Journal of Economic Theory and Econometrics, Vol. 26, No. 3, Sep. 2015, 35–62

Modeling Non-Normally Distributed Stock Portfolio Returns and Applications to Risk Management

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Abstract We utilize the copula function methodology to separate out the components which describe the marginal behavior of the return processes and the dependence structure between the random variables from the joint density. In order to reflect the non-ellipticity of the joint distribution and heavy tails in the extreme quantile of the marginal distributions of asset returns, we use the generalized Pareto distribution (GPD) as the margins and a variety of parametric copula functions along with a nonparametric copula function in the analysis. We select the optimal copulas from a variety of non-nested copulas based on the model selection criteria. In calculating the risk measures, we assume that the returns are jointly distributed to the parametric copulas as well as to the empirical copula. We then compare the result with that from the bivariate normal distribution. The results show that the VaR and ES computed from the copula function which takes the complicated and possibly nonlinear dependence structure into account performs better than the one based on the linear correlation-based normality assumption.

Keywords fat-tail behavior, Value-at-Risk, expected shortfall, generalized Pareto distribution, extreme value theory, copula function

JEL Classification G10

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Received September 2, 2014, Revised September 16, 2015, Accepted September 24, 2015

1. INTRODUCTION

Modeling and predicting financial asset returns have been important issues in empirical finance. A voluminous literature has focused on linear predictability through temporal and/or intertemporal linear relationship between variables. The modeling of the joint distribution of asset returns among other considerations is crucial in performing these tasks. Although the univariate and multivariate normality has been a critical assumption upon which the portfolio theory has been built, it is well documented that the normality assumption has been proved to be at odds by the data. Correlation-based approach assumes that scatterplot of multivariate Gaussian random variables is elliptical. If the ellipticity is met, tail independence has to be established except the case where the linear correlation coefficient is one.

However, we are now able to model general dynamic dependence in a stationary Markov chain in line with the development of copula function methods in finance. Copula function methodology is a way of separating the dependence structure and the marginal behavior from a joint density function of multivariate random variables. Risk measurement and risk management are the most frequent applications of copula methodology. Chen and Fan (2006) develop a method of combining parametric copulas with unspecified marginal distributions to obtain a copula-based semiparametric model and estimate the conditional quantile of portfolio of assets such as the value-at-risk (VaR). Embrechts et al. (2003b) construct optimal bounds for the VaR under various dependence structures based on the copula theory and show that the nonadditivity of the risk measure is attributed to the different dependence structures between the component random variables.

A group of research has been devoted to measure the temporal dependence between different classes of financial assets or between international financial markets and to assess implications for portfolio management decisions. Considering the fact that a traditional portfolio management decision is based on the linear correlation among the component assets, time-varying and/or asymmetric temporal dependence may have important implications for portfolio management decisions. Although we cannot exhaustively review the extant literature, we summarize those attempts as follows. Firstly, asymmetric dependence structure has been explored in establishing the dependence structure between the random variables from the joint density. Patton (2004) explores the economic and statistical significance of non-elliptical return distribution or the presence of asymmetric dependence of asset returns for asset allocation decisions via the copula theory. Patton (2006b) models the asymmetric dependence structure between the two exchange rates, the Deutsche mark and the Japanese yen against the U.S. dollar. In constructing joint distribution functions, he focuses on the possibility of misspecifying the marginal distributions via a variety of diagnostic tests. He then proposes the symmetrized Joe-Clayton copula to measure the asymmetric dependence structure of the data. His conclusion is that the model explaining skewness and asymmetric dependence outperforms the benchmark which assumes the bivariate normal distribution in the out-of-sample. Secondly, non-normality specification proves to be statistically pertinent in some studies. Chen et al. (2004) develop two statistical tests for the null hypotheses of the Gaussian copula dependence and of the Student's t copula. When they apply the tests to the multivariate equity returns and exchange rate returns data, they find statistically significant evidence against the Guassian copula but not against the Student's t copula. The results suggest an important implication for our study that we must take the non-normality structure of the tail area dependence into account in the management of the portfolio and the risk. Thirdly, non-linearity plays an important role in explaining the dependence structure between the random variables from the joint density in the following studies. Bouye et al. (2002) investigate a copula based empirical method to measure the nonlinear dynamic dependence structure between non-Gaussian bivariate random variables and apply the methodology to the financial return predictability. Bouye and Salmon (2009) develop the copula quantile regression method to measure the tail area dependence using exchange rate series and use the results to show that Forex markets are efficient in the sense that the exchange rates, either between rates or within rates, are dynamically independent. He uses the Kimeldorf-Sampson, the Joe-Clayton and the BB3 copula in estimating the copula-based quantile regression along with the Gaussian copula as the benchmark.

