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Hierarchic Process and Relevance of Dominated Strategies in the Long Run*

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Abstract Kim and Wong (2010) re-examine the robustness of the KMR process (Kandori, Mailath, and Rob 1993) showing that any strict Nash equilibrium for the base game can be selected as a unique long run equilibrium under the KMR process by adding a totally dominated strategy. In response to this negative result, we introduce a state dependent mutational process called the hierarchic process and show that the stochastic long run equilibria of such a process are generically robust with respect to adding or eliminating totally dominated strategies. However the proposed mutational process is not enough to justify eliminating strictly but not necessarily totally dominated strategies.

Keywords Dominated Strategies, Hierarchic Process, Long Run Equilibrium.

JEL Classification C72, C73

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1. INTRODUCTION

The problem of multiplicity of Nash equilibria has long been recognized and refinement criteria have been well developed for selecting Nash equilibria. However, these approaches are not useful in cases where there are multiple strict pure strategy Nash equilibria. Kandori, Mailath, and Rob (1993), Young (1993) and others developed equilibrium selection methods based on the stochastic process augmented with mutation that select among strict pure equilibria. Adding small mutational noises to a simple myopic dynamic, Kandori, Mailath and Rob (1993), Young (1993) provided criteria for selecting among multiple strict Nash equilibria for various classes of games. This process selects a risk dominant equilibrium as the unique long run equilibrium in a 2×2 coordination game.

A large literature modifies the KMR model and checks the robustness of such strong results. For example, Bergin and Lipman (1996) shows that any strict Nash equilibrium can be a limit of stochastic process with a relevantly chosen state dependent mutation rate. So KMR's prediction is shown to be sensitive to the choice of specific stochastic process. There are works challenging Bergin and Lipman (1996) using explicitly elaborate meaningful state dependent mutational processes and most of them select the risk dominant equilibrium as the long run equilibrium[1]. Thus if economically meaningful mutational processes are considered, then the KMR's long run prediction is still valid.

Recently Kim and Wong (2010) questioned the validity of KMR model with regard to the presence of strictly dominated strategies. We often delete strictly dominated strategies when we analyze strategic form games since they do not alter Nash equilibria. Kim and Wong asked the following question: Do the long run equilibria in the sense of KMR remain the same once strictly dominated strategies are removed? They show that for any symmetric normal form game, any strict Nash equilibrium can be selected as the unique long-run equilibrium by appropriately adding only one single totally dominated strategy which is strictly dominated by all original strategies (Theorem 1). Moreover, if instantaneous adjustment is assumed, then any convex combination of strict Nash equilibria with rational number weights can be realized as the long-run distribution by appropriately adding strictly dominated strategies (Theorem 2). In this sense, strictly dominated strategies matter in the selection of a long run equilibrium under KMR's state independent mutation process even though they will never occur in equilibrium. A recent attempt to generalize the KMR which is robust to the manipulation of dominated strategies has been made by Weidenholzer

[1] Van Damme and Weibull (2002) and Maruta (2002) are such examples.

(2012). Weidenholzer (2012) show that in the circular city model of local interactions the selection of 1/2 -dominant strategies remains when adding strictly dominated strategies if interaction is "decentral". However, Weidenholzer's critique is only valid in limited situations. For example, if the population size is relatively small, Kim and Wong' results are vaild.

In this paper, we try to generalize the KMR which is robust to the addition/deletion of totally dominated strategies. We introduce a state dependent mutational process with a payoff monotonicity property[2]. This process provides the same characterization of the long run behavior by KMR for 2×2 symmetric games. We examine the robustness of this process with regard to the manipulation of strictly dominated strategies. We show that the proposed process yields independence with respect to adding or eliminating totally dominated strategies. A distinct feature of our process is that when a strategy yields a higher payoff than another strategy, then the ratio between the mutation rate toward the first strategy and that of the second strategy goes to infinity as the mutation rate goes to zero. Such an assumption of increasing discrepancy of mutation rate is a crucial for our conclusion of irrelevance of dominant strategy (and the KMR-process violates this assumption). It is true that the generalization with a hierarchic ranking process is imperfect in that the mutation is not robust to the addition/subtraction of strictly dominated (but not totally dominated) strategies. We give an example showing that the proposed mutational process is not enough to justify eliminating strictly dominated strategies. We believe that this paper can be a starting point for investigating the robust mutational process with regard to the manipulation of dominated strategies.

