F(x,y,1/10)$, we obtain

$$F(0, -320.611, 1/10) = (5.75986 \times 10^{-140}, 1.19135 \times 10^{-14}).$$

The approximation error is sufficiently close to zero to make us believe that the quasi-solution $(x,y) = (0, -320.611)$ is a true solution to $F(x,y,1/10) = 0$. Without prior knowledge about the condition for the existence of solution to $F(x,y,a) = 0$, one would face the danger of taking the quasi-solution as an evidence for the existence of solutions.

DeMarzo and Eaves (1996) develop a path-following algorithm for computing GEI equilibrium by reformulating the pseudo-equilibrium approach in a constructive way. The algorithm is very sophisticated but may be hard to implement because whenever the algorithm comes near the singularity of the payoff matrix, it must switch the payoff matrix to a new one by creating an artificial asset. Schmedders (1998, 1999) develops a homotopy algorithm for computing GEI equilibrium by introducing a weighted utility function. The current paper exploits the utility weighting system of Schmedders (1998, 1999) in building a homotopy system for computing pre-GEI equilibrium. However, the pre-GEI-equilibrium-based homotopy system and the GEI-equilibrium-based homoyopy system of Schmedders (1998, 1999) may perform very differently in the presence of bad-asset-price problem. The difference between the two systems is revealed in Example 5.2 where the GEI economy fails to have GEI equilibrium due to the bad-asset-price problem. In this example, the former provides correct information on the existential failure while the latter converges to a quasi-solution. Kubler and Schmedders (2010) develops an algorithm to compute GEI equilibrium when the system of equililbrium-determining equations can be expressed as a semi-algebraic structure.

## 2. THE MODEL

The economy under study is a typical two-period GEI model described in Won (2016). Assets are traded in the first period (denoted by date 0) and make payoffs in the second period (denoted by date 1). Consumptions arise in both periods. The following provides a summary of notation used in Won (2016).

- $\mathcal{I} = \{1, 2, \ldots, I\}$ : the set of agents.
- $\mathcal{S} = \{1, \ldots, S\}$ : the set of events to be revealed in the second period.
- $\mathcal{L} = \{1, 2, \ldots, L\}$ : the set of consumption goods.
- $\mathcal{J} = \{1, 2, \ldots, J\}$ : the set of financial assets.
- $\ell = L(S+1)$ and $\ell_1 = LS$.
- $\mathbb{R}^\ell$ : the space of state-contingent consumptions.
- A vector $y \in \mathbb{R}^\ell$ has a decomposition $y = (y_0, y_1)$ where $y_0 = y(0) \in \mathbb{R}^L$ and $y_1 = (y(1), \ldots, y(S))$ is a collection of $S$ vectors in $\mathbb{R}^L$.
- $P = \mathbb{R}^\ell_+$, $P^\circ = \mathbb{R}^\ell_{++}$, $P_1 = \mathbb{R}^{\ell_1}_+$, and $P_1^\circ = \mathbb{R}^{\ell_1}_{++}$.
- $e = (e_1, \ldots, e_I)$ : the initial allocation.

Asset markets are incomplete, i.e., $J < S$. The asset payoff may depend on spot prices in each contingency of the second period. The asset structure is in the class of (primary) real assets. A real asset $j$ is a contract which promises to deliver in each $s \in \mathcal{S}$ a vector of commodities $a^j(s) = (a^j_1(s), \ldots, a^j_L(s)) \in R^L$. For each $s \in \mathcal{S}$, let $a(s)$ denote $L \times J$ matrix with $j$th columun $a^j(s)$. Real asset $j$ pays income $r^j_s(p_1) = p(s) \cdot a^j(s)$ in state $s$ that linearly depends on the spot price $p(s)$. The asset payoffs at $p = (p_0, p_1) \in P$ are summarized into the $S \times J$ matrix $R(p_1)$ which has $r^j_s(p_1)$ as the $(s, j)$ th element. Let $R_1(p_1)$ denote the $J \times J$ submatrix of $R(p_1)$ which consists of the first $J$ rows of $R(p_1)$ and $R_2(p_1)$ the $(S - J) \times J$ submatrix which constitutes the rest of $R(p_1)$. Then $R(p_1)$ is decomposed as

$$R(p_1) = \begin{bmatrix} R_1(p_1) \\ R_2(p_1) \end{bmatrix}$$

When a portfolio $\theta \in R^J$ is taken at an asset price $q \in \mathbb{R}^J$ at date 0, it costs $q \cdot \theta$ at date 0 and yields an income transfer $R(p_1) \cdot \theta \in \mathbb{R}^S$ at date 1.

Each $i \in \mathcal{I}$ is characterized by the common consumption set $P$, the initial endowment $e_i \in P$ of goods, and the preferences represented by a utility function $u_i : P \to \mathbb{R}$. Let $\mathcal{E}(e)$ denote the economy described above. For computational purpose, we make the following assumptions on $\mathcal{E}(e)$.

**Assumption 1 :** Each $u_i : P^\circ \to \mathbb{R}$ is strictly increasing, twice continuously differentiable ($C^2$) and satisfies the strict concavity, i.e., $v(D^2 u_i(x_i))v < 0$ for all $v \neq 0$ in $\mathbb{R}^\ell$ and $x_i \in P^\circ$.[3]
**Assumption 2 :** Each $e_i$ is in $P^\circ$.
**Assumption 3 :** The asset structure consists of real assets.

