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On the Interaction between Player Heterogeneity and Partner Heterogeneity in Two-way Flow Strict Nash Networks

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Abstract This paper brings together analyses of two-way flow Strict Nash networks under exclusive player heterogeneity assumption and exclusive partner heterogeneity assumption studied by Galeotti et al. (2006) and Billand et al. (2011) respectively. We provide a proposition that generalizes the results of these models by stating that: (i) Strict Nash network consists of multiple non-empty components as in Galeotti et al. (2006), and (ii) each non-empty component is a branching or B_i network as in Billand et al. (2011). This proposition requires that a restriction on link formation cost, which is called Uniform Partner Ranking, is satisfied.

Keywords Network Formation, Strict Nash Network, Two-way Flow Network, Branching Network, Agent Heterogeneity

JEL Classification C72, D85

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1. INTRODUCTION

A game-theoretic model of network formation assumes that agents form or remove costly link(s) for the best of their interests. It seeks to understand the shapes and properties of equilibrium networks that are outcomes of this assumption. The seminal work of Bala and Goyal (2000), BG henceforth, predicts that in Strict Nash equilibrium center-sponsored star is a unique non-empty equilibrium network, given that link formation is unilateral and information flow is two-way with no decay.[1]Such a simple form of network emerges as a unique non-empty Strict Nash network because several simple assumptions are adopted, including agent homogeneity. Naturally, this simplicity has spawned a vast literature that questions how agent heterogeneity may influence the properties of Strict Nash networks.

A class of models in this literature assumes that (i) heterogeneity resides in the diversity of link formation cost, and (ii) no decay is present. Within this class two strands are of the primary interest of this paper. The first one assumes that the diversity of link formation cost depends exclusively on the identity of link receiver (called exclusive partner heterogeneity onwards). The second one assumes that the diversity of link formation cost depends exclusively on the identity of link sender (called exclusive player heterogeneity onwards). This paper aims to bring together these two strands by (i) allowing the heterogeneity in link formation cost to depend on *both* link receiver and link sender, and (ii) generalizing the results on the properties of Strict Nash networks found in these two strands.

We briefly give an overview of the literature here. Concerning exclusive player heterogeneity, existence of Nash network and equilibrium characterization of Strict Nash networks (SNNs henceforth) are extensively studied by Galeotti et al. (2006) (See Proposition 3.1 in Galeotti et al. (2006)). Concerning exclusive partner heterogeneity, full equilibrium characterization and the existence of SNNs are extensively studied by Billand et al. (2011) and Billand et al. (2012). However, when the heterogeneity depends on both link sender and his partner (called two-way

[1]An example in this spirit is the telephone call in which the caller bears the entire connection cost, and the two parties do not mind sharing their private nonrival information. For further elaboration on related examples, see Bala and Goyal (2000). For further elaboration on the importance of the studies network formation, see, e.g., Bala and Goyal (2000), Jackson (2008) and Jackson (2007) .

heterogeneity henceforth), to the knowledge of the author little is known. We know from Galeotti et al. (2006) that SNN under two-way heterogeneity is minimal, and from Haller et al. (2007) that it does not always exist.

The only work that studies the shape of SNN under two-way heterogeneity assumption is the insider-outsider model of Galeotti et al. (2006) (See Proposition 4.1 in Galeotti et al. (2006)). It assumes that all agents can be partitioned into multiple insider groups. All agents in the same group shares an identical set of choices of link formation cost such that link formation cost is low if the link bridges agents in the same group, and link formation cost is high otherwise. If all agents in the network belong to the same insider group, this model is reduced to the original BG model (See Proposition 4.1 in BG), which assumes homogeneity. This marks a major difference between the insider-outsider model and the model in this paper, since it generalizes *both* the results of exclusive player heterogeneity model of Galeotti et al. (2006) and exclusive partner heterogeneity model of Billand et al. (2011), as well as the original BG model. Such a generalization is thus a major contribution of this paper to the literature.

Having the goal of bridging the above two strands in the literature in mind, Proposition 1 in this paper generalizes the equilibrium characterization of SNNs found in the Proposition 3.1 in Galeotti et al. (2006), Proposition 1 in Billand et al. (2011), as well as Proposition 4.1 in BG which assumes homogeneity. To relate the results of Proposition 1 in this paper with the above three propositions, we elaborate on the known properties of SNN with no decay as follows:

1. Given that value of information and link formation cost are homogeneous, SNN is a center-sponsored star. This property is found in Proposition 4.1 in BG.

2. Given that value flows freely and link formation cost assumes exclusive player heterogeneity, SNN is a disconnected network of which each non-empty component is a center-sponsored star. This property is found in Proposition 3.1 in Galeotti et al. (2006).

3. Given that both value and link formation cost assume exclusive partner heterogeneity, non-empty SNN has a unique component that is a branching or B_{i_0}, where i_0 is the lowest cost agent. This property is found in Proposition 1 in Billand et al. (2011).

Consequently, the literature confirms that (1) exclusive player heterogeneity *cannot* alter the shape of SNN, yet it splits the connected SNN in BG into many components, and (2) conversely, exclusive partner heterogeneity *cannot* increase the quantity of components in SNN, yet it transforms the shape of SNNs into a larger class that contains branching and B_i networks. Naturally, this raises the question of whether both properties of SNN - disconnectedness and every non-empty component being branching or B_i - can be preserved when the two heterogeneities are allowed to interact. Proposition 1 in this paper confirms that indeed this is the case, given that link formation cost satisfies a certain restriction called *Uniform Partner Ranking*. This result holds true even when value flows freely. Figure 1 illustrates an SNN predicted by this proposition.