The rest of the paper is organized as follows. In Section 2, we develop a stochastic process that is based on the hierarchic ranking and show that it is difficult to keep strictly dominated strategies from affecting long run prediction of the model, but totally dominated strategies do not affect the selection of a long run equilibrium. Section 3 provides our concluding remarks. Section 4 provides proofs.

2. HIERARCHIC PROCESS AND DOMINATED STRATEGIES

Suppose that a finite number of players are repeatedly matched to play a stage game and adjust their behavior over time. A set of players $\{1,2,...,N\}$ is called a population. The stage game is a finite and symmetric two player

[2] As proposed in Myerson (1978), mistakes that are associated with larger payoff losses are less likely, and thus state dependent mutation rates appear more reasonable.

normal form game $G = \{S_1, S_2; u_1, u_2\}$, where the finite set $S = \{s_1, s_2, \cdots, s_I\}$, the payoff function $u : S \times S \to \Re$ such that: $S = S_1 = S_2 = S$ and $u_i(s_i, s_j) = u_j(s_j, s_i) = u(s_i, s_j)$ for all $s_i, s_j \in S$. We can identify G with the pair (S, u). We assume that actions are taken in discrete time. At the beginning of the period, each agent chooses a strategy. Let $z_t(s)$ be the number of players that adopt $s \in S$ at t. Then the state space is $Z_{S,N} = \{z \in \mathbb{N}^S : \sum_{s \in S} z(s) = N\}$.

We denote by $\triangle(S)$ the set of all mixed strategies.[3] Each $z \in Z_{S,N}$ corresponds to the element $x = z/N \in \triangle(S)$. Hence, given a z, the expected payoff for using a $s \in S$ is:

$$u(s, z/N) = u(s, x) = \sum_{s_i \in S} x(s_i) u(s, s_i) \ . \tag{1}$$

For each $s \in S$, we define the set:

$$B_s = \{x \in \triangle(S) : \forall_{s' \in S} \quad u(s, x) \geq u(s', x)\} \ . \tag{2}$$

A *best response selection* is a function $b : \triangle(S) \to S$ such that

$$x \in B_{b(x)} \qquad \text{for all } x \in \triangle(S) \ . \tag{3}$$

We define the *deterministic dynamics* over the population state space $Z_{S,N}$ by the best response selection:[4]

$$z_{t+1}(b(z_t/N)) = N \ , \tag{4}$$

$$z_{t+1}(s) = 0 \qquad \text{for all } s \neq b(z_t/N) \ . \tag{5}$$

Under this deterministic dynamics, any strict Nash equilibrium is a stationary point. When there are multiple Nash equilibria, there is a difficulty in determining the outcome of this game. KMR (1993) introduce noise to the system to resolve this indeterminacy as follows. They assume that each player's strategy flips with probability ε in each period (i.i.d. across players and over time). Let $g(z, \varepsilon) : Z_{S,N} \to \triangle(S)$ be the function such that

$$g(z, \varepsilon)(s') = \varepsilon \qquad \text{if } s' \neq b(z/N) \ , \tag{6}$$

$$g(z, \varepsilon)(s) = 1 - \sum_{s' \in S \setminus \{s\}} \varepsilon \qquad \text{if } s = b(z/N) \ . \tag{7}$$

[3]For any set X, by $\triangle(X)$ we denote the set of probability measures on X. For each $x \in \triangle(X)$, by $x(a)$ we denote the probability of a for any $a \in X$. We write $x \gg 0$ to indicate that $x(a) > 0$ for all $a \in X$.

[4]Sometimes, by abuse of notation, we will write "$b(i) = j$" to indicate that "under this deterministic dynamic, if $z_t = i$, then $z_{t+1} = j$."

Thus, $g(z,\varepsilon)$ defines a unique element in $x \in \triangle(S)$, which in turn generates a unique probability measure $y \in \triangle(Z_{S,N})$. This relation is denoted by $y = \Phi(x)$. Clearly, $x \gg 0$ implies $y \gg 0$. Now, the *stochastic dynamics* are described by the irreducible stationary Markov process, in which the transition probabilities over states in $Z_{S,N}$ are given by the Markov matrix $P = P(\varepsilon)$, where $P = [p_{z,z'}]_{z,z' \in Z_{S,N}}$ and

$$p_{z,z'} = \text{Prob}(z_{t+1} = z' | z_t = z) \text{ and } p_{z,z'} = (\Phi(g(z,\varepsilon)))(z') . \tag{8}$$

