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Simple Characterizations of Potential Games and Zero-sum Equivalent Games

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Abstract

We provide several tests to determine whether a game is a potential game or whether it is a zero-sum equivalent game—a game which is strategically equivalent to a zero-sum game in the same way that a potential game is strategically equivalent to a common interest game. We present a unified framework applicable for both potential and zero-sum equivalent games by deriving a simple but useful characterization of these games. This allows us to re-derive known criteria for potential games, as well as obtain several new criteria. In particular, we prove (1) new integral tests for potential games and for zero-sum equivalent games, (2) a new derivative test for zero-sum equivalent games, and (3) a new representation characterization for zero-sum equivalent games.

Keywords Potential Games, Zero-Sum Games, Zero-Sum Equivalent Games

JEL Classification C72, C73

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1. INTRODUCTION

We provide several tests to determine whether a game is a potential game or whether it is a zero-sum equivalent game—a game which is *strategically equivalent* to a zero-sum game in the same way that a potential game is strategically equivalent to a common interest game (see Definition 1 and also Section 11.2 in Hofbauer and Sigmund, 1998). We present a unified framework applicable for both potential and zero-sum equivalent games by deriving a simple but useful characterization of these games. In particular, we propose (1) new integral tests for potential games and for zero-sum equivalent games, (2) a new derivative test for zero-sum equivalent games, and (3) a new representation characterization for zero-sum equivalent games. We also re-derive known criteria for potential games, such as Monderer and Shapley (1996), Ui (2000) and Sandholm (2010), as well as obtain several new criteria.

An advantage of our approach is that our new integral tests can be applied to normal form games with continuous strategy sets as well as those with finite strategy spaces, whether payoff functions are discontinuous or not. Many popular games with continuous strategy sets, such as Bertrand competition games and Hotelling games, have discontinuous payoff functions. It is well-known that games with continuous strategy sets and discontinuous payoff pose special challenges such as the existence of Nash equilibria (see, for example, Reny, 1999). Our integral test provides a useful tool to study this class of games. In the case of finite strategy sets our test reduces to the test in Sandholm (2010).

The integral test for potential games is also easier to implement than the cycle condition in Monderer and Shapley (1996)'s Theorem 2.8 (see Remark 1). For, say, a two-player game our integral test requires checking the values of a function at two different points, while the cycle condition requires checking the values of a function at four different points. For finite strategy sets, Hino (2011) and Sandholm (2010)'s algorithms checking for potential games have complexity $O(n^2)$ and the integral test has the same complexity.

We also study in detail zero-sum equivalent games and provide integral and derivative tests as well as representations of those games. While the derivative test for potential games is well-known (Monderer and Shapley, 1996 Theorem 4.5), the derivative test for zero-sum equivalent games is new and provides an easy and convenient way to check if a game is zero-sum equivalent when the payoff function is sufficiently smooth (Proposition 3). The usefulness of this test is illustrated in Example 2 where we analyze contest games. Finally, we provide a representation characterization (Proposition 4) which generalizes to zero-sum equivalent games the result in Ui (2000).

In the existing literature, conditions for potential games, such as Monderer and Shapley (1996), Ui (2000) and Sandholm (2010), are regarded as distinct and derived by different methods (see, e.g., the discussion in Section 3 in Sandholm, 2010). Our result provides a unified framework to understand and generalize (also to zero-sum equivalent games) these conditions.

2. MAIN RESULTS

We follow the setup in Hwang and Rey-Bellet (2017) where we provide general decomposition theorems for $n-$player games. Let N be the set of all player indexed by $i \in N = \{1, \cdots, n\}$. Let $S = S_1 \times \cdots \times S_n$ be the space of all strategy profiles where S_i is the set of strategies for the i^{th} player. Let m_i be a finite measure on S_i and m be the product measure $m := m_1 \times \cdots \times m_n$. We denote by $u^{(i)}$ the payoff function for the i^{th} player, where $u^{(i)} : S \to \mathbb{R}$ is a (measurable) function. For fixed n and S, a game is uniquely specified by the vector-valued function $u := (u^{(1)}, u^{(2)}, \cdots, u^{(n)})$. We use the notation $g(s_{-i})$ for a function which does not depend on its i-th argument. If the payoff for the i^{th} player has the form $u^{(i)}(s) = g^{(i)}(s_{-i})$ then her payoff does not depend on her own strategy (also called a *passive* game). It is easy to see that if two game payoffs differ by a passive game for each player, then they have the same Nash equilibria and best response functions—these are called *strategically equivalent*.