This paper aims to quantify the tail area dependence and use this information to evaluate the tail related risk measures from the semi-parametric estimates of the marginal distribution and the copula function. The aims of this paper are two folds. Firstly, the copula function methodology is briefly introduced and is utilized to separate the parts which describe the marginal behavior of the individual stock returns and the dependence structure between the bivariate stock returns from the joint density of a stationary Markov chain process. We then extend our discussion to the nonlinear specification of dependence structures by using non-Gaussian copulas which as far as we know is an underdeveloped area of research. During the process, we do not constrain ourselves to a specific parametric copula a priori. We will assume a variety of copula functions along with the benchmark Gaussian copula. Secondly, we then proceed to evaluating market risk using the VaR and the expected shortfall (ES). In calculating the risk measures, we assume that the two stock returns are jointly distributed to the parametric copulas as well as to the empirical copula. We then compare the results with that from the bivariate normal distribution. The results show that the VaR calculated from the copula function which takes the complicated dependence structure into account performs better than the one based on the linear correlation-based normality assumption.

Our contributions to these findings are threefold. Firstly, we investigate a variety of dependence measures between the risky assets and the related risk measures under the non-normality and the non-linearity assumption. In order to reflect the non-ellipticity of the joint distribution and heavy tails in the extreme quantile of the marginal distributions of asset returns, we use the generalized Pareto distribution (GPD) as the margins and the copula functions in the analysis. Secondly, we approximate the conditional distribution of the bivariate random variables using conditional copulas. Further, we compute the portfolio VaR based on the copulas. We select the optimal copulas from a variety of non-nested copulas based on the model selection criteria. Thirdly, we use a semi-parametric method for estimating the market risk from an estimated copula functions. We combine a parametric assessment of the VaR with the fitted tail distribution to form a semi-parametric evaluation of the VaR. By sampling from the tail of the distribution, the level of statistical precision is elevated in evaluating the VaR.

The rest of the paper is organized as follows. We proceed with utilizing the copula function methodology to separate out the components which describe the marginal behavior of the individual returns and the dependence structure between the bivaraite returns from the joint density of the portfolio return process. In order to reflect heavy tails in the extreme quantile of the marginal distributions of asset returns, we use the GPD as the margins and a variety of parametric copula functions along with a nonparametric copula function in the analysis. We select the optimal copulas from a variety of non-nested copulas based on the model selection criteria and fit copulas to the data in section 2. We then use this information to evaluate the tail related risk measures such as the VaR and ES from the semi-parametric estimates of the marginal distributions and the copula function in section 3. Section 4 concludes the discussion.

Table 1: Descriptive statistics

	Microsoft	Boeing
A. Returns		
Mean	0.001	0.000
Standard Deviation	0.023	0.019
Skewness	-0.460	-0.296
Kurtosis	17.127	9.891
Minimum	-0.361	-0.194
Maximum	0.189	0.144
ARCH(20) LM test	557.18 (0.00)	460.75 (0.00)
Q(20) on the rate changes	47.89 (0.00)	37.46 (0.00)
Jarque-Bera test	38193.85 (0.00)	10557.53 (0.00)
B. Squared returns		
Q(20) on the squared rate changes	1118.41 (0.00)	948.73 (0.00)