Clearly, all elements in the matrix P are strictly positive, and thus there is a unique stationary probability distribution $\mu = \mu(\varepsilon)$ for the Markov matrix $P = P(\varepsilon)$, i.e. $\mu P = \mu$. The limit distribution μ^* is defined by $\mu^* = \lim_{\varepsilon \to 0} \mu(\varepsilon)$ if it exists. We call the above stochastic process with state independent mutation rates the *KMR process*. KMR (1993) consider 2×2 symmetric games and obtain a surprisingly strong result that stochastic dynamics tend to select the risk dominant equilibrium that is relatively robust to mutations.

In this paper, we develop a stochastic process that is state dependent and payoff monotone, and examine the robustness of the process with regard to eliminating or adding strictly dominated strategies. The novelty of our stochastic process is as follows. When a strategy yields a higher payoff than another strategy, the ratio of the mutation rate toward the better strategy to that of the other strategy goes to infinity as the mutation rate goes to zero. Thus, players are less likely to choose a strategy with more damages. In this sense, our process is in line with that of Myerson (1978). For formal presentation, we give the following definitions of rankings.

Definition 1. *A ranking is a function $r : S \times \triangle(S) \to [0, 1, \cdots, I-1\}$ such that there is a best response selection $b : \triangle(S) \to S$ such that for all $s \in S$ and all $x \in \triangle(S)$: $r(s,x) = 0$ implies $s = b(x)$.*[5]

The following class of ranking function is of our special interest.

Definition 2. *A proper ranking is a ranking r such that for all $x \in \triangle(S)$:*

$$u(s,x) > u(s',x) \text{ implies } r(s,x) < r(s',x) \qquad \text{for all } s,s' \in S. \tag{9}$$

(Note that when $u(s,x) = u(s',x)$, we can have $r(s,x) = r(s',x)$ or $r(s,x) < r(s',x)$.)

[5] Thus, by definition for every x there is exactly one s satisfying $r(s,x) = 0$, and the ranking function is not an onto function.

The KMR mutation process can be modeled in terms of a ranking function as follows; A *KMR ranking* is a ranking r such that for all $x \in \triangle(S)$:

$$r(s', x) = 1 \qquad \text{for exactly } I-1 \text{ number of } s' \in S. \tag{10}$$

It is clear that the KMR ranking is not a proper ranking. When a given game has only two strategies, the notions of a KMR ranking and a proper ranking are the same. However, in general, they are not.

A ranking puts strategies into different classes, for which we will assign different mutation rates in our stochastic process. Roughly speaking, under a KMR ranking, at any state $z \in Z_{S,N}$, the strategies are grouped into only two classes: the best-response class, and the non best-response class. Under a proper ranking, strategies can be grouped into more than two classes: the first-best response class C_1, the second-best response class C_2, the third-best response class C_2, $\cdots$, and the least-best response class. Note that this approach incorporates the property of payoff-monotonicity in Maruta (2002), but with a simpler formulation[6].

In the following stochastic process, we will assume that the mutation rate of strategies depends on which class it belongs. In particular, the rate of mutating to playing a second-best strategy is much larger than the rate of mutating to playing a third-best strategy. Moreover the discrepancy rates will increase as the total mutation rate ε goes to 0.

For any ranking r, and any population size N, the *stochastic dynamics* is the irreducible stationary Markov process, where the transition probabilities over states in $Z_{S,N}$ are given by the Markov matrix $P = P(\varepsilon, r)$, where $P = [p_{z,z'}]_{z,z' \in Z_{S,N}}$ are defined by (8) with respect to the function $g(z, r, \varepsilon)$ where

$$g(z,r,\varepsilon)(s') = \exp\{-1/\varepsilon^{r(s',z/N)}\} \qquad \text{if } r(s',(z/N)) \neq 0\ , \tag{11}$$

$$g(z,r,\varepsilon)(s) = 1 - \sum_{s' \in S \setminus \{s\}} \exp\{-1/\varepsilon^{r(s',z/N)}\} \qquad \text{if } r(s,(z/N)) = 0\ . \tag{12}$$

It is clear that if r is a KMR ranking, then this stochastic dynamics becomes the one described by KMR (1993) (where it is replaced the mutation rate ε by $\exp\{-1/\varepsilon\}$). The unique stationary probability distribution $\mu = \mu(\varepsilon, r)$ for the Markov matrix $P = P(\varepsilon, r)$ exists, and we sometimes denote it by $\mu(\varepsilon, r)$ to emphasize its dependence on ε and r.