Definition 1. *We have:*
(i) A game u is a ***potential game*** *if there exists a function v and functions $g^{(i)}$'s such that*

$$\begin{aligned}&(u^{(1)}(s), u^{(2)}(s), \cdots, u^{(n)}(s))\\ &= (v(s), v(s), \cdots, v(s)) + (g^{(1)}(s_{-1}), g^{(2)}(s_{-2}), \cdots, g^{(n)}(s_{-n})), \mathit{for\,all\,s}.\end{aligned}$$

(ii) A game u is a ***zero-sum equivalent game*** *if there exists functions $v^{(i)}$'s with $\sum_i v^{(i)} = 0$ and functions $g^{(i)}$'s such that*

$$\begin{aligned}&(u^{(1)}(s), u^{(2)}(s), \cdots, u^{(n)}(s))\\ &= (v^{(1)}(s), v^{(2)}(s), \cdots, v^{(n)}(s)) + (g^{(1)}(s_{-1}), g^{(2)}(s_{-2}), \cdots, g^{(n)}(s_{-n})), \mathit{for\,all\,s}.\end{aligned}$$

The definition of a potential game in Monderer and Shapley (1996) is that u is a potential game if there exists a function v such that $u^{(i)}(s_i, s_{-i}) - u^{(i)}(\widetilde{s}_i, s_{-i}) = v(s_i, s_{-i}) - v(\widetilde{s}_i, s_{-i})$ for all $s_i, \widetilde{s}_i, s_{-i}$ and all i. This is easily shown to be equivalent to Definition 1.

The next proposition is simple but important since it recasts the definitions of potential and zero-sum equivalent games without reference to unknown functions v or $v^{(i)}$ in Definition 1. This will provide the key ingredient to establish our criteria.

Proposition 1 (Characterization). *We have:*
(i) A game u is a potential game if and only if there exist functions $g^{(i)}$'s such that for all i, j

$$u^{(i)}(s) - g^{(i)}(s_{-i}) = u^{(j)}(s) - g^{(j)}(s_{-j}) \tag{1}$$

for all s.
(ii) A game u is a zero-sum equivalent game if and only if there exist functions $g^{(i)}$'s such that

$$\sum_{i=1}^{n} \left[u^{(i)}(s) - g^{(i)}(s_{-i})\right] = 0 \tag{2}$$

for all s.

Proof. The "only if" parts are trivial. Conversely, let us assume that there exist $g^{(i)}$'s which satisfy the conditions (1) or (2). Then, if we write

$$\begin{aligned}
&(u^{(1)}(s), u^{(2)}(s), \cdots, u^{(n)}(s)) \\
&= (u^{(1)}(s) - g^{(1)}(s_{-1}), u^{(2)}(s) - g^{(2)}(s_{-2}), \cdots, u^{(n)}(s) - g^{(n)}(s_{-n})) \\
&+ (g^{(1)}(s_{-1}), g^{(2)}(s_{-2}), \cdots, g^{(n)}(s_{-n}))
\end{aligned}$$

we see that u is a potential game if (1) holds and that u is a zero-sum equivalent game if (2) holds. □

For our integral test, we introduce the following operator.

Definition 2. *For an integrable function $h{:}S \to \mathbb{R}$, we define T_i by*

$$T_i h(s) := \frac{1}{m_i(S_i)} \int h(s) dm_i(s_i).$$

Note that T_i and T_j commute and that we have the identity

$$(I - T_i)(I - T_j) = I - (T_i + (I - T_i)T_j), \tag{3}$$

where I is the identity operator. And, by induction,

$$\prod_{l=1}^{n}(I-T_l)=I-(T_1+\sum_{j=2}^{n}\prod_{l=1}^{j-1}(I-T_l)T_j)\,. \tag{4}$$

Note as well for any h, T_ih does not depend on s_i. We next prove our integral tests.