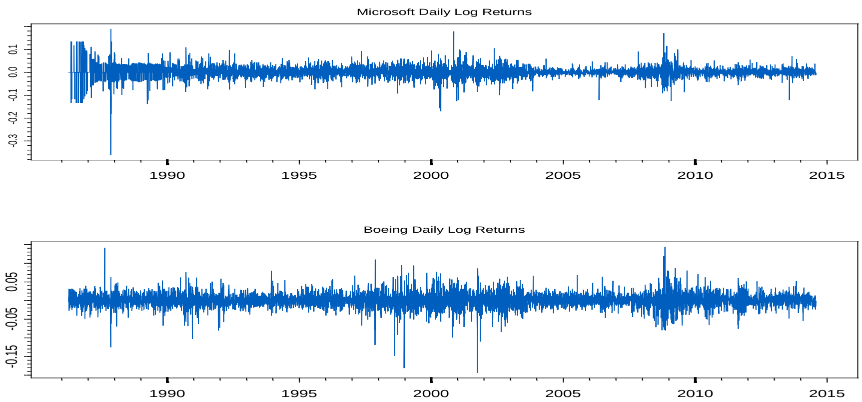
Note: The ARCH(20) test reports the Engle (1982) Lagrange multiplier test for ARCH(q) effects for q lags. The entries in the parentheses are the p-values. The Q(20) stands for the modified Ljung-Box Q-statistic for up to twentieth-order serial correlation in the stock returns and the squared stock returns.

2. METHODOLOGY AND ESTIMATION

2.1. MARGINAL MODEL

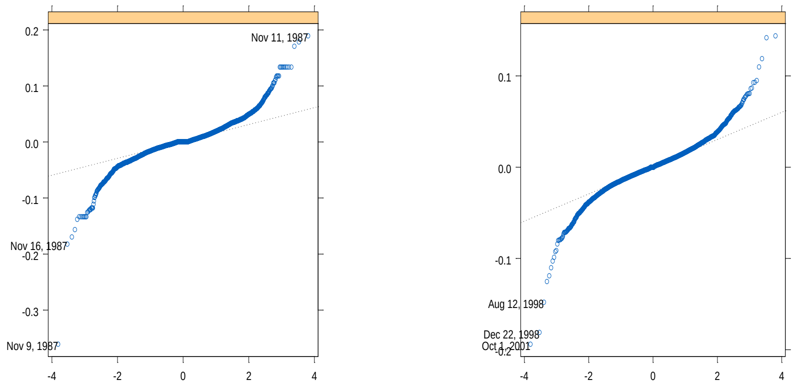
We use the daily log returns on the Microsoft and Boeing stock over the period from March 13, 1986 to July 31, 2014, yielding 7,157 observations. Figure 1 shows the daily log returns on the two stocks, the Microsoft and Boeing. To test whether the returns data come from the standard normal distribution, we take the ordered sample and plot against the standard normal distribution as the reference. Since the qq-plots curve down at the left and up at the right as we observe in Figure 2, the marginal distribution of the returns should reflect the non-normal fat-tailed behavior. The descriptive statistics of the returns in Table 1 also confirm the assertion. The Jarque-Bera test statistic for the Microsoft returns is 38,193.85 (p-value=0.00) and for the Boeing returns is 10,557.53 (p-value=0.00), and the null hypotheses that the individual returns are normally distributed are strongly rejected.

Figure 1: Daily log returns on the Microsoft and Boeing



Note: The figures show the daily log returns on the Microsoft and Boeing stock over the period from March 13, 1986 to July 31, 2014.