We give the existence of the limit distribution of the stationary probability distribution as ε goes to zero, and the following Lemma 1 generalizes Theorem

[6]While Maruta's approach (Maruta 2002) requires computing the detailed amount of payoffs in determining the mutation rates for different strategies, ours only needs to know the relative performance of different strategies.

1 in KMR (1993). We say that μ^* is the *long run equilibrium* for the stochastic process.

Lemma 1. *For any ranking r, the long run equilibrium* $\mu^* = \mu^*(r,N)$ *by:*

$$\mu^* = \lim_{\varepsilon \to 0} \mu(\varepsilon, r) \tag{13}$$

exists and is unique.

Proof: See the Appendix.

Kim and Wong (2010) showed that the long run prediction of the KMR process based on state independent mutation rates was sensitive to the manipulation of eliminating strictly dominated strategies by Theorem 1 and Theorem 2. In fact, they showed that any strict Nash equilibrium can be selected as the unique long-run equilibrium by appropriately adding only one single strategy which is strictly dominated by all original strategies known as a totally dominated strategy. In this sense, Kim and Wong (2010) casts doubt on the validity of the long run characterization of the KMR process.

In this paper, we will show that a reasonable stochastic process is robust to the manipulation of a totally dominated strategy. We pay attention to the following special proper ranking.

Definition 3. *A hierarchic ranking is a ranking such that for all* $x \in \triangle(S)$ *and all* $s \in S$*:*

$$\begin{aligned} &r(s,x) = 1 \qquad \text{for } [(s \neq b(x) \ \& \ (u(s,x) = u(b(s),x))] \ , \\ &r(s,x) = \#\{s' \in S : u(s',x) > u(s,x)\} \qquad \text{for } u(s,x) < u(b(s),x) \ . \end{aligned} \tag{14}$$

where b is the best response selection, as given in Definition 1.

We call the stochastic process with state dependent mutation rates based on a hierarchic ranking the "*hierarchic process*". In what follows, we show that the hierarchic process is robust with respect to the elimination of totally dominated strategies. A strategy is *totally dominated* when it is strictly dominated by all other strategies. Formally, a strategy $s \in S$ is *totally dominated* if s is strictly dominated by all $s' \in S \backslash \{s\}$.

Theorem 1. *Let game* $G = \{S,u\}$ *be obtained from a game* $G' = \{\tilde{S}, \tilde{u}\}$ *by eliminating one totally dominated strategy* s^**. Suppose that:*

$$(s' \neq s) \ \Rightarrow \ (u(s,z/N) \neq u(s',z/N)) \quad \text{for all } s,s' \in S \text{ and all } z \in Z_{S,N}, \tag{15}$$

and let the mutation process be hierarchic. Let μ be the long run equilibrium for G (on $Z_{S,N}$) and μ' be the long run equilibrium for G' (on $Z_{\tilde{S},N}$). Then, $\mu'(z) = \mu(z)$ for all $z \in Z_{S,N}$, and $\mu'(z) = 0$ for all $z \notin Z_{S,N}$.[7]

Condition (15) holds generically for all possible payoffs. The proof of Theorem 1 uses the fact that ratio between the mutational rate moving to a non-totally dominated strategy and that of a totally dominated strategy unboundedly expands as the overall mutation rate goes to zero[8], because a hierarchical ranking of a totally dominated strategy is always the lowest. This property is indispensable in our proof. Indeed, for a strictly dominated (but not totally) strategy, this property does not necessarily hold, and thus the ratio of mutation rates among strategies can be bounded (rather than explosive). As a result, this discrepancy property cannot rule out the impact of such a strictly dominated strategy to long run equilibria.