The following notation is used in defining the budget set.

$$p \,\square\, (x_i - e_i) = \begin{bmatrix} p(0) \cdot (x_i(0) - e_i(0)) \\ p(1) \cdot (x_i(1) - e_i(1)) \\ \vdots \\ p(S) \cdot (x_i(S) - e_i(S)) \end{bmatrix}, \; p \,\square_1\, (x_i - e_i) = \begin{bmatrix} p(1) \cdot (x_i(1) - e_i(1)) \\ \vdots \\ p(S) \cdot (x_i(S) - e_i(S)) \end{bmatrix},$$

and

$$W(p_1, q) = \begin{bmatrix} -q \\ R(p_1) \end{bmatrix}.$$

For a given pair $(p, q) \in P \times \mathbb{R}^J$, agent $i$ has the budget constraint and demand

[3]The function $u_i$ is strictly increasing if for any $x, x'$ in $P$ with $x - x' \in P$ and $x \neq x'$, $u_i(x) > u_i(x')$.

correspondence in the economy $\mathcal{E}(e)$ defined by

$$\mathcal{B}_i(p,q,e_i) = \left\{(x_i,\theta_i) \in P \times \mathbb{R}^J \,:\, p\,\Box\,(x_i - e_i) \leq W(p_1,q)\cdot\theta_i\right\},$$
$$\xi_i(p,q,e_i) = \left\{(x_i^*,\theta_i^*) \in P \times \mathbb{R}^J \,:\, (x_i^*,\theta_i^*) \in \arg\max\left\{u_i(x);(x_i,\theta_i) \in \mathcal{B}_i(p,q,e_i)\right\}\right\}.$$

On the other hand, agent $i$ has the demand correspondence in the Arrow-Debreu complete-market economy defined by

$$\chi_i(p,p\cdot e_i) = \{x_i^* \in P \,:\, x_i^* \in \arg\max\{u_i(x);\, p\cdot(x_i - e_i) \leq 0\}\}.$$

Equilibrium of the economy $\mathcal{E}(e)$ is defined as follows.

**Definition 1 :** A list $(p,q,x,\theta) \in P^\circ \times \mathbb{R}^J \times P^I \times \mathbb{R}^{IJ}$ is a *GEI equilibrium* of $\mathcal{E}(e)$ if it satisfies the conditions

$(i)$ $(x_i,\theta_i) \in \xi_i(p,q,e_i)$ for every $i \in \mathcal{I}$,

$(ii)$ $\sum_{i\in\mathcal{I}}(x_i - e_i) = 0$, and

$(iii)$ $\sum_{i\in\mathcal{I}}\theta_i = 0$.

The list $(p,q,x,\theta)$ is a *full-rank GEI equilibrium* if $R(p_1)$ has rank $J$.

Let $C$ denote the set of spot prices at which $R(p_1)$ fails to have full rank

$$C = \{p_1 \in P_1^\circ : |R(p_1)^\mathsf{T} R(p_1)| = 0\},$$

where $B^\mathsf{T}$ indicates the transpose of the matrix $B$ and $|A|$ the determinant of the square matrix $A$. For each positive integer $\tau$, we define a set

$$C_\tau = \left\{p_1 \in P_1^\circ \,\middle|\, |R(p_1)^\mathsf{T} R(p_1)| < \frac{1}{\tau^2}\right\}.$$

It holds that $C_1 \supset C_2 \supset \cdots$ and $C = \bigcap_{\tau=1}^{\infty} C_\tau$, i.e., $C_\tau$ coincides in the limit with $C$. For a positive integer $\tau$, we define sets

$$C^J = \{p_1 \in P_1^\circ : |R_1(p_1)| = 0\},$$
$$C_\tau^J = \{p_1 \in P_1^\circ : -1/\tau < |R_1(p_1)| < 1/\tau\}.$$

The set $C^J$ includes prices which make the submatrix $R_1(\cdot)$ singular while $C_\tau^J$ includes prices which make the determinant of $R_1(\cdot)$ small.

Le $\Phi_\tau$ be a function in $P_1^\circ$ defined by

$$\Phi_\tau(p_1) = \begin{cases} \phi_\tau(|R_1(p_1)|), & \text{if } p_1 \in C_\tau^J \\ \dfrac{1}{|R_1(p_1)|}, & \text{if } p_1 \in P_1^\circ \setminus C_\tau^J \end{cases}$$

where for each $x \in \mathbb{R}$, $\phi_\tau(x) = \tau^2 x$. For each $\tau > 0$, we define two matrices

$$V_\tau^1(p_1) = \begin{bmatrix} I_J \\ \Phi_\tau(p_1) R_2(p_1) R_1^*(p_1) \end{bmatrix}$$

and

$$V_\tau^2(p_1) = \begin{bmatrix} I_J \\ \phi_\tau(|R_1(p_1)|) R_2(p_1) R_1^*(p_1) \end{bmatrix},$$

where $I_J$ is the $J \times J$ identity matrix and $A^*$ is the adjoint of the matrix $A$. The matrix $V_\tau^1(\cdot)$ will become the payoff matrix for the pre-GEI budget set while $V_\tau^2(\cdot)$ the payoff matrix for the test budget set. Both matrices are built to have full rank on the price domain $P^\circ$.

For each $(p, \tau)$ and each $k = 1, 2$, we introduce an artificial budget set $\mathcal{B}_{i,\tau}^k(p, e_i)$ and demand correspondence $\xi_{i,\tau}^k(p, e_i)$ of agent $i$.

$$\mathcal{B}_{i,\tau}^k(p, e_i) = \left\{ (x_i, \theta_i) \in P \times \mathbb{R}^J \, : \, p \cdot (x_i - e_i) \leq 0, \; p \, \Box_1 (x_i - e_i) = V_\tau^k(p_1) \cdot \theta \right\},$$
$$\xi_{i,\tau}^k(p, e_i) = \left\{ x_i^* \in P \, : \, (x_i^*, \theta_i^*) \in \arg\max\{u_i(x); (x_i, \theta_i) \in \mathcal{B}_{i,\tau}^k(p, e_i)\} \right\}.$$

The set $\mathcal{B}_{i,\tau}^1(p, e_i)$ denotes the pre-GEI budget set and $\mathcal{B}_{i,\tau}^2(p, e_i)$ the test budget set of agent $i$.

Based on the characterization of the new budget sets, we provide the notions of pre-GEI equilibrium and test equilibrium for $\mathcal{E}(e)$.[4]

**Definition 2 :** A pair $(p, x) \in P^\circ \times P^I$ is a *pre-GEI equilibrium* (*test equilibrium*, resp.) of $\mathcal{E}(e)$ for a positive integer $\tau$ if it satisfies the conditions

(*i*) $x_i \in \xi_{i,\tau}^1(p, e_i)$ ($x_i \in \xi_{i,\tau}^2(p, e_i)$, resp.) for every $i \in \mathcal{I}$, and

(*ii*) $\sum_{i \in \mathcal{I}} (x_i - e_i) = 0$.

[4]The definition of pre-GEI and test equilibrium is specialized in the current framework for computational purpose. For their full definition, see Won (2016).