To further elaborate on this result, we remark that the Uniform Partner Ranking condition, UPR henceforth, is a sufficient but not necessary condition to predict that every non-empty component of SNN is a branching or B_i network. Indeed, given that a set of agents are better off being in the same component, the existence of a *common best partner* is sufficient to warrant that the component is a branching or B_i network, where common best partner is defined as an agent that generates a lowest link formation cost to every other agent in the component if chosen as a partner. This fact is formally stated as Lemma 4, which becomes the most important building block of the main results. In Proposition 1, UPR condition is simply a restriction on link formation cost that guarantees that a common best partner exists in every non-empty component of SNN. This in turn guarantees that every non-empty component of SNN is a branching or B_i network. To illustrate this point, Example 3 shows that a connected network that is a B_i or branching network, which can be supported as SNN by value and link formation cost that assume exclusive partner heterogeneity according to Proposition 1 in Billand et al. (2011), can also be supported as SNN by a set of value and link formation cost that merely warrants the existence of a common best partner among all agents in the network.

The paper proceeds as follows. The model and relevant notations are described in Section 2. In Section 3 we introduce a lemma that shows that if a common best partner exists then a component of SNN is a branching or B_i network. In Section 4 this lemma is put in use to establish Proposition 1. Finally, Section 5 concludes.

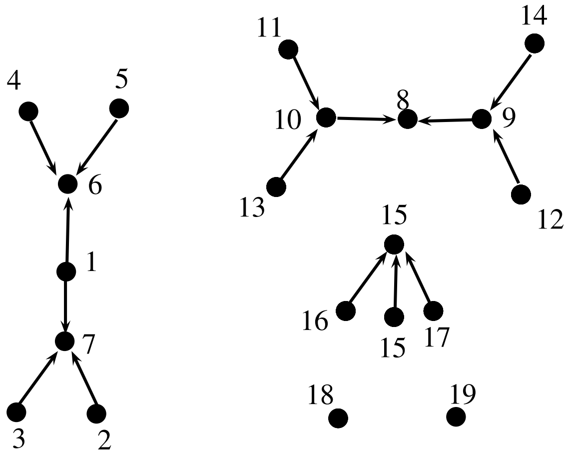

Figure 1: An example of SNN. It consists of two empty components and three non-empty components. Each non-empty component is a branching or B_i network. The non-empty component on the left is B_i, while the other two on the right are branching. Note that an arrow from i to j represents the fact that j sponsors the link to i.

2. THE MODEL

Let $N = \{1, ..., n\}$ be the set of all agents and let i and j be typical members of this set. Each agent possesses a nonrival distinct piece of information that is valuable both to himself and any other agent who has an entry to it. There are two ways to which a pair of agents can have an entry to each other's information: there is a pairwise link between i and j or there is a series of links where the two ends are i and j. To ease the comparison with other existing models, the notations in what follows are primarily based upon Billand et al. (2011).

Link establishment and individual's strategy. Link establishment is costly and one-sided. A strategy of i is $g_i = \{g_{i,j} : j \in N, j \neq i\}$, where $g_{i,j} = 1$ if i establishes a link with j and $g_{i,j} = 0$ otherwise. If $g_{i,j} = 1$, it is said that i *accesses* j or j receives a link from i. Since all links form the network, we write $g = \{g_i : i \in N\}$ to represent both a strategy profile and a network.

Network representation. In this paper a node depicts an agent, and an arrow from j to i represents that j receives a link from i. If all arrows are removed, the network representation merely represents how information flows among agents. This structure of information flow is denoted by $\bar{g} = \left\{\bar{g}_{i,j} : i,j \in N, i \neq j\right\}$, where $\bar{g}_{i,j} = 1$ if $g_{i,j} = 1$ or $g_{j,i} = 1$ or both, and $\bar{g}_{i,j} = 0$ otherwise.

Information flow. Information of j flows to i directly through a link between i and j, regardless to who sponsors it. Alternatively, information of j flows to i through a *chain*. Formally, a chain between i and j ($i \neq j$) is a sequence $j_0, ..., j_m$ such that $\bar{g}_{j_l, j_{l+1}} = 1$ for $l = 0, ..., m-1$ and $j_0 = i$ and $j_m = j$. In this case, it is said that i *observes* j and vice versa. We assume, following the convention in the literature, that i observes himself. The set of all agents that i observes is denoted by $N_i(g)$

Cost heterogeneity Let $\mathcal{C} = \{c_{i,j} : i,j \in N, i \neq j\}$ be a *cost structure*, $\mathcal{C}$ is said to assume cost homogeneity if $c_{i,j} = c$ for all $i \neq j$, and cost heterogeneity if it holds true that $c_{i,j} \neq c_{k,l}$ for some $c_{i,j}, c_{k,l} \in \mathcal{C}$. Cost heterogeneity can be further classified as follows. If $c_{i,j} = c_i$ for all i ($c_{i,j} = c_j$ for all j), $\mathcal{C}$ is said to assume exclusive player (partner) heterogeneity. If $c_{i,j} \neq c_i$ for an i and $c_{i,j} \neq c_j$ for a j, $\mathcal{C}$ is said to assume two-way heterogeneity. Similarly, Let $\mathcal{V} = \{V_{i,j} : i,j \in N, i \neq j\}$ be a *value structure*. If $V_{i,j} = V_i$ for all i ($V_{i,j} = V_j$ for all j), $\mathcal{V}$ is said to assume exclusive player (partner) heterogeneity. If $V_{i,j} \neq V_i$ for an i and $V_{i,j} \neq V_j$ for a j, $\mathcal{V}$ is said to assume two-way heterogeneity.