We now give such an example that even under the hierarchic process, keeping or eliminating a strictly but not totally dominated strategy will affect the long run equilibrium. Consider the following example:

$$\text{Game III} \quad \begin{array}{cccc} & A & B & C \\ A & (8,8) & (0,4) & (-K,-2K) \\ B & (4,0) & (6,6) & (-2K,0) \\ C & (-2K,-K) & (0,-2K) & (-3K,-3K) \end{array} \tag{16}$$

Clearly, for a sufficiently large positive number K, strategy C is strictly dominated by B, but only weakly dominated by A. For a sufficiently large K, we have A as a best response to any mixed strategy $x \in \triangle(S)$ with $x(C) \geq 1/N$. We divide $\triangle(S)$ into 3 zones:

$$\begin{aligned} &\text{Zone } 1 = \{z \in Z_{\tilde{S},N} : z(B) = N\} \\ &\text{Zone } 2 = \{z \in Z_{\tilde{S},N} : z(B) = z(C) = 0 \text{ and } B \text{ is the response to } z\,\} \\ &\text{Zone } 3 = \{z \in Z_{\tilde{S},N} : \ A \text{ is the response to } z\,\}. \end{aligned} \tag{17}$$

[7]Here we identify $Z_{S,N}$ with the subset $\{z \in Z_{\tilde{S},N} : z(s^*) = 0\}$ of $Z_{\tilde{S},N}$.

[8]The key feature that generates results is that the sum of mutation probabilites of strategies ranked lower than the second diminishes exponentially relative to the second best no matter how many such strategies exist. In this sense, the hierarchic ranking process generalizes the class of mutational process with payoff monotonicity propery, e.g., Maruta (2002). It would be an interesting to characterize sufficient and necessary conditions under which Thereom 1 holds.

The hierarchic ranking for Game III is as follows.

$$\begin{array}{cccc} & & z \text{ in} & \\ & \text{Zone 1} & \text{Zone 2} & \text{Zone 3} \\ r(A,z) & 1 & 1 & 0 \\ r(B,z) & 0 & 0 & 1 \\ r(C,z) & 1 & 2 & 2 \end{array} \tag{18}$$

Note that moving from state B into Zone 3 requires only one mutation (from B into $z = (N-1)/N)B + (1/N)C)$), so $\Pr\{B \to \text{Zone } 3\} \geq \exp\{-1/\varepsilon\}$. Moving from state A into Zone 2 requires at least two mutations, so $\Pr\{A \to \text{Zone } 2\} = \mathscr{O}([\exp\{-1/\varepsilon\}]^2)$. Therefore, it follows that (AA) is the unique long run equilibrium for Game *III*. This contrasts sharply with the case, in which for Game *I* the unique long run equilibrium is (BB).

3. CONCLUDING REMARKS

Kim and Wong (2010) show that the KMR process with state independent mutation rates is not robust to the manipulation of adding or eliminating dominated strategies. Thus, an evolutionary approach with mutations must be carefully applied to the study of the selection of Nash equilibria when dominated strategies are involved in stage games.

The main result of this paper shows that if a state dependent mutation process called the hierarchic process is applied, then a certain class of strictly dominated strategies known as totally dominated strategies does not affect the long run equilibrium since manipulating such strategies does not alter the basins of attraction of steady states. Hence, the stochastic process based on a hierarchic ranking selects the same outcome, risk dominant equilibria, independent of the presence of totally dominated strategies. In this sense, we confirm that the state dependent mutation process with the payoff monotonicity in line with Myerson (1978) is a reasonable mutation process for economically meaningful long run prediction. Whether there is a reasonable mutation process that yields the independence with regard to any strictly dominated strategies remains open.

4. APPENDIX: PROOFS

We recall several notions from Freidlin and Wentzell (1984) and Kim and Wong (2010). A *directed graph* h on Z is a finite set collected order pairs of elements in Z (denoted by $(i \to j)$).[9] For any $z \in Z$, a *z-tree* is a direct graph h on Z such that every state except z has a unique successor and there are no closed loops. Denote by H_z the set of all z-trees by H_z.

Consider a ranking r, and an mutation rate μ, and the corresponding Markov matrix $P = P(\varepsilon, r)$. We can state Lemma 1 in Kim and Wong (2010) as follows.

Lemma 2. (Kim and Wong (2010), Lemma 1) *Let μ be the stationary probability distribution, i.e. $\mu P = \mu$. Then: μ is proportional to the vector $q = (q_z)_{z\in Z}$, i.e., $\mu_z = q_z / \sum_{i\in Z} q_i$, where*

$$q_z = \sum_{h\in H_z} P_h, \qquad \textit{where } P_h = \Pi_{(i\to j)\in h} P_{ij}. \tag{19}$$

Let $\mathscr{K}$ be the set of all functions $K(\varepsilon)$ in the form of

$$K(\varepsilon) = (\exp\{-1/\varepsilon^{n_1}\}) \cdots (\exp\{-1/\varepsilon^{n_K}\}) \ , \tag{20}$$

where $n_1, \cdots, n_K$ are strictly positive integers. Clearly, for all $K, G \in \mathscr{K}$, if $K \neq G$, then either $\lim_{\varepsilon\to 0}(K/G) = 0$ or $\lim_{\varepsilon\to 0}(G/K) = 0$.