As shown later, pre-GEI equilibrium always exists under Assumptions 1–3, and both full-rank GEI and pre-GEI equilibria are equivalent in real terms. The notions of equilibrium in Definition 2 are equivalent to the following Cass-trick-based definition which turns out to be useful in verifying the existence of pre-GEI and test equilibria and developing algorithms for computing pre-GEI equilibrium.

**Definition 2′ :** A pair $(p,x) \in P^\circ \times P^I$ is a *pre-GEI equilibrium* (*test equilibrium*, resp.) of $\mathcal{E}(e)$ for a positive integer $\tau$ if it satisfies the conditions

$(i)$ $x_1 \in \chi_1(p, p \cdot e_i)$,

$(ii)$ $x_i \in \xi^1_{i,\tau}(p,e_i)$ ($x_i \in \xi^2_{i,\tau}(p,e_i)$, resp.) for every $i \neq 1$, and

$(iii)$ $\sum_{i \in \mathcal{I}}(x_i - e_i) = 0$.

A pre-GEI equilibrium $(p,x)$ for some $\tau > 0$ turns out to be a GEI equilibrium when $p$ lies outside $C^J_\tau$. This fact is exploited in finding out GEI equilibrium from computing pre-GEI equilibrium.

## 3. PRE-GEI EQUILIBRIUM AND FULL-RANK GEI EQUILIBRIUM

This section states the first two theorems of Won (2016) without proof. They are specialized in the current framework for later reference. The first theorem presents the existence of pre-GEI equilibrium while the second one a sufficient condition for pre-GEI equilibrium to be full-rank GEI equilibrium.

**Theorem 1 :** For each positive integer $\tau$, there exists a pre-GEI equilibrium in the economy $\mathcal{E}(e)$.

**Theorem 2 :** If $(p,x)$ is a pre-GEI equilibrium of $\mathcal{E}(e)$ with $p_1 \in P^\circ_1 \setminus C^J_\tau$, then there exists $(q,\theta) \in \mathbb{R}^J \times \mathbb{R}^{IJ}$ such that $(p,q,x,\theta)$ in $P^\circ \times \mathbb{R}^J \times P^I \times \mathbb{R}^{IJ}$ is a full-rank GEI equilibrium of $\mathcal{E}(e)$ with $1_S = (1,1,\ldots,1) \in \mathbb{R}^S$ as the equilibrium state prices of agent 1.

## 4. ALGORITHMS FOR COMPUTING PRE-GEI EQUILIBRIUM

In this section, various forms of algorithms are developed based on the prototype system of equations which determine pre-GEI equilibrium. The prototype system consists of the first-order conditions for utility maximization with the artificial payoff $V_\tau^1$ and the market clearing conditions. It can be directly encoded into an algorithm to compute a pre-GEI equilibrium. As illustrated later in Section 5, the prototype system of equations is encoded into the FindRoot algorithm of Mathematica to compute pre-GEI equilibrium. The direct computation of pre-GEI equilibrium through the prototype system of equations usually needs a choice of the initial point from which the underlying algorithm starts to search for a solution. Since the initial point is hard to choose properly especially in a high-dimensional space, the underlying algorithm may fail to converge to a solution to the prototype system of equations.

Homotopy path-following algorithms can avoid the difficulty with the initial-point choice problem. To discuss them, we need to modify the prototype system in a path-following friendly way. To give a brief idea of the path-following procedure, let $g(x)$ and $f(x)$ be two functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ where $g(x)$ is the start system and $f(x)$ is the target system.[5] The homotopy path-following algorithm is a sophisticated way of computing a solution to $f(x) = 0$. For a parameter $\tau$ in $[0, 1]$, we define a linear homotopy $H(x, \tau)$ by

$$H(x, \tau) = (1 - \tau)g(x) + \tau f(x).$$

The linear homotopy $H(x, \tau)$ represents a continuous deformation from $g(x)$ to $f(x)$. Since $H(x, 0) = g(x)$ and $H(x, 1) = f(x)$, $H(x, 0) = 0$ and $H(x, 1) = 0$ give the same solutions as $g(x) = 0$ and $f(x) = 0$, respectively. In particular, the start system $g(x)$ is chosen such that

i) it is easy to solve $g(x) = 0$, and

ii) $g(x) = 0$ has a unique solution.

To implement the homotopy path-following algorithm, we need to take a finite set of points $\tau_0, \tau_1, \ldots, \tau_n$ in the parameter space $[0, 1]$ which are strictly increasing in the index with $\tau_0 = 0$ and $\tau_n = 1$. The path-following procedure starts to solve the start system $H(x, \tau_0) = g(x) = 0$ and then uses its solution as the initial point to solve the system $H(x, \tau_1) = 0$. In general, for each $k = 0, 1, \ldots, n - 1$, the solution to $H(x, \tau_k)$ is fed as the initial point to solve the next system $H(x, \tau_{k+1}) = 0$ during the algorithmic implementation. Finally, the solution of $H(x, \tau_{n-1}) = 0$ is taken as the initial point for the target system $H(x, \tau_n) = f(x) = 0$. The path-following algorithm has two advantages relative to the one-shot algorithm which attempts to compute pre-GEI equilibrium from the prototype system of equations:

[5]The systems $g(x)$ and $f(x)$ are assumed to be sufficiently smooth.

a) The penultimate system $H(x, \tau_{n-1})$ is much likely to behave as the target system $f(x)$. As the algorithm computes the solution to $H(x, \tau_{n-1}) = 0$ at the $(n-1)$ step, it automatically feeds a great initial point into the target system.

b) Initial points which emerge during the algorithmic implementation can be reset as frequently as needed by adjusting the number $n$ of $\tau$-points in [0, 1].