The payoffs. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be such that $\pi(x,y)$ is strictly increasing in x and strictly decreasing in y. The payoff of player i is given by:

$$\pi_i(g) = \pi\left(\sum_{j \in N_i(\mathbf{g}) \setminus \{i\}} V_{i,j}, \sum_{j \in N_i(\mathbf{g}) \setminus \{i\}} \mathbf{g_{i,j}} c_{i,j}\right) \tag{1}$$

Where the first term is interpreted as the total benefits that i receives in this network, while the second term represents the total link formation costs that he bears. A special case is when the above payoff is linear in both terms. More Formally,

$$\pi_i^L(g) = \sum_{j \in N_i(\mathbf{g}) \setminus \{i\}} V_{i,j} - \sum_{j \in N_i(\mathbf{g}) \setminus \{i\}} \mathbf{g_{i,j}} c_{i,j} \tag{2}$$

Network-related Notations. Recall from the above that a chain from i to j is a sequence of distinct players $j_0, ..., j_m$ such that $\bar{g}_{j_l, j_{l+1}} = 1$ for $l = 0, ..., m-1$ and $j_0 = i$ and $j_m = j$, a path is defined similarly except that link sponsorship matters. That is, a path from j to i is a sequence $j_0, ..., j_m$ such that $g_{j_l,j_{l+1}} = 1$ for $l = 0,...,m-1$ and $j_0 = i$ and $j_m = j$. A cycle is defined in the same fashion as a chain, except that $j_0 = i$ and $j_m = i$ and all other players in the sequence are distinct. We use these notations to define the following terms. A network is connected if there is a chain for every distinct $i, j \in N$. A subnetwork of g is a network g' such that $g' \subset g$. g' is a component of g if it is a maximal connected subnetwork of g. The set of all agents in a component g' is denoted by $N(g')$. A component is said to be minimal if it contains no cycle. In a minimal component, every distinct pair of agents is connected by a unique chain so that a removal of a link $g_{i,j}$ disconnects the component into two components - one containing i and the other one containing j. The resulted disconnected components are denoted by $D_i(g_{i,j})$ and $D_j(g_{i,j})$ respectively.

Consider an agent i, $I_i(g)$ and $O_i(g)$ are defined as the set of all links of i that are not sponsored by i and the set of all links that i establishes respectively. If $|I_i(g)| = |O_i(g)| = 0$, then i is said to be a singleton. If every agent in a network is a singleton, then the network is an empty network. Observe that a singleton is, by definition, an empty component. If either $|I_i(g) = 1|$ or $|O_i(g)| = 1$ (but not both), then i is said to be a terminal agent.

Some important patterns of networks. There are some patterns of networks that are often referred to, since they emerge as Strict Nash Equilibria. They are defined here[2]. For a generic set $X \subset N$, let $Q_X = X \cup \{j' |$ there exists a path from j' to j where $j \in X\}$. X is said to be a contrabasis of a network g if it is a minimal set with respect to the property that $Q_X = N$. X is said to be an i-point contrabasis if every $j \in X$ accesses i. If i is a point contrabasis of g and $|I_i(g)| \geq 2$ but $|I_j(g)| < 2$ for all $j \neq i$ and $j \in N$, g is said to be a B_i *network*. Observe that in B_i network i is the only agent that receives more than one link [3]. By contrast, if there exists a unique i such that $|I_i(g)| = 0$ and $|I_j(g)| \leq 1$ for all $j \neq i$, then the network is said to be a *branching network* rooted at i. Figure 2 and Figure 3 illustrate some B_i and branching networks respectively.

Certain important forms of branching network and B_i network are worth mentioning. A *star* is a network such that there exists an agent i such that $\bar{g}_{i,j} = 1$ for all $j \neq i$ and $\bar{g}_{j,k} = 0$ for all $j, k \neq i$. A star is a *center-sponsored star* if $g_{i,j} = \bar{g}_{i,j} = 1$ for all $j \neq i$. A star is a *periphery-sponsored*

[2]The definitions of B_i network and branching network are borrowed from Billand et al. (2011)

[3]Intuitively, B_i is a network such that every agent in the network can be reached through a path to an agent that accesses i.

star if $g_{j,i} = \bar{g}_{i,j} = 1$ for all $j \neq i$. Note that center-sponsored star is a form of branching rooted at i, while periphery-sponsored star is a form of B_i network.

Strict Nash Equilibrium. Let g_{-i} denote a strategy profile of all agents except i, ie., $g_i \cup g_{-i} = g$. A best response of an agent i is g_i such that $\pi_i(g_i \cup g_{-i}) \geq \pi_i(g_i' \cup g_{-i})$ for every g_i' that is a strategy of i. A strategy profile or a network g is Nash if every agent plays his best response. A Nash network is a Strict Nash network if the best response of every agent is unique.