Proposition 2 (Integral Tests). *We have:*
(i) A game u is a potential game if and only if

$$(I-T_i)(I-T_j)(u^{(i)}-u^{(j)})=0\,. \tag{5}$$

for all i, j.
(ii) A game u is a zero-sum equivalent game if and only if

$$\sum_{i=1}^{n}\prod_{l=1}^{n}(I-T_l)u^{(i)}=0\,. \tag{6}$$

Proof. Suppose that a game is a potential game (or a zero-sum equivalent game). Equation (5) (or (6)) follows from equation (1) (or equation (2)) in Proposition 1. Thus, only if parts in (i) and (ii) hold. For the if part in (i), suppose that (5) holds. Recall that N is the set of all players and let $i\in N$ be fixed. Note that

$$I=(I-T_i)+T_i=\sum_{\substack{M\subset N\\ M\ni i}}\prod_{l\notin M}\prod_{k\in M}T_l(I-T_k)+T_i$$

Thus,

$$u=(\sum_{\substack{M\subset N\\ M\ni 1}}\prod_{l\notin M}\prod_{k\in M}T_l(I-T_k)u^{(1)},\cdots,\sum_{\substack{M\subset N\\ M\ni n}}\prod_{l\notin M}\prod_{k\in M}T_l(I-T_k)u^{(n)})+(T_1u^{(1)},\cdots,T_nu^{(n)})$$

Then, we have

$$\begin{aligned}\prod_{l\notin M}\prod_{k\in M}T_l(I-T_k)u^{(i)}&=\prod_{l\notin M}\prod_{\substack{k\in M\\ k\neq i,j}}T_l(I-T_k)(I-T_i)(I-T_j)u^{(i)}\\&=\prod_{l\notin M}\prod_{\substack{k\in M\\ k\neq i,j}}T_l(I-T_k)(I-T_i)(I-T_j)u^{(j)}=\prod_{l\notin M}\prod_{k\in M}T_l(I-T_k)u^{(j)}\end{aligned} \tag{7}$$

which shows that for all $i, j \in M$ such that $|M| > 2$,

$$\prod_{l \notin M} \prod_{k \in M} T_l (I - T_k) u^{(i)} = \prod_{l \notin M} \prod_{k \in M} T_l (I - T_k) u^{(j)}.$$

Thus, we can define

$$\xi_M := \prod_{l \notin M} \prod_{k \in M} T_l (I - T_k) u^{(i)}$$

for any $i \in M \subset N$. Then, we have

$$u^{(i)} = \sum_{\substack{M \subset N \\ M \ni i}} \prod_{l \notin M} \prod_{k \in M} T_l (I - T_k) u^{(i)} + T_i u^{(i)} = \sum_{\substack{M \subset N \\ M \ni i}} \xi_M + T_i u^{(i)} = \sum_{\substack{M \subset N \\ M \neq \emptyset}} \xi_M - \sum_{\substack{M \not\ni i \\ M \neq \emptyset}} \xi_M + T_i u^{(i)}$$

and $\sum_{\substack{M \not\ni i \\ M \neq \emptyset}} \xi_M$ does not depend on s_i, because ξ_M does not depend on s_l for $l \notin M$. Since neither $T_i u^{(i)}$ depends on s_i, u is a potential game.