### 4.1. PROTOTYPE SYSTEM FOR PRE-GEI EQUILIBRIUM

For a price $(p,q)$ and an integer $\tau > 0$, the following relations represent the first-order conditions for utility maximization in the artificial economy with the payoff matrix $V_\tau^1(p_1)$.

$$\begin{aligned} \frac{\partial u_i(x^i)}{\partial x_s^i} &= \lambda_s p(s) \quad \text{for each } s = 0,1,\dots,S \\ \lambda_0 q &= \lambda_s V_\tau^1(p_1) \quad \text{or} \quad \lambda W_\tau^1(p_1,q) = 0, \\ p \,\square\, (x^i - e^i) &= W_\tau^1(p_1,q)\cdot \theta^i, \end{aligned}$$

where $\lambda \in \mathbb{R}_{++}^{S+1}$ indicates the Lagrangian multiplier and

$$W_\tau^1(p_1,q) = \begin{bmatrix} -q \\ V_\tau^1(p_1) \end{bmatrix}.$$

Now we take the first good as a numeraire in each state, i.e., the price of the first good is normalized to 1 in each state, i.e., $p_1(s) = 1$ for all $s \in \{0\} \cup \mathcal{S}$. Then we have $\lambda_s = \partial u^i(x^i)/\partial x_1^i(s)$ for each $s = 0,1,\dots,S$.

The first-order conditions are expressed as.

$$\begin{aligned} f_1(p,x) &\equiv \Big(\frac{\partial u_i(x^i)}{\partial x^i(s)} - \frac{\partial u_i(x^i)}{\partial x_1^i(s)} p(s)\Big)_{i\in\mathcal{I}_1,\, s\in\{0\}\cup\mathcal{S}}, \\ f_2\left(p,q,x,(\theta_i)_{i=1}^{I-1};\tau\right) &\equiv \left(\frac{\partial u_i(x^i)}{\partial x_{(1)}^i}\cdot W_\tau^1(p_1,q), \left(p\,\square\,(x^i-e^i) - W_\tau^1(p_1,q)\cdot\theta^i\right)^{\mathsf{T}}\right)_{i\in\mathcal{I}_1}, \\ f_3(p,q,x;\tau) &\equiv \left(\Big(\frac{\partial u_I(x^I)}{\partial x^I(s)} - \frac{\partial u_I(x^I)}{\partial x_1^I(s)}p(s),\ s\in\{0\}\cup\mathcal{S}\Big), \frac{\partial u_I(x^I)}{\partial x_{(1)}^I}\cdot W_\tau^1(p_1,q)\right), \end{aligned}$$

where $x_{(1)}^i = (x_1^i(0),\dots,x_1^i(S))$ and $\mathcal{I}_1 = \{1,\dots,I-1\}$. Relation $f_3(p,q,x;\tau) = 0$ represents the first-order conditions for agent $I$ with the budget constraints missing. The market clearing condition is expressed by the function

$$f_4(x) = \sum_{i=1}^{I}(x^i - e^i).$$

The prototype system of equations for pre-GEI equilibrium is defined as

$$F\left(p,q,x,(\theta_i)_{i=1}^{I-1};\tau\right) = \Big(f_1(p,x), f_2\left(p,q,x,(\theta_i)_{i=1}^{I-1};\tau\right), f_3(p,q,x;\tau), f_4(x)\Big),$$

which consists of $(\ell-(S+1))(I-1)+(J+S+1)(I-1)+(\ell-(S+1))+J+\ell$ functions and endogenous variables $p \in \mathbb{R}^{\ell-(S+1)}$, $q \in \mathbb{R}^J$, $x \in \mathbb{R}^{\ell I}$, and $(\theta^1,\dots,\theta^{I-1}) \in \mathbb{R}^{J(I-1)}$. The prototype system $F\left(p,q,x,(\theta_i)_{i=1}^{I-1};\tau\right) = 0$ determines pre-GEI equilibrium $(p,x)$ where agent $i$ makes the portfolio choice $\theta^i$ at the asset price $q$. When $L = J = I = 2$ and $S = 3$, the prototype system consists of 24 equations.

The prototype system can be directly encoded into algorithms such as Gröbner-basis-based algorithms and FindRoot of Mathematica to compute pre-GEI equilibrium. FindRoot is employed in Examples 5.1 and 5.2 of the next section. It can be also transformed into formulas for homotopy path-following algorithms. The following subsections present various algorithms based on the homotopy path-following procedure.

### 4.2. HOMOTOPY SYSTEM FOR PRE-GEI EQUILIBRIUM

The prototype system $F$ which determines pre-GEI equilibrium can be transformed into an implementable form for the path-following procedure by exploiting the utility weighting method of Schmedders (1998). To do this, for each $i = 1,\dots,I-1$ who consumes $x^i$ and holds a portfolio of assets $\theta^i$, we introduce a weighted utility

$$v_i(x^i,\theta^i,t;\varepsilon) \equiv t\,u_i(x^i) - \frac{1}{2}(1-t)\|\theta^i\|^2 - \varepsilon\frac{(1-t)}{2}\|x^i - e^i\|^2,$$

where $\varepsilon > 0$, $t \in [0,\,1]$ and $\|\cdot\|$ stands for the Euclidian norm.

Note that $u_i(x^i,\theta^i,1;\varepsilon) = u_i(x^i)$ and for each $l = 1,\dots,L$ and $s = 0,\dots,S$,

$$\frac{\partial v_i(x^i,\theta^i,t;\varepsilon)}{\partial x_l^i(s)} = t\,\frac{\partial u_i(x^i)}{\partial x_l^i(s)} - \varepsilon(1-t)(x_l^i(s) - e_l^i(s)).$$

For each $i = 1,\dots,I-1$, the first-order conditions for utility maximization with

$v_i$ is represented by the following functions.

$$f_1^w(p,x,t;\varepsilon) = \Big(\frac{\partial v_i}{\partial x^i(s)} - \frac{\partial v_i}{\partial x_1^i(s)} p(s)\Big)_{i\in\mathcal{I}_1,\, s\in\{0\}\cup\mathcal{S}},$$

$$f_2^w\left(p,q,x,(\theta_i)_{i=1}^{I-1},t\,;\tau,\varepsilon\right) = \Bigg(\frac{\partial v_i}{\partial x_{(1)}^i}\cdot W_\tau^1(p_1,q) - (1-t)\theta^i, \\ \left(p\,\Box\,(x^i-e^i) - W_\tau^1(p_1,q)\cdot\theta^i\right)^\mathsf{T}\Bigg)_{i\in\mathcal{I}_1}.$$