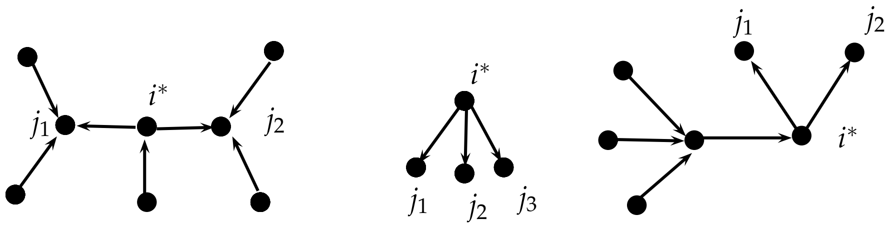

Figure 2: Three B_{i^*} networks. Observe that the set of all agent js in each network is a contrabasis. That is, for every agent that is not a j, there is a path from that agent to an agent j. Observe further that the middle network is a periphery-sponsored star.

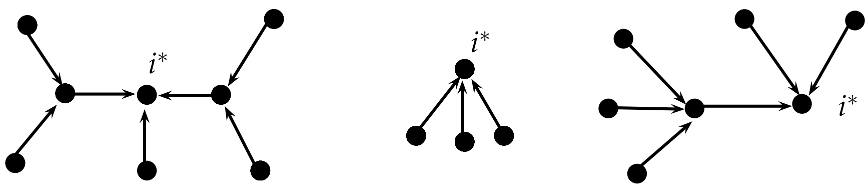

Figure 3: Three branching networks rooted at i^*. Observe that i^* is the only agent that receives no links. Observe further that the middle network is a center-sponsored star.

2.1. COST STRUCTURE AND THE UNIFORM PARTNER RANKING CONDITION

In the main results section, a proposition that fully characterizes the shapes of SNNs are proven, given that the cost structure $\mathcal{C}$ satisfies a condition called *Uniform Partner Ranking*. This section serves to define this condition.

Definition 1 (Better Partner)**.** *Consider a set $X \subset N$ and agents $j,k \in X$, j is* ***at least as good a partner*** *as k with respect to the set X if $c_{i,j} \leq c_{i,k}$ for any $i \in X, i \neq j \neq k$. Moreover, if the inequality is strict then j is said to be a* ***better partner*** *than k with respect to the set X.*

Intuitively, this definition states that in a set $X \subset N$ j is at least as good a partner as k with respect to the set X if every agent in X, except j and k, 'ranks' j as an equivalent or superior partner to k in terms of link formation cost. The definition *Uniform Partner Ranking* below simply adds that $X = N$ and that any distinct pair j,k can be ranked.

Definition 2 (Uniform Partner Ranking)**.** *A cost structure $\mathcal{C}$ is said to satisfy* ***Uniform Partner Ranking*** *condition if for any distinct pair $j,k \in N$ it holds true that j is at least as good a partner as k or k is at least as a good a partner as j with respect to the set N.*

Remarks 1. *For the sake of simplicity, let $I(N) = \{i_0, i_1, ..., i_{n-1}\}$ be an ordered rearrangement of the set N such that i_x is at least as good a partner as i_y with respect to the set $I(N)$ for any $x < y$. This notation is used throughout this paper.*

Example 1 and 2 below show that exclusive partner heterogeneity assumption as in Billand et al. (2011) and exclusive player heterogeneity as in Galeotti et al. (2006) satisfy UPR condition.

Example 1. *Let $c_{i,j} = c_j$ (ie., we assume exclusive partner heterogeneity as in Billand et al. (2011)). Specifically, let $N = \{1,..,5\}$ and $\mathcal{C} = \{c_1 = 5,\ c_2 = 4,\ c_3 = 3,\ c_4 = 2,\ c_5 = 1\}$, then clearly $\mathcal{C}$ satisfies Uniform Partner Ranking Property since $I(N) = \{5,4,3,2,1\}$*

Example 2. *Let $c_{i,j} = c_i$ (ie., we assume exclusive player heterogeneity as in Galeotti et al. (2006)). Specifically, let $N = \{1,..,5\}$ and $\mathcal{C} = \{c_1 = 5,\ c_2 = 4,\ c_3 = 3,\ c_4 = 2,\ c_5 = 1\}$. Clearly, $\mathcal{C}$ satisfies Uniform Partner Ranking Property since every $j,k \in N$ is such that j is at least as good a partner as k and k is at least as good a partner as j.*

Contrary to the above two examples, the following example shows a cost structure that satisfies neither exclusive player heterogeneity nor exclusive partner heterogeneity, yet satisfies our Uniform Partner Ranking Condition.

Example 3. *Consider the cost structure represented by the matrix in Table 1. It is easy to see that exclusive player heterogeneity is not satisfied since $c_{1,2} \neq c_{1,3}$, ie., $1 \neq 2$. Also, exclusive partner heterogeneity is not satisfied since $c_{1,2} \neq c_{3,2}$, ie., $1 \neq 2$. However, this cost structure satisfies Uniform Partner Ranking. Indeed, $c_{i,j} < c_{i,j+1}$ for any $i \neq j, j+1$. That is, j is a better partner than $j+1$ for every i and $I(N) = \{1, 2, 3, 4, 5\}$.*

$$\begin{array}{cc} & \text{Partner } j \\ \text{Player } i & \begin{bmatrix} - & 1 & 2 & 3 & 4 \\ 1 & - & 2 & 3 & 4 \\ 1 & 2 & - & 3 & 4 \\ 1 & 2 & 3 & - & 4 \\ 1 & 2 & 3 & 4 & - \end{bmatrix} \end{array}$$

Table 1: Cost Structure in Example 3. Note that this cost structure violates both exclusive player heterogeneity and exclusive partner heterogeneity, yet it satisfies Uniform Partner Ranking.