Figure 2: The normal quantile plots for the log returns

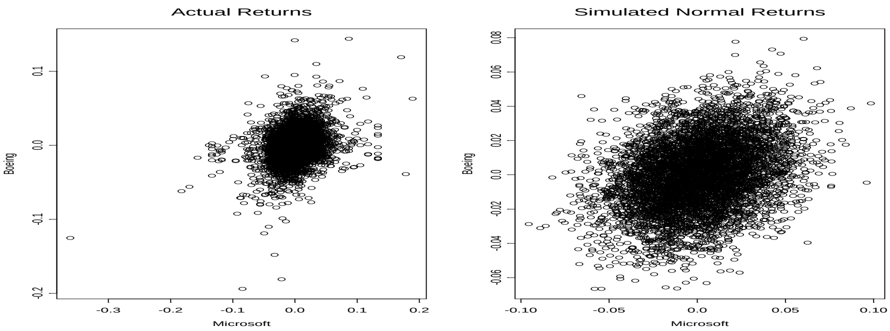


Note: The figures show the qq-plots of the ordered returns against the standard normal distribution as the reference distribution. Since the distributions are heavy-tailed, the plots curve down at the left and up at the right.

In applying the copula method to the modeling of the conditional bivariate distribution of the daily log returns on the Microsoft and Being, we need to specify the models for the marginal models of each return and the model for the conditional copula. The appropriate marginal models for the stock returns should reflect heavy-tailed behavior of the processes. As Chen et al. (2004) and Patton (2006b) shows, however, the degrees of freedom parameters of the two marginal distributions and the copula should be the same to use a bivariate Student's t-distribution. However, the degrees of freedom parameter of the Microsoft return is 4.52 with standard error of 0.24 and the degrees of freedom parameter of the Boeing returns is 5.98 with standard error of 0.35. Since the degrees of freedom parameters of the two marginal distributions are significantly different from each other, the use of the Student's t copula is inappropriate. In addition to this restriction, although the Student's t copula can accommodate the tail area dependence, it does not allow for asymmetric tail dependence. Also, in specifying the joint density of the returns, it seems to be inappropriate to use the bivariate normal distribution when the two margins deviate from the Gaussian normality.

To prove that this is the case, we generate bivariate return processes from the normal distribution calibrated to the data. We compare the generated data with the actual returns in Figure 3. The left panel of the figure depicts the scatter plot of the returns from the actual data and the right panel shows the generated data from the bivariate normal distribution calibrated to the actual returns. As can be seen from the figure, the artificially generated data from the calibrated bivariate normal distribution do not seem to show joint negative and positive extremes more often than the actual returns. This leads to the observation that the simulated data in the first and third quadrant are more compactly distributed than the actual returns. That is, the simulated data do not fully represent the tail dependence structure observed in the actual data. As is observed in the right panel of Figure 3, the normally distributed margins and bivariate normal distribution are inappropriate to model the extreme returns on both tails of the distribution and the positive dependence in the regions of the distribution. If we model the joint distribution as the bivariate normal distribution calibrated to the data, then the artificially generated data from this specification replicate the actual data in the middle of the distribution, but do not seem to fit the actual data which represent the tail dependence structure. We assume the existence of a positive dependence between the bivariate random variables when large values of a random variable are more likely to be followed by large values of the other random variable than when the bivariate random variables are independent. A visual inspection of the plot shows a denser cloud on extreme returns of the main diagonal. This indicates the stronger tail area dependence in the actual returns than in the simulated bivariate normal returns.

Figure 3: Plot of actual returns vs. simulated returns from the calibrated bivariate normal distribution