Proof of Lemma 1. By Lemma 2 the stationary probability distribution μ is proportional to vector q. Each q_z is a linear combination of elements $K_{z,i} \in \mathscr{K}$:

$$q_z = \sum_{i=0}^{M_z} a_{z,i} K_{z,i}, \tag{21}$$

where $a_{z,i}$'s are constant terms that are independent of ε, and each pair $K_{z,i}, K_{z,i'}$ are distinct for distinct i, i'. Define $\mathscr{K}^* = \cup_{z\in Z}\{K_{z,i} : 0 \leq i \leq M_z\}$. Choose the unique "maximal" element K^* from $\mathscr{K}^*$, i.e. $K^* \in \mathscr{K}^*$ is such that $\lim_{\varepsilon\to 0}(G/K^*) = 0$ for all $G \in \mathscr{K}^* \setminus \{K^*\}$. Then for each $z \in Z$,

$$\text{if there is some } i \text{ such that } K_{z,i} = K^*, \text{ define } a_z^* = a_{z,i}, \tag{22}$$

$$\text{otherwise, define } a_z^* = 0. \tag{23}$$

Using (21), we obtain:

$$\mu_z^* = \lim_{\varepsilon\to 0} \frac{q_z}{\sum_i q_i} = \frac{a_z^*}{\sum_i a_i^*} \ . \tag{24}$$

Q.E.D.

[9]Then we call $(i \to j)$ an *edge*, and say that j is a *successor* of i. A *loop* is a finite sequence $i_1, \cdots, i_n \in Z$ such that $i_1 \to i_2 \to \cdots \to i_n \to i_1$.

Proof of Theorem 1. We consider game $G = \{S, u\}$ and Game $G' = \{\tilde{S}, \tilde{u}\}$ as given in the theorem, where $\tilde{S} = S \cup \{s^*\}$. For game G, the population state set is $Z = Z_{S,N}$, and for game G', the population state set is $\tilde{Z} = Z_{\tilde{S},N}$. We define $Z^* = \{z \in \tilde{Z} : z(s^*) = 0\}$, and we will identify Z as Z^* in the natural manner.

Let $b : \triangle(S) \to S$ be a best response selection for G, and let $\tilde{b} : \triangle(\tilde{S}) \to \tilde{S}$ be a best response selection for G'. By (15) for every $z \in Z$ there is a unique best response in S, hence we have $b(z/N) = \tilde{b}(z/N)$ for all $z \in Z = Z^*$.

Let r be the hierarchic ranking for G, $g(z, r, \varepsilon)(s)$ be the mixed strategy probability, and $P_{z,z'}$ be the corresponding transitional probability over Z; similarly, $\tilde{r}$, $\tilde{g}(z, \tilde{r}, \varepsilon)(s)$ and $\tilde{P}_{z,z'}$ denote the corresponding terms for $\tilde{G}$ and $\tilde{Z}$.

Note that (15) ensures that:

$$\text{a) } r(s,z) = \tilde{r}(s,z) < I \quad \text{for all } s \in S \text{ and all } z \in Z = Z^* \,, \tag{25}$$

$$\text{b) } \tilde{r}(s^*, x) = I \quad \text{for all } x \in \tilde{Z} \backslash Z^* \,. \tag{26}$$

Hence, for all $z \in Z = Z^*$, we have:

$$\text{a) if } s \neq b(z) \text{ and } s \in S, \text{ then } g(z, r, \varepsilon)(s) = \tilde{g}(z, \tilde{r}, \varepsilon)(s) \geq \exp\{-1/\varepsilon^I\} \,, \tag{27}$$

$$\text{b) if } s = b(z), \text{ then } g(z, r, \varepsilon)(s)/\tilde{g}(z, \tilde{r}, \varepsilon)(s) \to 1 \text{ as } \varepsilon \to 0 \,, \tag{28}$$

$$\text{c) } \tilde{g}(z, \tilde{r}, \varepsilon)(s^*) = \exp\{-1/\varepsilon^{I+1}\}. \tag{29}$$

For game G, we choose the set H of all trees in Z, choose the set H_z of all z-trees for all $z \in Z$, and choose the probabilities q_z and P_h as given in (19). Similarly, we choose the corresponding terms $\tilde{H}$, $\tilde{H}_z$, $\tilde{q}_z$ and $\tilde{P}_h$ for the game G'.