Note that if $\mathcal{C}$ satisfies the Uniform Partner Ranking Condition, then the agent i_0 can be considered as a *common best partner* among the set of agents N, in the sense that every agent (except i_0) agrees that i_0 is the partner that incurs the lowest link formation cost. In more formal terms,

Definition 3. *Let $X \subset N$ be a set of agents, then $i^* \in X$ is said to be a* ***common best partner*** *among all agents in X if $c_{ii^*} \leq c_{ij}$ for all $i, j \in X$ and $i \neq j \neq i^*$.*

It is worth mentioning that a common best partner is not necessarily unique. Indeed, a special case is when $\mathcal{C}$ satisfies exclusive player heterogeneity assumption. Since link formation cost does not depend on the identity of the partner, all agents are common best partners with respect to the set N.

3. USEFUL LEMMAS

Proposition 1 in the main results section makes use of the fact that a component of SNN is a branching or B_i network, given that a common best partner exists in the component. The purpose of this section, therefore, is to establish this fact as a lemma. In what follows we build up three lemmas that leads to the establishment of this fact as Lemma 4. All proofs are relegated to the appendix.

Lemma 1 (SNN is minimal). *A component in an SNN is minimal.*

Next, Lemma 2 and 3 ascertain that an agent that receives more than one link is a common best partner.

Lemma 2. *In an SNN, if i accesses j, then $c_{i,j} < c_{i,k}$ for any agent k that is contained in $D_j\left(g_{i,j}\right)$ and $k \neq j$.*

The proof is trivial and is omitted. Intuitively, if i decides to access an agent in $D_j\left(g_{i,j}\right)$, then he chooses an agent that incurs the lowest link formation cost among all agents in $D_j\left(g_{i,j}\right)$. The fact that our equilibrium prediction criterion is SNN further necessitates that this agent is unique and the above inequality is strict.

Lemma 3. *In a non-empty component of SNN, if a common best partner among all agents in the component exists, then this component contains at most one agent that receives more than one link. Moreover, this agent is a unique common best partner among all agents in the component.*

Finally, in what follows the main lemma of this paper is introduced. It characterizes the shape of a non-empty component in SNN given that a common best partner exists in the component.

Lemma 4. *A non-empty component in SNN is a branching or B_{i^*}, given that a common best partner (denoted by i^*) among all agents in this component exists.*

4. MAIN RESULTS - EQUILIBRIUM CHARACTERIZATION

In Lemma 4, the existence of a common best partner in a non-empty component of SNN guarantees that the component is a branching or B_{i^*}. Proposition 1 below makes use of this lemma. It imposes UPR condition to guarantee that every non-empty component contains a common best partner. This in turn guarantees that every non-empty component of SNN is a branching or B_{i^*}

Proposition 1. *Let $\mathcal{C}$ satisfy Uniform Partner Ranking Condition, $V_{i,j}$ flow freely, and the payoff function satisfy Equation 1, then every non-empty component in SNN is a branching or B_{i^*}. Conversely, given that the payoff satisfies Equation 2, a network of which each non-empty component is a branching or B_i network can be supported as SNN by a pair of $\mathcal{V}$ and $\mathcal{C}$, where $\mathcal{C}$ satisfies Uniform Partner Ranking Condition.*

As shown in Example 1 and 2, exclusive player heterogeneity and exclusive partner heterogeneity in link formation cost satisfy UPR condition. Consequently, Proposition 1 in this paper is a generalization of Proposition 1 in Billand et al. (2011) and Proposition 3.1 in Galeotti et al. (2006). A comparison with both propositions is noteworthy. Comparing with Proposition 3.1 in Galeotti et al. (2006), the only difference is that the set of networks that are candidates for non-empty components of SNN in Proposition 1 of this paper are much larger than that of Proposition 3.1 in Galeotti et al. (2006), which predicts that each non-empty component is a center-sponsored star. Comparing with Proposition 1 in Billand et al. (2011), the only difference is that Proposition 1 in Billand et al. (2011) predicts that a non-empty SNN is connected, while Proposition 1 in this paper predicts that multiplicity of non-empty components and singletons may exist in SNN.

Two additional observations are worth elaborating. First, Proposition 1 in this paper shows that value heterogeneity plays a relatively less important role in predicting the properties of SNN than cost heterogeneity does. Intuitively, when an agent decides whether to enter a non-empty component, he simply weighs the sum of value of information of the entire component against the link formation cost with a common best partner. Consequently, value heterogeneity does not alter his choice of partner. This entails that even when $V_{i,j} = V$, the shape of each non-empty component in SNN remains a branching or B_i network if UPR is satisfied. Moreover, network disconnectedness and singletons may still exist in SNN due to cost heterogeneity. Naturally, if $c_{i,j}$ is sufficiently high, the benefits of i in accessing the component of j may not be sufficient to cover the link formation cost borne by i. This causes SNN to have multiple non-empty components.