For the if part in (ii), from (4) we obtain

$$\sum_{i=1}^{n} \prod_{l=1}^{n} (I - T_l) u^{(i)} = \prod_{l=1}^{n} (I - T_l) \sum_{i=1}^{n} u^{(i)} = 0$$

$$\text{if and only if } \sum_{i=1}^{n} u^{(i)} = T_1 \sum_{i=1}^{n} u^{(i)} + \sum_{j=2}^{n} \prod_{l=1}^{j-1} (I - T_l) T_j \sum_{i=1}^{n} u^{(i)}.$$

Again, observe that $T_1 \sum_{i=1}^{n} u^{(i)}$ does not depend on s_1 and $\prod_{l=1}^{j-1} (I - T_l) T_j \sum_{i=1}^{n} u^{(i)}$ does not depend on s_j. Thus from Proposition 1, u is a zero-sum equivalent game. □

Remark 1. **(The cycle condition)** The integral test can be compared to the well-known cycle condition of Monderer and Shapley (1996) (Theorem 2.8). Consider a two player game. The cycle condition requires the following four variable function, $\Phi(s_1, s_2, \widetilde{s_1}, \widetilde{s_2})$, to be identically zero

$$\Phi(s_1, s_2, \widetilde{s_1}, \widetilde{s_2}) := \left[u^{(1)}(\widetilde{s_1}, s_2) - u^{(1)}(s_1, s_2) \right] + \left[u^{(2)}(\widetilde{s_1}, \widetilde{s_2}) - u^{(2)}(\widetilde{s_1}, s_2) \right]$$
$$+ \left[u^{(1)}(s_1, \widetilde{s_2}) - u^{(1)}(\widetilde{s_1}, \widetilde{s_2}) \right] + \left[u^{(2)}(s_1, s_2) - u^{(2)}(s_1, \widetilde{s_2}) \right],$$

while our integral test requires that the following two variable function, $\Psi(s_1, s_2)$, to be identically zero

$$\begin{aligned}\Psi(s_1,s_2) :=& u^{(1)}(s_1,s_2) - \frac{1}{|S_1|}\int u^{(1)}(s_1',s_2)ds_1' - \frac{1}{|S_2|}\int u^{(1)}(s_1,s_2')ds_2' \\ &+ \frac{1}{|S_1||S_2|}\int u^{(1)}(s_1',s_2')ds_1'ds_2' - u^{(2)}(s_1,s_2) + \frac{1}{|S_1|}\int u^{(2)}(s_1',s_2)ds_1' \\ &+ \frac{1}{|S_2|}\int u^{(2)}(s_1,s_2')ds_2' - \frac{1}{|S_1||S_2|}\int u^{(2)}(s_1',s_2')ds_1'ds_2'\end{aligned}$$

Thus, the cycle condition requires checking the values of a function of four variables, while the integral test for potential games requires checking the values of a function of two variables—this implies a significant reduction of the computational complexity. For instance, if we numerically compare two functions at n different points, the number of equalities to be checked under our test is order n^2, while this number under the cycle condition test becomes order n^4 (see the related discussion on p.200 in Hino, 2011). Note as well that in Hwang and Rey-Bellet (2017) we prove a cycle-like condition for games which are zero-sum equivalent.

If S is a finite set and m is the counting measure, then the integral test for potential games becomes the condition by Sandholm (2010). For the convenience of the reader, we provide a two-player version.

Corollary 1 (Sandholm, 2010)**.** *A two player game with payoff matrices* (A,B) *is a potential game if and only if*

$$\begin{aligned}&A_{ij} - \frac{1}{|S_1|}\sum_i A_{ij} - \frac{1}{|S_2|}\sum_j A_{ij} + \frac{1}{|S_1|}\frac{1}{|S_2|}\sum_{i,j} A_{ij} \\ &= B_{ij} - \frac{1}{|S_1|}\sum_i B_{ij} - \frac{1}{|S_2|}\sum_j B_{ij} + \frac{1}{|S_1|}\frac{1}{|S_2|}\sum_{i,j} B_{ij}\,.\end{aligned}$$

For the derivative test one needs to assume that strategy sets S_i consist of intervals and that payoff functions $u^{(i)}$ are twice continuously differentiable on S. An elementary fact from calculus is that if function g is twice continuously differentiable, then

$$\frac{\partial^2 g}{\partial s_i \partial s_j}(s) = 0 \text{ if and only if } g(s) = G(s_{-i}) + K(s_{-j})$$

for some G and K. From this, it is easy to derive a derivative test for potential games (Monderer and Shapley, 1996, Theorem 4.5). We also provide a similar test for zero-sum equivalent games.