As later discussed, the functional form of $f_2^w$ is not yet convenient because it makes it hard to start the homotopy path-following algorithm at $t=0$. To develop an easy start system for the homotopy path-following algorithm, we slightly perturb the pricing relations in $f_2^w$ such that

$$\hat{f}_2\left(p,q,x,(\theta_i)_{i=1}^{I-1},t\,;\tau,\varepsilon\right) = \Bigg(t\frac{\partial v_i}{\partial x_{(1)}^i}\cdot W_\tau^1(p_1,q) - (1-t)\theta^i, \\ \left(p\,\Box\,(x^i-e^i) - W_\tau^1(p_1,q)\cdot\theta^i\right)^\mathsf{T}\Bigg)_{i\in\mathcal{I}_1}.$$

The homototy system for pre-GEI equilibrium is defined as

$$\hat{F}\left(p,q,x,(\theta_i)_{i=1}^{I-1},t\,;\tau,\varepsilon\right) = \Big(f_1^w(p,x,t;\varepsilon), \hat{f}_2\left(p,q,x,(\theta_i)_{i=1}^{I-1},t\,;\tau,\varepsilon\right), \\ f_3(p,q,x\,;\tau), f_4(x)\Big).$$

To see how pre-GEI equilibrium is computed through the homotopy system $\hat{F}$, we check the properties of $\hat{F}$. For $t=1$, it holds that

$$f_1^w(p,x,1;\varepsilon) = f_1(p,x)$$
$$\hat{f}_2\left(p,q,x,(\theta_i)_{i=1}^{I-1},1\,;\tau,\varepsilon\right) = f_2\left(p,q,x,(\theta_i)_{i=1}^{I-1};\tau\right).$$

Thus, the homotopy system $\hat{F}\left(p,q,x,(\theta_i)_{i=1}^{I-1},1\,;\tau,\varepsilon\right)$ coincides with the prototype system $F\left(p,q,x,(\theta_i)_{i=1}^{I-1};\tau\right)$. Consequently, pre-GEI equilibrium can be computed from the system $\hat{F}\left(p,q,x,(\theta_i)_{i=1}^{I-1},1\,;\tau,\varepsilon\right)=0$. To compute pre-GEI equilibrium via a homotopy path-following algorithm, we need to solve the start system $\hat{F}\left(p,q,x,(\theta_i)_{i=1}^{I-1},0\,;\tau,\varepsilon\right)=0$. The second subsystem gives

$$\hat{f}_2\left(p,q,x,(\theta_i)_{i=1}^{I-1},0\,;\tau,\varepsilon\right) = \Big(\theta^i,\ \left(p\,\Box\,(x^i-e^i) - W_\tau^1(p_1,q)\cdot\theta^i\right)^\mathsf{T}\Big)_{i\in\mathcal{I}_1} = 0.$$

This relation is immediately solved as $\theta^i = 0$ and $p \square (x^i - e^i) = 0$ for each $i = 1, \ldots, I-1$. On the other hand, $f_1^w(p, x, 0; \varepsilon) = 0$ gives

$$x^i(s) - e^i(s) - (x_1^i(s) - e_1^i(s))p(s) = 0.$$

For each $s = 0, \ldots, S$, we take the inner product between the right-hand side of the previous relation and $x^i(s) - e^i(s)$. The result combined with $p \square (x^i - e^i) = 0$ yields $\|x^i(s) - e^i(s)\|^2 = 0$ for each $s$ or $x^i = e^i$ for each $i = 1, \ldots, I-1$. By the market clearing condition $f_4(x) = 0$, we obtain $x^I = e^I$. By plugging it into the first relation of $f(p, q, x; \tau)$, we obtain a unique $p$ which satisfies

$$\frac{\partial u_I(e^I)}{\partial x^I(s)} - \frac{\partial u_I(e^I)}{\partial x_1^I(s)} p(s) = 0.$$

Finally, $q$ is obtained by solving the linear system

$$\frac{\partial u_I(e^I)}{\partial x_{(1)}^I} \cdot W_\tau^1(p_1, q) = 0$$

Thus, the start system is solved straightforwardly to produce the unique solution $(p, q, x, \theta)$.

## 4.3. CASS-TRICK-BASED HOMOTOPY SYSTEM FOR PRE-GEI EQUILIBRIUM

The Cass trick is used to prove the existence of pre-GEI equilibrium in Won (2016). The proof procedure can be translated into a homotopy path-following algorithm by replacing the utility $u_i$ by the weighted utility $v_i$. It is worth recalling that the Cass trick sets one agent to behave as if he were in complete asset markets. Specifically, the homotopy system for pre-GEI equilibrium consists of the first-order conditions for one unconstrained agents and for the other constrained agents. Won (2016) treats the first agent as the unconstrained one for the Cass trick. The first-order conditions for agent 1 are given

$$g_1(p, x^1, \lambda^1) \equiv \left(Du^1(x^1) - \lambda^1 p,\ p \cdot (x^1 - e^1)\right) = 0,$$

where for a function $f$, $Df$ indicates the gradient of $f$. To introduce the homotopy system, for each $\tau > 0$, constrained agent $i \neq 1$ is set to maximize the weighted utility $v_i(x^i, \theta^i, t; \varepsilon)$ subject to the modified budget constraints

$$p \cdot (x^i - e^i) = 0,$$
$$p \cdot \square_1(x^i - e^i) = V_\tau^1(p_1) \cdot \theta^i.$$

The first-order conditions for the penalized utility function $v_i(\cdot\,;\varepsilon)$ are summarized as

$$\begin{aligned}&g_i(p,x^i,\theta^i,\mu^i,\lambda^i,t\,;\tau,\varepsilon)\\&\quad\equiv\big(t\,Du^i(x^i)-\varepsilon(1-t)(x^i-e^i)-\lambda^i p-(0_L,\mu^i(1)p(1),\ldots,\mu^i(S)p(S)),\\&\qquad-(1-t)(\theta^i)^\intercal+\mu^i V^1_\tau(p_1),\ p\cdot(x^i-e^i),\ p\,\square_1(x^i-e^i)-V^1_\tau(p_1)\cdot\theta^i\big)=0,\end{aligned}$$

where $0_L$ is a zero in $\mathbb{R}^L$ and $\mu^i\in\mathbb{R}^S_{++}$ stands for the Lagrangian multiplier for the second-period budget constraints. The market clearing conditions and the price normalization lead to the relations $f_4(x)=0$ and $g(p)\equiv p_1(0)-1=0$, respectively. The aforementioned conditions are built into the homotopy system for pre-GEI equilibrium

$$\begin{aligned}&G(p,x,(\theta^i,\mu^i)_{i\in\mathcal{I}_1},\lambda^i,t\,;\tau,\varepsilon)\\&\qquad\equiv\Big(Du^1(x^1)-\lambda^1 p,\ \big(g_i(p,x^i,\theta^i,\mu^i,\lambda^i,t\,;\tau,\varepsilon)\big)_{i\in\mathcal{I}_1},\ f_4(x),g(p)\Big),\end{aligned}$$

where $\mathcal{I}_1=\mathcal{I}\setminus\{1\}$ indicates the set of $I-1$ constrained agents.[6] Pre-GEI equilibrium of the economy $\mathcal{E}(e)$ is obtained by computing the solution to the target system

$$G(p,x,(\theta^i,\mu^i)_{i\in\mathcal{I}_1},\lambda^i,1\,;\tau,\varepsilon)=0.$$