The last observation is that UPR is a sufficient but not necessary condition for every non-empty component of SNN being a branching or B_i. Indeed, if a set of agents can be predicted to be in the same component and a common best partner within this set of agents exists, Lemma 4 predicts that the component is a branching or B_i network. Example 5 below accentuates this point.

Example 4. *Let the payoff satisfies Equation 2. If $c_{i,j} < V_{i,j}$ for all distinct pairs i, j and i^* is a unique Common Best Partner among all agents, then an SNN is a fully connected branching or B_{i^*} network. This shape of SNN is similar to that of Proposition 1 in Billand et al. (2011), which assumes exclusive partner heterogeneity on both value and link formation cost.*

The proof of this example is trivial and is omitted. Intuitively, $c_{i,j} < V_{i,j}$ guarantees that an agent i is better off forming a link with another agent j regardless to the identity of j, if the two agents are not connected. This in turn guarantees that all agents belong to the same component. The existence of a common best partner among all agents further predicts that SNN is a branching or B_{i^*}, which is a consequence of Lemma 4.

5. CONCLUDING REMARKS

This paper establishes a proposition that aims to understand the interaction between exclusive player heterogeneity and exclusive partner heterogeneity and how this interaction influences the properties of SNN in the two-way flow model with no decay, which was orginally studied by Bala and Goyal (2000). The main conclusions are:

1. Even if $c_{i,j} \neq c_i$ and $c_{i,j} \neq c_j$ so that two-way heterogeneity is assumed, a non-empty component of SNN can be characterized. Given that all agents, except one agent, in the component agree on who the link receiver is that incurs the lowest link formation cost, this component of SNN is a branching or B_i network. Consequently, in this paper the prediction of the shape of a non-empty component in SNN is similar to that of Billand et al. (2011), which assumes exclusive partner heterogeneity. This conclusion is formally stated as Lemma 4.

2. As a result of the conclusion above, if all agents in the network agree on which agent is at least as good a partner, as measured by a lower link formation cost, than which, then it can be concluded that every non-empty component in this SNN is a branching or B_i network. This restriction is called Uniform Partner Ranking, and the prediction of SNN is formally stated as Proposition 1 in this paper.

3. Finally, value heterogeneity plays a relatively less important role in predicting the shape of each component in SNN than cost heterogeneity does. Intuitively, when an agent i decides whether to form a link in order to access a component, he weights the benefits of accessing the component against his link formation cost with the lowest-cost partner in the component. Therefore, it is concluded that value heterogeneity does not alter his choice of partner.

Naturally, a question that remains is how the shape and properties of SNN may be predicted in the absence of Uniform Partner Ranking. This becomes a potential research question to explore.

APPENDIX

A. PROOFS OF LEMMAS

Proof of Lemma 1. Suppose not. Consider a cycle in a non-minimal component. Observe that all agents in it have at least two chains through which they observe one another. In this cycle, consider an agent who establishes at least one link. If he removes the link, the component remains unbroken so that he still observes all other agents in the component. Thus, he is better off removing the link since it reduces his link formation cost, a contradiction. □

Proof of Lemma 3. In the proof of the lemma below, we adopt the following notion: j is called a best partner of i in $D_j\left(g_{i,j}\right)$ if $c_{i,j} \leq c_{i,k}$ for every $k \neq j \in N\left(D_j\left(g_{i,j}\right)\right)$.

Let k be an agent that receives more than one link. Let j_1 be an agent that accesses k. By Lemma 2 we know that k is the best partner of j_1 in $D_k\left(g_{j_1,k}\right)$. Let j_2 be another agent that accesses k. By Lemma 2 we know that k is also the best partner of j_2 in $D_k\left(g_{j_2,k}\right)$. Observe that $N\left(D_k\left(g_{j_2,k}\right)\right) \cup N\left(D_k\left(g_{j_1,k}\right)\right) = N\left(g'\right)$. Thus, k is a common best partner among all agents in the component if k receives more than one link.

We now prove that k is a unique common best partner among all agents in the component if k receives more than one link. Suppose not, let k' be another common best partner. Without loss of generality let us assume that k' is contained in $D_k\left(g_{j_2,k}\right)$. Consequently, j_2 is indifferent between accessing k and k'. This entails that this network is not SNN, a contradiction.

Finally, we prove that k is the only agent that receives more than one link. Suppose not, let k' be another agent that receives more than one link. This follows that k' is a common best partner. But this is contradictory to the above proof that k is a unique common best partner. □

Proof of Lemma 4. By Lemma 3, we know that a component of SNN has at most one agent that receives more than one link. Consequently, to complete this proof it suffices to show that: 1) if a component contains no agent that receives more than one link, this component is a branching and, 2) if a component contains exactly one agent that receives more than one link, this component is a B_{i^*} network.

Proof of the first part: if a component contains no agent that receives more than one link, then this component is a branching. We prove by contradiction.

Suppose that the component is not a branching. Recall that in a branching network an agent that receives no link is unique. Consequently, our presupposition that the component is not a branching results in two cases: (1) an agent that receives no link does not exist, and (2) an agent that receives no link exist but is not unique.