Proposition 3 (**Derivative Tests**). *Assume that the strategy sets are intervals. Then we have:*
(i)(Monderer and Shapley, 1996) If u is twice-continuously differentiable, the game u is a potential game if and only if for all i, j

$$\frac{\partial^2 u^{(i)}}{\partial s_i \partial s_j}(s) = \frac{\partial^2 u^{(j)}}{\partial s_i \partial s_j}(s). \tag{8}$$

for all s.
(ii) If u is n-times continuously differentiable, the game u is zero-sum equivalent if and only if

$$\sum_{i=1}^{n} \frac{\partial^n u^{(i)}}{\partial s_1 \partial s_2 \cdots \partial s_n}(s) = 0. \tag{9}$$

for all s.

Proof. Again, from Proposition 1 "only parts" easily follow. For "if" parts, (i) follows from the remark before Proposition 3. For (ii), we observe that

$$\frac{\partial^n \sum_{i=1}^{n} u^{(i)}}{\partial s_1 \partial s_2 \cdots \partial s_n}(s) = 0 \quad \text{if and only if}$$

$$\sum_{i=1}^{n} u^{(i)}(s) = g^{(1)}(s_{-1}) + g^{(2)}(s_{-2}) + \cdots + g^{(n)}(s_{-n}).$$

□

Finally, our last results are alternative representations which are useful to identify games.

Proposition 4 (**Representation**). *We have:*
(i) (Ui, 2000) A game u is a potential game if and only if there exist functions w and $h^{(i)}$'s such that

$$u^{(i)}(s) = w(s) + \sum_{l \neq i} h^{(l)}(s_{-l}). \tag{10}$$

for all s.
(ii) A game u is a zero-sum equivalent game if and only if there exist a constant c, functions $w^{(i)}$'s and $h^{(i)}$'s such that $\sum_i w^{(i)}(s) = c$ and

$$u^{(i)}(s) = w^{(i)}(s) + \sum_{l \neq i} h^{(l)}(s_{-l}).$$

for all s.

Proof. Observe that for the "if" part in (i)

$$u^{(i)}(s)-u^{(j)}(s)=\sum_{l\neq i}h^{(l)}(s_{-l})-\sum_{l\neq j}h^{(l)}(s_{-l})=h^{(j)}(s_{-j})-h^{(i)}(s_{-i}).$$

If we let $g^{(i)}(s_{-i}):=-h^{(i)}(s_{-i})$, then the asserted claim follows from Proposition 1. For the "if" part in (ii), we have

$$\sum_{i=1}^{n}u^{(i)}(s)=c+\sum_{i=1}^{n}\sum_{l\neq i}h^{(l)}(s_{-l})=c+\sum_{l=1}^{n}\sum_{i\neq l}h^{(l)}(s_{-l})=\sum_{l=1}^{n}(\frac{c}{n}+(n-1)h^{(l)}(s_{-l})),$$

and if we let $g^{(l)}(s_{-l}):=\frac{c}{n}+(n-1)h^{(l)}(s_{-l})$, then the assertion follows from Proposition 1. Conversely, let u be a potential game. Then from Proposition 1, there exist function $g^{(i)}$'s satisfying (1). Then we write

$$u^{(i)}(s)=\underbrace{u^{(i)}(s)-g^{(i)}(s_{-i})+g^{(i)}(s_{-i})+\sum_{l\neq i}g^{(l)}(s_{-l})}_{=:w(s)}-\sum_{l\neq i}g^{(l)}(s_{-l}).$$

and $h^{(l)}(s_{-l}):=-g^{(l)}(s_{-l})$. Similarly, if u is a zero-sum equivalent, from Proposition 1, there exist function $g^{(i)}$'s satisfying (2). Then,

$$u^{(i)}(s)=\underbrace{u^{(i)}(s)-g^{(i)}(s_{-i})+g^{(i)}(s_{-i})-\frac{1}{n-1}\sum_{l\neq i}g^{(l)}(s_{-l})}_{=:w^{(i)}(s)}+\frac{1}{n-1}\sum_{l\neq i}g^{(l)}(s_{-l}).$$