We define H^* to be set of all $h \in \tilde{H}$ satisfying the property that for all $i, j \in \tilde{Z}$, if $(i \to j) \in h$, then:

$$\text{a) if } i \in Z^*, \text{ then } j \in Z^*, \tag{30}$$

$$\text{b) if } i \in \tilde{Z} \backslash Z^*, \text{ then } j = \tilde{b}(i). \tag{31}$$

Clearly, for each $h \in H^*$, there is a unique tree $f(h) \in H$ such that $h|_{Z \times Z} = f(h)$. In addition, this mapping $f : H^* \to H$ is one-to-one and onto.

Claim 1. *If $h \in \tilde{H} \backslash H^*$, then there is an $h' \in H^*$ such that $(\tilde{P}_h/\tilde{P}_{h'}) \to 0$ as $\varepsilon \to 0$.*

Proof: (Case 1) Suppose that h violates (30) at some $i \in Z^*$, say $(i \to j) \in h$ but $j(s^*) > 0$. Then, by (29) and the definitions of $\tilde{P}_h$ and $\tilde{P}_{i,j}$, we have:

$$\tilde{P}_h = \mathcal{O}(\exp\{-1/\varepsilon^{I+1}\}) \,. \tag{32}$$

Now, we choose any $h' \in H^*$. By (27), (28), and the definitions of $\tilde{P}_h$ and $\tilde{P}_{i,j}$, we have:

$$[\exp\{-1/\varepsilon^I\}]^{\#(Z^*)} = \mathscr{O}(\tilde{P}_{h'}) \ . \tag{33}$$

Therefore, we have $(\tilde{P}_h/\tilde{P}_{h'}) \to 0$ as $\varepsilon \to 0$.

(Case 2) Suppose that h satisfies (30) for all $i' \in Z^*$, but violates (31) at some $i \in \tilde{Z}\backslash Z^*$. Hence, we have $i \in \tilde{Z}\backslash Z^*$, and $(i \to j) \in h$, but $j \neq \tilde{b}(i)$. Then we can choose a unique $h' \in H^*$ such that for all $i' \in Z^*$, we ahve: $(i' \to j') \in h$ if and only $(i' \to j') \in h'$ for all $j' \in \tilde{Z}$. Then $(\tilde{P}_h/\tilde{P}_{h'}) = (\tilde{P}_{i,\tilde{b}(i)}/\tilde{P}_{i,j}) \leq (\tilde{P}_{i,\tilde{b}(i)}/(1-\tilde{P}_{i,\tilde{b}(i)})) \to 0$ as $\varepsilon \to 0$. This establishes Claim 1.

Claim 2. *If $h \in H^*$ and $h' = f(h) \in H$, then $(\tilde{P}_h/P_{h'}) \to 1$ as $\varepsilon \to 0$.*

Proof: This follows immediately from (27) and (28).

Now, we consider the terms $q_z = \sum_{h \in H_z} P_z$ (where $z \in Z$), and the terms $\tilde{q}_z = \sum_{h \in \tilde{H}_z} \tilde{P}_z$ (where $z \in \tilde{H}$).

First, note that if $z \in \tilde{Z}\backslash Z^*$, then each z-tree violates (30) at some i. Therefore, by Claim 1, we have:

$$\text{if } z \in \tilde{Z}\backslash Z^*, \text{ then } \frac{\tilde{q}_z}{\sum_{z' \in \tilde{Z}} \tilde{q}_{z'}} \to 0 \text{ as } \varepsilon \to \infty \ . \tag{34}$$

Next, it follows from Claim 2 that

$$\text{if } z \in Z^*, \text{ then } \frac{\tilde{q}_z}{q_z} \to 1 \text{ as } \varepsilon \to \infty. \tag{35}$$

Therefore, by Lemma 2 the long run distribution μ for G and that μ' for G' satisfy $\mu(z) = \mu'(z)$ for all $s \in Z = Z^*$, and $\mu'(z) = 0$ for all $z \in \tilde{Z}\backslash Z^*$. Q.E.D.

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