For the first case, since an agent that receives no link does not exist, it follows that every agent receives exactly one link. Consider a terminal agent i'. By our presupposition i' receives a link from an agent. Let this agent be $j+1$. Similarly, $j+1$ receives a link from another agent. Thus, this logic repeats infinitely. It follows that this network has an infinite amount of agents. A contradiction.

For the second case in which an agent that receives no link exists but is not unique, consider agents x and y who receive no link. Since x and y are in the same component, there is a chain between x and y. Let the sequence of agents in this chain be $x, j_1, j_2, j_3, j_4, ..., j_K, y$ respectively. Since x receives no links, it is the case that x accesses j_1. Since it is assumed that every agent receives at most one link, it is the case that j_1 accesses j_2, j_2 accesses j_3, ..., and j_K accesses y, which is a contradiction to our presupposition that y receives no link.

Proof of the second part: if a component contains an agent that receives more than one link, then this component is a B_{i^*} network. Let this component be g'. By Lemma 3, we know that in such a component there is only one agent that receives more than one link, and this agent is i^*, a unique common best partner. Therefore, what remains to be proven is that $I_{i^*}(g')$ is a contrabasis of this component. To do so, our strategy of proof consists of showing that: (i) if $j \in I_{i^*}(g')$, $I_{i^*}(g')\backslash\{j\}$ is not a contrabasis, and (ii) for each agent l, $l \notin I_{i^*}(g')$, there exists a path from l to j where $j \in I_{i^*}(g')$.

For (i), to show that $I_{i^*}(g')\backslash\{j\}$ is not a contrabasis, it suffices to show that there exists no path from $j \in I_{i^*}(g')$ to j' for any $j' \in I_{i^*}(g')\backslash\{j\}$, which is proven in what follows. Consider the chain between j' and j. Observe that this chain, which is $\{j, i^*, j'\}$, is unique since g' is minimally connected. Moreover, this chain is not a path since both j and j' access i^*. Therefore, there exists no path from $j \in I_{i^*}(g')$ to j' for any $j' \in I_{i^*}(g')\backslash\{j\}$. This in turn guarantees that $I_{i^*}(g')\backslash\{j\}$ is not a contrabasis.

For (ii), which is to show that for each agent l, $l \notin I_{i^*}(g')$, there exists a path from l to j where $j \in I_{i^*}(g')$, we divide our proofs into two cases: (a) l is such that $l \in D_k(\bar{g}_{j,k})$ where $\bar{g}_{j,k} = 1$, and (b) l is such that $l \notin D_k(\bar{g}_{j,k})$.

For the case (a), consider first an agent l that is a terminal agent. We aim to show that the chain between l and j is a path from l to j. That is, the sequence $\{j_0, ..., j_m\}$ is such that $j_0 = j$, $j_m = l$ and $\bar{g}_{j_{p-1},j_p} = 1$ and $g_{j_{p-1},j_p} = 1$ for any $p = 1, .., m$. To prove by contradiction, let there exist an agent j_p such that $\bar{g}_{j_{p-1},j_p} = 1$ but $g_{j_{p-1},j_p} = 0$, which entails that $g_{j_p,j_{p-1}} = 1$. Consider the link $g_{j_p,j_{p-1}}$. Observe that $D_{j_{p-1}}\left(g_{j_p,j_{p-1}}\right)$ contains both agent j_{p-1} and i^*. Thus, j_p has a strictly positive deviation by accessing i^* instead of j_{p-1}. A contradiction.

We now turn to prove the case in which l is not a terminal agent. Observe that if l is not a terminal agent, l is contained in a chain between l' to j, where l' is a terminal agent and $j \in I_{i^*}(g)$. Since this chain, $\{j_0, ..., l, .., j_m\}$ where $j_0 = j$ and $j_m = l'$, is a path, it follows that $\{j_0, ..., l\}$ is also a path. Hence, every chain between l and j is a path from l to j.

Next, let us prove case (b), which is to show that there exists a path from $l \notin D_k\left(\bar{g}_{j,k}\right)$ to $j \in I_{i^*}(g')$. If $l = i^*$, clearly, there is a path from i^* to j since j accesses i^*. Thus, if a path from $l \neq i^*$ to i^* is proven to exist, the fact that j accesses i^* guarantees that a path from l to j ,via i^*, exists.

We prove this claim using the same analogy found in the above proof of case (a). If l is a terminal agent, it holds true that every chain between i^* and l is a path from l to i^*. By virtue of minimal connectedness we know that a chain between i^* and l exists and is unique. This in turn guarantees that a path from l to i^* exists if l is a terminal agent. In the case that l is not a terminal agent, we know that l is contained in a path from l' to i^* where l' is a terminal agent. This in turn guarantees that a path from l to i^* exists. This completes the proof.

□

B. PROOF OF PROPOSITION 1

Proof. The first part: SNN consists of non-empty components that are branching or B_i network, if UPR is satisfied. Since UPR is satisfied, we know that all agents can be permuted into the set $\{i_0, i_1, ..., i_{n-1}\}$ such that i_x is at least as good a partner as i_y with respect to the set $I(N)$ for any $x < y$. Consequently, in every non-empty component of SNN there exists i^{x^*} such that $x^* \leq y$ for any i_y that is in that non-empty component. Naturally, i^{x^*} is i^*, a common best partner in the component. This fact, which guarantees that every non-empty component of SNN has a common best partner, together with Lemma 4 guarantee that every non-empty component is a branching or B_{i^*}.