Observe that $\sum_{i=1}^{n}\frac{1}{n-1}\sum_{l\neq i}g^{(l)}(s_{-l})=\sum_{l=1}^{n}g^{(l)}(s_{-l})$. We also have $h^{(l)}(s_{-l})=\frac{1}{n-1}g^{(l)}(s_{-l})$. From these "only if" parts follow. □

The first part of Proposition 4 is closely related to Theorem 3 in Ui (2000). It is identical for two-player games and easily seen to be equivalent in general. Proposition 4 provides a useful tool to verify if a game is a potential game or a zero-sum equivalent. For example, if $u^{(i)}(s)=w^{(i)}(s)+h^{(i)}(s_i)$, as is often the case in economics models with quasi-linear utility functions where benefit and cost functions are separable, one ignores the cost term depending on his own strategy to determine if the game is potential or zero-sum equivalent.

Figure 1 summarizes the relationships between various conditions developed in this paper. All our conditions are derived from Proposition 1. We first derive

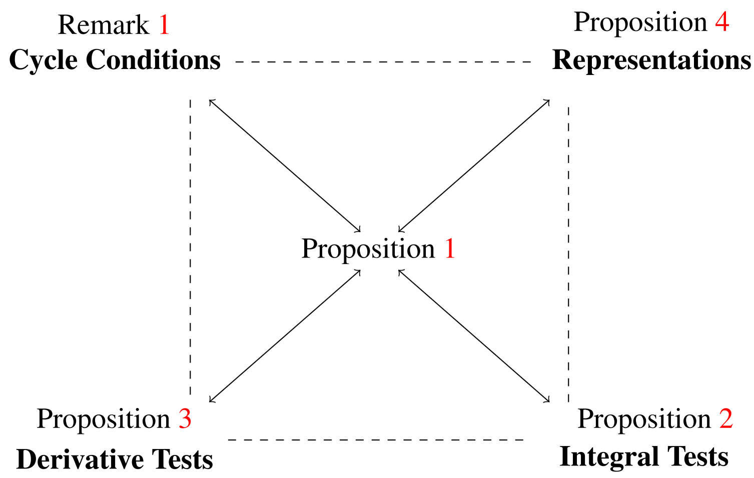

Figure 1: **Relationships between various conditions.**

the integral tests from Proposition 1 (Proposition 2). We then derive the derivative tests (Proposition 3) and derive the representation characterizations (Proposition 4). A cycle condition for zero-sum equivalent games appears in Hwang and Rey-Bellet (2017).

3. EXAMPLES

We illustrate our results with two simple examples. First we discuss the integral test for potential games.

Example 1. (Integral test for potential games) Consider a two-player game where the strategy sets are two intervals S_1 and S_2 with Lebesgue measures $|S_1|$ and $|S_2|$, respectively, and the payoffs are $u^{(1)}(s_1,s_2)$ and $u^{(2)}(s_1,s_2)$. By definition the game is a potential game if the payoff has the form $u^{(1)}(s_1,s_2) = v(s_1,s_2)+g^{(1)}(s_2)$ and $u^{(2)}(s_1,s_2) = v(s_1,s_2)+g^{(2)}(s_1)$. Then it is easy to check that we have the equality

$$\begin{aligned} &u^{(1)}(s_1,s_2) - \frac{1}{|S_1|}\int u^{(1)}(s_1,s_2)ds_1 - \frac{1}{|S_2|}\int u^{(1)}(s_1,s_2)ds_2 + \frac{1}{|S_1||S_2|}\int u^{(1)}(s_1,s_2)ds_1ds_2 \\ =&u^{(2)}(s_1,s_2) - \frac{1}{|S_1|}\int u^{(2)}(s_1,s_2)ds_1 - \frac{1}{|S_2|}\int u^{(2)}(s_1,s_2)ds_2 + \frac{1}{|S_1||S_2|}\int u^{(2)}(s_1,s_2)ds_1ds_2 \,. \end{aligned} \tag{11}$$