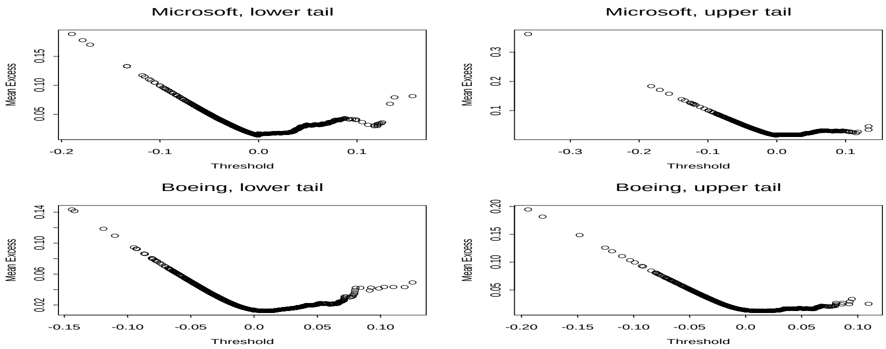


Note: The left panel of the figure depicts the scatter plot of the returns from the actual data and the right panel shows the generated data from the bivariate normal distribution calibrated to the actual returns. The artificially generated returns from the calibrated bivariate normal distribution do not seem to show joint negative and positive extremes more often than the actual returns.

We have shown that the normal distribution and the Student's t distribution are inappropriate for the marginal models due to the ellipticity of the joint distribution and heavy tails in the extreme quantiles of the marginal distributions of asset returns. As an alternative to the approaches in the extant literature, we use the generalized Pareto distribution (GPD) as the margins due to Zivot and Wang (2006) in the copula-based analysis.

We take the extreme value theory (EVT) based approach and use the generalized Pareto distribution (GPD) as the marginal model. The EVT offers a fully parametric method for estimating the tails of the marginal distribution. The most complicated thing in estimating the GPD parameters via the maximum likelihood method is to separate out the tail area from the center of the marginal distribution by selecting the optimal threshold. There are two approaches to choosing a high threshold. Danielsson et al. (2001), Danielsson and de Vries (1998), Goldie and Smith (1987) and Hall (1990) use a subsample bootstrap procedure for determining the optimal threshold value. Alternatively, McNeil and Saladin (1997), McNeil and Frey (2000) and Zivot and Wang (2006) use the sample mean excess function plot. Theoretically, it would be ideal if we could fit the GPD marginal model to the excesses over a threshold which are included in the tail area of the distribution and not pertained in the center of the distribution. We can minimize the bias of the GPD parameter estimates by selecting a high threshold. Otherwise, we can reduce the variance of the parameter estimates by including a sufficient number of exceedances of the returns over a threshold. This is where our subjective judgment kicks in. Following the sample mean excess function plot procedure, we choose a threshold and fit the GPD marginal model to the excesses over a threshold. When we locate a threshold value, we trade off the bias against the efficiency of the GPD parameter estimates.

Figure 4: Sample mean excess function plots



Note: The figures present the sample mean excess function plot against a variety of threshold values. We choose the threshold value where the slope of the curve changes from negative to positive. The excesses over the selected threshold are fitted to obtain the GPD marginal models. In our sample of data, the sample mean excess function plot changes the slope of the curve from negative to positive at the upper threshold value of 0.023 and 0.0165 for the Microsoft and Boeing, respectively.

Figure 4 presents the sample mean excess function plot against a variety of threshold values. As suggested by the extant literature, we locate the threshold value to obtain the excesses over a threshold with which the GPD marginal models are estimated. For example, the sample mean excess function plot changes the slope of the curve from negative to positive at the upper threshold value of 0.023 and 0.0165 for the Microsoft and Boeing, respectively. Thus, the observations above the upper tail threshold value are fitted to the generalized Pareto distribution. For the Microsoft, we include 14.21% with the upper threshold of 0.023, and 14.11% of the total of 7,156 observations with the lower threshold of -0.020. For the Boeing, we include 15.99% with the upper threshold of 0.0165, and 14.62% of the total of 7,156 observations with the lower threshold of -0.0165. The tail shape parameters on both side lobes are estimated via the MLE procedure and reported in Tables 2 and 3. Since the estimates of the tail shape parameter ξ are greater than zero in both returns, the tails of the marginal models of the returns are fat-tailed.