To conduct the path-following procedure, we need to check a solution of the start system $G(p,x,(\theta^i,\mu^i)_{i\in\mathcal{I}_1},\lambda^i,0\,;\tau,\varepsilon)=0$. It is straightforward to see that it has a unique solution $(p,x,(\theta^i,\mu^i)_{i\in\mathcal{I}_1},\lambda^i)$ which satisfies

$$x=e,\ \theta=0,\ \lambda^1=\frac{\partial u^1(e^1)}{\partial x^1_1(0)},\ p=\lambda^1 Du^1(e^1),\ \text{and}\ (\lambda^i,\mu^i)=0\ \text{for all}\ i\in\mathcal{I}_1.$$

The two systems $G$ and $\hat{F}$ are built on the basis of the first-order conditions for the weighted utility. Both aim at computing pre-GEI equilibrium but are of distinct form. The former exploits the presence of an unconstrained agent in GEI equilibrium while the latter treats agents symmetrically in GEI equilibrium. The formal distinction is reflected in the way of determining asset prices in the homotopy system which can affect computational performance. Asset prices are indirectly determined through state prices of the unconstrained agent in the system $G$ while they are directly computed from the system $\hat{F}$. Note that the size $J$ of asset prices is less than the size $S$ of state prices in incomplete markets.

[6]Note that the budget constraint of the unconstrained agent is omitted from the homotopy function due to Walras' law.

The former has a larger number of endogenous variables than the latter. For example, when $I = J = L = 2$ and $S = 3$, the former has 31 equations while the latter has 24 equations. The difference in the size of the two systems may not be crucial to computational efficiency in the two-period GEI model. However, the difference can affect computational efficiency significantly as the economy persists in longer periods. In particular, state prices enter the homotopy system of multi-period GEI models in a complicated manner that they can undermine computational efficiency.

### 4.4. HOMOTOPY SYSTEM FOR GEI EQUILIBRIUM

The homotopy systems for pre-GEI equilibrium can be easily translated into a homotopy system which can directly compute GEI equilibrium. For instance, let $\tilde{F}(\cdot, t; \varepsilon)$ denote the system $\hat{F}(\cdot, t; \tau, \varepsilon)$ where the artificial payoff $V_\tau^1(\cdot)$ is replaced by the original payoff $R(\cdot)$. Similarly, let $\widetilde{G}(\cdot, t; \varepsilon)$ denote the system $G(\cdot, t; \tau, \varepsilon)$ where the artificial payoff $V_\tau^1(\cdot)$ is replaced by the original payoff $R(\cdot)$. Both $\tilde{F}(\cdot, t; \varepsilon)$ and $\widetilde{G}(\cdot, t; \varepsilon)$ can be encoded into a path-following algorithm to compute GEI-equilibrium directly. In particular, the system $\widetilde{G}(\cdot, t; \varepsilon)$ is a slight modification of the homotopy system $H_{e^u}(\cdot, t)$ of Schmedders (1998, 1999).[7]

A main difference between the homotopy systems for pre-GEI equilibrium and GEI equilibrium lies in the property of the payoff matrices $R(\cdot)$ and $V_\tau^1(\cdot)$. The matrix $R(\cdot)$ need not have full rank while $V_\tau^1(\cdot)$ always has full rank on the price domain. The difference affects the performance of algorithms when they are stuck near bad prices where the rank of $R(\cdot)$ drops suddenly. From a theoretical viewpoint, path-following algorithms can fail to converge in the homotopy system for GEI equilibrium when they meet the bad-price problem. For each $\tau$, the homotopy systems for pre-GEI equilibrium are not supposed to undergo such computational failure because the artificial payoff $V_\tau^1$ has full rank on the price domain.

## 5. EXAMPLES

This section gives two examples in which attempts are made to compute pre-GEI equilibrium (or full-rank GEI equilibrium) through algorithms presented in the previous section. The first example (Example 5.1) considers an economy

[7] Both differs in price normalization. Another difference is that $\varepsilon$ is set to 1 in $H_{e^u}(\cdot, t)$ while $\widetilde{G}(\cdot, t; \varepsilon)$ is specified to depend on $\varepsilon > 0$. I find that the size of $\varepsilon$ may matter to computational efficiency.

with two agents who have separable preferences with the same relative risk aversion. Computational outcomes show that the economy has a pre-GEI equilibrium identified as a full-rank GEI equilibrium. The second example (Example 5.2) illustrates a two-agent economy with logarithmic utility and distinct utility weight between consumption goods. First, the example provides an economy which fails to have GEI equilibrium. Surprisingly, an algorithm for computing GEI equilibrium directly converges to a quasi-solution. Then the economy is slightly perturbed to have GEI equilibrium where the asset payoffs are almost collinear. The example provides a tough test bed for the performance of homotopy systems developed in the previous section. To implement the homotopy systems, we use the Mathematica function `LinearHomotopyRF` of Awange et al. (2010) which applies the Newton-Raphson method to compute solutions to homotopy functions. The prototype system $F$ is implemented through `FindRoot` of Mathematica.