Our integral test asserts that *if equation (11) holds, the game is actually a potential game*. By the symmetry of the formula in s_1 and s_2, one also sees that if the payoffs have the form $u^{(1)}(s_1,s_2) := v(s_1,s_2)+\phi(s_1)$ and $u^{(2)}(s_1,s_2) := v(s_1,s_2)+\psi(s_2)$, then the condition (11) holds and thus the game is a potential game. More explicitly, we can write

$$(u^{(1)}(s_1,s_2),u^{(2)}(s_1,s_2))$$
$$= (v(s_1,s_2)+\phi(s_1)+\psi(s_2), v(s_1,s_2)+\psi(s_2)+\phi(s_1)) - (\psi(s_2),\phi(s_1))$$

which shows that u is a potential game. This provides the characterization of potential games in Proposition 4. Although somewhat trivial, this example illustrates our integral test in the simplest possible setting.

Next we use our derivative test for a class of contest games.

Example 2. (Contest games) Suppose that $S_1 = S_2 = (0,\infty)$ and consider the following contest game (see, e.g., Konrad, 2009). For f positive, define

$$u^{(1)}(s_1,s_2) = \frac{f(s_1)}{f(s_1)+f(s_2)}v - c_1(s_1),\; u^{(2)}(s_1,s_2) = \frac{f(s_2)}{f(s_1)+f(s_2)}v - c_2(s_2)\,. \tag{12}$$

We set $p^{(1)}(s_1,s_2) := f(s_1)/(f(s_1)+f(s_2))$ and $p^{(2)}(s_1,s_2) := 1-p^{(1)}(s_1,s_2)$ which are the probabilities of winning a prize of value v. Here, s_i is the amount of resources invested in the contest to obtain the prize while $c_i(s_i)$ is its associated cost.

Our derivative test for zero-sum equivalent games (see Proposition 3) asserts that when the payoffs are differentiable, a game is equivalent to a zero-sum game if we have the equality

$$\frac{\partial^2 u^{(1)}}{\partial s_1 \partial s_2}(s_1,s_2) + \frac{\partial^2 u^{(2)}}{\partial s_1 \partial s_2}(s_1,s_2) = 0\,.$$

Indeed we have

$$\frac{\partial^2 u^{(1)}}{\partial s_1 \partial s_2} + \frac{\partial^2 u^{(2)}}{\partial s_1 \partial s_2} = v\frac{\partial^2 p^{(1)}}{\partial s_1 \partial s_2} + v\frac{\partial^2 p^{(2)}}{\partial s_1 \partial s_2} = 0$$

from $p^{(1)}(s_1,s_2)+p^{(2)}(s_1,s_2) = 1$. If $f(s_i) = {s_i}^{\alpha}$ where $\alpha \le 1$ and $c_i(s_i) = s_i$, the game in (12) admits a pure strategy Nash equilibrium (Konrad, 2009).

4. DISCUSSION

We developed systematic ways of studying potential games and zero-sum equivalent games. We provided simple characterizations for potential games and zero-sum equivalent games (Proposition 1), from which we obtained new integral tests (Proposition 2), and a new derivative test for zero-sum equivalent games (Proposition 3). Our methods are general and require few assumptions on the game structure; for example, discontinuous payoff function games can be studied by the integral tests.

The advantage of the integral tests lies in that it can be applied to games with discontinuous payoff functions, as we mentioned earlier. Discontinuous payoff functions are often used in modeling competitive activities such as auctions and contests. The disadvantage of the integral tests is that it is sometimes difficult to evaluate integrals, and hence implementing the tests may be harder. Since differentiation is easier than integration in general, the derivative tests have an advantage in that it can be implemented easier, with the disadvantage in limited applicability; that is, the derivative tests can only be applied to games with differentiable payoff functions.

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