The second part: If the payoff satisfies Equation 2, a network of which each non-empty component is a branching or B_i network can be supported as SNN by a pair of $\mathcal{V}$ and $\mathcal{C}$, where $\mathcal{C}$ satisfies UPR. We first introduce the following notations: an agent j is said to be an l-th neighbor of i if the chain between j and i consists of l links, the set of all l-th neighbor of i is denoted by $N^l(i;g)$.

Let there be $\bar{K}'$ components in the network, enumerate all components as $g^1, g^2, ..., g^J, ..., g^{\bar{K}'}$. Enumerate all agents as $i_0, i_1, ..., i_{n-1}$ with the restriction that agents $i_{|\sum_{j=1}^{k'-1} N(g^j)|}, ..., i_{|\sum_{j=1}^{k'} N(g^j)|-1}$ belong to the component $g^{k'}$.

We further impose the following restriction on how agents $i_{|\sum_{j=1}^{k'-1} N(g^j)|}$, $..., i_{|\sum_{j=1}^{k'} N(g^j)|-1}$ in $g^{k'}$ are enumerated. To ease the notational cumbersomeness, let $k'' = |\sum_{j=1}^{k'-1} N(g^j)|$ so that the sequence of agents above becomes $i_{k''}, ..., i_{k''+|N(g^{k'})|-1}$. Let $i_{k''}$ be the agent at which the component is rooted if it is a branching. Alternatively, let $i_{k''}$ be the point contrabasis if the component is a $B_{i_{k''}}$ network. By the definition of minimal component, we know that there is a unique chain between $i_{k''}$ and i_j for all $i_j \in N\left(g^{k'}\right)$ and $i_j \neq i_{k''}$. Let the longest chain between $i_{k''}$ and i_j consists of L links. It follows that the set $i_{k''} \cup N^1(i_{k''};g) \cup N^2(i_{k''};g) ... \cup N^L(i_{k''};g)$ contains all agents in $g^{k'}$. Let $L_1, ..., L_L$ be the number of agents in $N^1(k'';g), N^2(k'';g), ..., N^L(k'';g)$ respectively. Consequently, the sequence $i_{k''}, ..., i_{k''+|N(g^{k'})|-1}$ becomes $i_{k''}, ..., i_{k''+L_1+L_2+...L_L}$. Onwards, we impose the restriction that $i_{k''+L_1+...L_{j-1}+1}, ..., i_{k''+L_1+...L_j}$ are j^{th} neighbors of $i_{k''}$.

We now turn to identify the cost structure and value structure. Let $V_{i_x i_y} = \bar{V}$ if i_x, i_y belongs to the same component. If i_x, i_y do not belong to the same component, let $V_{i_x i_y} = \underline{V}$, where $\underline{V}$ is such that $\underline{V}|N(g)| < c_{i_x,i_y}$ and $c_{i_x,i_y} = \bar{V} - (|N(g)| - y)\epsilon + x\epsilon$ where ϵ is sufficiently small so that $c_{i_x,i_y} > 0$ for all pairs of i_x, i_y. Observe that c_{i_x,i_y} is increasing in y, which entails that $c_{i_x,i_y} < c_{i_x,i_{y+1}}$. Hence, i_y is at least as good as a partner as i_{y+1} with respect to the set of all agents for any $y = 0, ..., n-2$. Consequently, the cost structure $\mathcal{C}$ satisfies UPR condition. Observe further that, since $|N(g)| < |N\left(g^{k'}\right)|$, the fact that $\underline{V}|N(g)| < c_{i_x,i_y}$ guarantees that $\underline{V}|N\left(g^{k'}\right)| < c_{i_x,i_y}$ where $g^{k'}$ is the component that i_y belongs. This in turn guarantees that i_x does not have a positive deviation by accessing

i_y, if i_y does not belong to the same component as i_x.

Finally, we prove that if i_x, i_y belong to the same component, then all links that i_x forms constitute his unique best response. Consider the agents $i_{k''}, \ldots, i_{k''+|N(g^{k'})|-1}$, which belong to the component $g^{k'}$. Recall that we put on the restriction that $i_{k''+L_1+..L_{j-1}+1}, \ldots, i_{k''+L_1+..L_j}$ are j^{th} neighbors of $i_{k''}$. Thus, if i_x accesses i_y and i_x is a j-th neighbor of $i_{k''}$, we know that i_y is a $(j+1)$-th neighbor of $i_{k''}$. Hence, $x < y$. Similarly, if i_x accesses i_y and i_y accesses i_z, we know that $x < y < z$. This fact and the assumption that $c_{i_x,i_y} = \bar{V} - (|N(g)| - y)\epsilon + x\epsilon$ guarantee that, if i_x accesses i_y then $c_{i_x,i_y} < c_{i_x,i_{y'}}$ for all $i'_y \in N\left(D_{i_y}\left(g_{i_x i_y}\right)\right)$ and $i_{y'} \neq i_y$. Moreover, since the benefits that i_x receives from accessing i_y is $\bar{V}|N\left(D_{i_y}\left(g_{i_x,i_y}\right)\right)| > c_{i_x,i_y} = \bar{V} - (|N(g)| - y)\epsilon + x\epsilon$, the benefit from each link that he forms exceeds the corresponding link formation cost. It follows that all links that i_x forms constitute his unique best response.

□

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