The two examples are a GEI economy with $L = J = I = 2$ and $S = 3$ where agent $i$ has a utility function with relative risk aversion $\gamma^i$ represented by

$$u_i(x) = \frac{\beta_0^i}{1-\gamma^i}\left(\alpha_0^i x_1(0)^{1-\gamma^i} + (1-\alpha_0^i)x_2(0)^{1-\gamma^i}\right) \\ + \sum_{s=1}^{3} \frac{\beta_s^i}{1-\gamma^i}\left(\alpha_s^i x_1(s)^{1-\gamma^i} + (1-\alpha_s^i)x_2(s)^{1-\gamma^i}\right),$$

where $\beta_0^i = \sum_{s=1}^3 \beta_s^i$ and $0 < \alpha_s^i < 1$ for each $s = 0,1,2,3$. The weight $\alpha_s^i$ indicates the utility weight between consumption goods in state $s$ while the ratio $\beta_s^i/\beta_0^i$ can be interpreted as agent $i$'s belief that state $s'$ occurs in the second period.

The two assets are forward contracts. The first and second asset pay one unit of the first and second good in each state $s = 1,2,3$, respectively. The payoff matrix is given

$$R(p_1) = \begin{bmatrix} p_1(1) & p_2(1) \\ p_1(2) & p_2(2) \\ p_1(3) & p_2(3) \end{bmatrix}.$$

**Example 5.1 :** It is assumed in the example that the two agents have the same relative risk aversion 2.5, i.e., $\gamma^1 = \gamma^2 = 2.5$, homogeneous beliefs represented by $\beta^1 = \beta^2 = (3,1,1,1)$ and distinct utility weight $\alpha^1 = (1/3,1/3,1/3,1/3)$ and

$\alpha^2 = (2/3, 2/3, 2/3, 2/3)$. They have the initial endowments

$$e^1 = \begin{bmatrix} e_1^1(0) & e_1^1(1) & e_1^1(2) & e_1^1(3) \\ e_2^1(0) & e_2^1(1) & e_2^1(2) & e_2^1(3) \end{bmatrix} = \begin{bmatrix} 7 & 6 & 10 & 10 \\ 4 & 6 & 8 & 13 \end{bmatrix},$$

$$e^2 = \begin{bmatrix} e_1^2(0) & e_1^2(1) & e_1^2(2) & e_1^2(3) \\ e_2^2(0) & e_2^2(1) & e_2^2(2) & e_2^2(3) \end{bmatrix} = \begin{bmatrix} 5 & 10 & 5 & 10 \\ 5 & 10 & 5 & 10 \end{bmatrix}.$$

Price and payoff related notion has the following special form in this example.

$$P = \mathbb{R}^8_+, \quad P_1^\circ = \mathbb{R}^6_{++} \quad \text{and} \quad C = \Big\{ p_1 \in P_1^\circ : \frac{p_1(1)}{p_2(1)} = \frac{p_1(2)}{p_2(2)} = \frac{p_1(3)}{p_2(3)} \Big\},$$

and

$$\begin{aligned} C^2 &= \{p_1 \in P_1^\circ : p_1(1)p_2(2) = p_1(2)p_2(1)\}, \\ C^2_\tau &= \{p_1 \in P_1^\circ : |p_1(1)p_2(2) - p_1(2)p_2(1)| < 1/\tau\} \text{ for each } \tau, \end{aligned}$$

$$V^1_\tau(p_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \Phi_\tau(p_1)(p_1(3)p_2(2) - p_1(2)p_2(3)) & \Phi_\tau(p_1)(p_1(1)p_2(3) - p_1(3)p_2(1)) \end{bmatrix},$$

$$V^2_\tau(p_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \phi_\tau(p_1)(p_1(3)p_2(2) - p_1(2)p_2(3)) & \phi_\tau(p_1)(p_1(1)p_2(3) - p_1(3)p_2(1)) \end{bmatrix},$$

where $\phi_\tau(|R_1(p_1)|) = \tau^2(p_1(1)p_2(2) - p_1(2)p_2(1))$ and

$$\Phi_\tau(p_1) = \begin{cases} \tau^2(p_1(1)p_2(2) - p_1(2)p_2(1)), & \text{if } p \in C^2_\tau \\ \dfrac{1}{p_1(1)p_2(2) - p_1(2)p_2(1)}, & \text{if } p \in P_1^\circ \setminus C^2_\tau \end{cases}.$$

Assumptions 1–3 hold trivially in the example. Thus by Theorem 1, the economy has pre-GEI equilibrium for each $\tau > 0$. For $\tau = 1,000$, the four computational systems $F$, $\hat{F}$, $G$ and $\tilde{F}$ discussed in the previous section are implemented separately to compute pre-GEI equilibrium. `FindRoot` is applied to the prototype system $F$ and the path-following procedure is conducted in the other three systems. It is worth recalling that the homotopy system $\tilde{F}$ holds the original payoff matrix $R(\cdot)$. As summarized below, the four algorithms successfully compute a pre-GEI equilibrium.

| | $F(\cdot;\tau)$ | $\hat{F}(\cdot;\tau,1)$ | $G(\cdot;\tau,1)$ | $\tilde{F}(\cdot;1)$ |
|---|---|---|---|---|
| Convergence | OK | OK | OK | OK |

Successfully implemented algorithms produce a pre-GEI equilibrium with the following outcomes.[8]

$$\begin{aligned}
p &= \big((1,2.0357),(1,0.9036),(1,1.4370),(1,0.8101)\big),\\
q &= (0.492,0.5244),\\
x^1 &= \big((5.1022,5.0662),(5.7655,7.9225)),(6.5179,7.4395),(10.6562,15.2965)\big),\\
x_2 &= \big((6.8978,3.9338),(10.2345,8.0775),(8.4821,5.5605),(9.3438,7.7035)\big),\\
\theta^1 &= (11.3106,-10.8549),\ \theta^2 = -\theta^1.
\end{aligned}$$

The fact that $|R_1(p_1)| = 0.5334 > 1/1,000$ shows that the pre-GEI equilibrium price $p$ lies outside $C^2_{1,000}$. By Theorem 2, $(p,q,x,\theta)$ is a full-rank GEI equilibrium of the economy. It is worth noting that the path-following algorithm works well in the $\tilde{F}$ with the original payoff because it is far from being singular in equilibrium.

**Example 5.2 :** In this example, the two agents have a logarithmic utility function, homogeneous beliefs represented by $\beta^1 = \beta^2 = (3,1,1,1)$, and distinct utility weights $\alpha^1 = (1/3,1/3,2/3,1/3)$ and $\alpha^2 = (2/3,1/3,2/3,1/3)$. They have the initial endowments

$$e^1 = \begin{bmatrix} e^1_1(0) & e^1_1(1) & e^1_1(2) & e^1_1(3) \\ e^1_2(0) & e^1_2(1) & e^1_2(2) & e^1_2(3) \end{bmatrix} = \begin{bmatrix} \frac{4}{10} & \frac{4}{10} & \frac{8}{10}+\delta & \frac{6}{10} \\ 1 & 1 & \frac{5}{10} & \frac{12}{10} \end{bmatrix},$$

$$e^2 = \begin{bmatrix} e^2_1(0) & e^2_1(1) & e^2_1(2) & e^2_1(3) \\ e^2_2(0) & e^2_2(1) & e^2_2(2) & e^2_2(3) \end{bmatrix} = \begin{bmatrix} \frac{6}{10} & \frac{6}{10} & \frac{12}{10}+\delta & \frac{4}{10} \\ 1 & 1 & \frac{5}{10} & \frac{8}{10} \end{bmatrix},$$

where $\delta$ is a nonnegative number.

When $\delta = 0$, the economy has no GEI equilibrium.[9] In this case, it is naturally anticipated that algorithms with the original payoff matrix may diverge because it ultimately encounters the bad-price problem. Surprisingly, the path-following algorithm converges and produce a quasi-solution when it is implemented in the systems $\tilde{G}(\cdot,t;1)$ and $H_{e^u}(\cdot,t)$. The convergence result is quite confusing. In fact, algorithms can converge to a limit point which does not actually exist and produce a 'quasi-solution'. (The case is analogous to a sequence

[8] Numbers in equilibrium outcomes are rounded to the 4th decimal place.

[9] Nonexistence can be verified by applying GroebnerBasis of Mathematica to the system which is obtained by replacing $V^1_\tau(\cdot)$ by the original payoff $R(\cdot)$. The modified system consists of the first-order conditions for maximizing the original utility with the original payoffs.

without the limit point.) However, the bad-price problem with incomplete markets does not fall into the category of quasi-limit-point problems because individual asset holdings go unbounded at bad prices. For $\tau = 1,000$, the system $\tilde{G}(\cdot, t; 1/1,000)$ produces a pre-GEI equilibrium with $|R_1(p_1)| = 0$, i.e., $p_1 \in C^2_{1,000}$. This result is consistent to the existential failure. For $\tau = 10,000$, the system $\hat{F}(\cdot, t; 1/10)$ works reasonably as well by producing a pre-GEI equilibrium with $|R_1(p_1)| = 8.9 \times 10^{-6}$, i.e., $p_1 \in C^2_{10,000}$. Note that pre-GEI equilibrium cannot be GEI equilibrium in both cases because pre-GEI equilibrium prices fall in the critical price domain.

Now let's perturb slightly the endowments of agent 1 in the previous economy by setting $\delta = 1/1,000$. Assumptions 1–3 of Won (2016) hold trivially in the example. By Theorem 1, the economy has pre-GEI equilibrium for each $\tau > 0$. For $\tau = 10,000$, As in Example 5.1, the four computational systems $F$, $\hat{F}$, $G$ and $\tilde{F}$ are implemented separately to compute pre-GEI equilibrium. `FindRoot` is applied to the prototype system $F$ and the path-following procedure is conducted in the other three systems. As summarized below, the four algorithms successfully compute a pre-GEI equilibrium where the first three algorithms are implemented at $\tau = 10,000$.

| | $F(\cdot;\tau)$ | $\hat{F}(\cdot;\tau,1/5)$ | $G(\cdot;\tau,1/100)$ | $\tilde{F}(\cdot;1)$ |
|---|---|---|---|---|
| Convergence | OK | OK | OK | OK |

The economy has a pre-GEI equilibrium with the following outcomes.[10]

$$\begin{aligned}
p &= \big((1,0.4834),(1,1),(1,1.001),(1,1)\big),\\
q &= (0.6633,0.6635),\\
x^1 &= \big((0.3112,1.2875),(0.4089,0.8177)),(0.9502,0.4746),(0.5422,1.0844)\big),\\
x_2 &= \big((0.6888,0.7125),(0.5911,1.1823),(1.0518,0.5254),(0.4578,0.9156)\big),\\
\theta^1 &= (-297.433,297.26),\ \theta^2 = -\theta^1.
\end{aligned}$$

The following shows that the pre-GEI equilibrium price $p$ lies outside $C^2_{10,000}$.

$$|R_1(p_1)| = \begin{bmatrix} 1 & 1 \\ 1 & 1.001 \end{bmatrix} = 0.001 > \frac{1}{10,000}.$$

By Theorem 2, $(p,q,x,\theta)$ is a full-rank GEI equilibrium of the economy. It is worth noting that the asset holding of agent 1 is very large relative to consumption choices because the asset payoffs are almost collinear.

[10]Numbers in equilibrium outcomes are rounded to the 4th decimal place.

## 6. CONCLUSION

The paper provides several algorithms for computing pre-GEI equilibrium in the two-period GEI model. Full-rank GEI equilibrium is characterized as pre-GEI equilibrium in real terms. A merit of the pre-GEI approach for computing GEI equilibrium lies in the fact that the algorithms do not encounter the existential failure when they are supposed to cross prices inducing the singularity of the payoff matrix during their implementation. To develop algorithms, the paper starts with the prototype system $F$ of equations which characterize pre-GEI equilibrium. The prototype system can be directly encoded into algorithms or can be reformulated into homotopy path-following algorithms. The path-following procedures are developed in Section 4 by exploiting the utility weighting system of Schmedders (1998, 1999). Algorithms are implemented in Example 5.1 and 5.2. In particular, Example 5.2 provides a tough test bed for the capability of the algorithms developed in Section 4 to overcome the computational failure and the quasi-solution problem in the presence of the bad-price problem.

The current paper is restricted to the case of the two-period GEI model. An immediate concern will be an extensions of the pre-GEI equilibrium approach to the GEI model with longer or infinite horizon. The paper pays no attention to the computational efficiency of algorithms measured by computing speed. The issue of computational efficiency will emerge when the prototype system for computing pre-GEI equilibrium is very large and complicated due to the longer horizon of the stochastic finance economy.

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