Table 2: GPD fit to the daily log returns on Microsoft: $G_{\xi,\beta(u)}(y) = 1-(1+\xi y/\beta(u))^{-1/\xi}\ for\ \xi \neq 0$ or $G_{\xi,\beta(u)}(y) = 1-exp(-y/\beta(y))\ for\ \xi = 0$

	Upper Tail Estimate		
	Value	S.E.	t-ratio
ξ	0.1193	0.0343	3.48
β	0.0152	0.0007	20.76
	Lower Tail Estimate		
	Value	S.E.	t-ratio
ξ	0.1739	0.0357	4.87
β	0.0146	0.0007	20.38

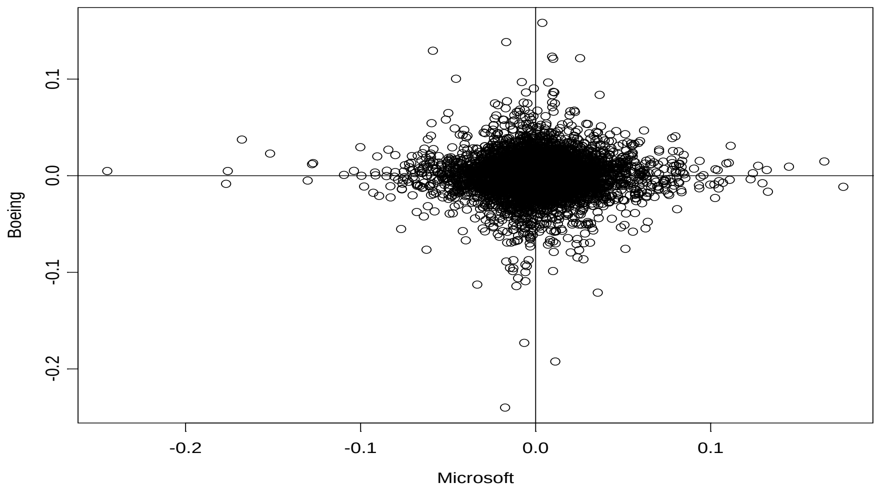
Note: The tail shape parameters on both side lobes are estimated via the MLE procedure and reported in the table. Since the estimates of the tail shape parameter ξ are greater than zero on both side, the tails of the marginal models of the returns are fat-tailed.

Table 3: GPD fit to the daily log returns on Boeing

	Upper Tail Estimate		
	Value	S.E.	t-ratio
ξ	0.1001	0.0353	2.83
β	0.0112	0.0005	20.84
	Lower Tail Estimate		
	Value	S.E.	t-ratio
ξ	0.1838	0.0395	4.65
β	0.0109	0.0006	19.40

Note: For the Boeing returns, the estimates of the tail shape parameter ξ are greater than zero on both side. Thus, the tails of the marginal models of the returns are fat-tailed.

Figure 5: Calibration result of the semiparametric GPD model



Note: The figure depicts the pairs of simulated returns from the estimated GPD margins. Although the two GPD margins generate the extreme values in the tail area, the co-movements in the extreme quantile of the joint distribution fail to replicate the extreme returns in tandem and the dependence structure of the bivariate returns.

We check for adequacy of the two GPD marginal models in describing both the fat-tailed behavior and the tail area dependence. We artificially generate the pairs of returns from the estimated GPD margins. Figure 5 depicts the pairs of simulated returns. The two GPD margins successfully generate the extreme values in the tail area, however, do not exhibit the co-movements in the extreme quantile of the joint distribution as closely as we expect. That is, we fail to replicate the extreme returns in tandem and the dependence structure of the bivariate returns.

Our findings suggest that the tail dependence structure in the bivariate returns observed in the left panel of Figure 3 should be included in the bivariate modeling of the returns. Our strategy is to combine the two GPD marginal models to describe fat-tailed behavior and the copula model to specify the dependence structure in the tail area. This is the subject which we now turn to in the next section.

2.2. COPULA MODEL

A comprehensive treatment of the copula methodology can be found in Joe (1997), Cherubini et al. (2004), and Nelson (2006). According to the Sklar′s theorem, copulas are joint distribution functions of standard uniform marginal distributions.

Theorem (Sklar's theorem) Let X and Y be two random variables on a probability space with their joint and marginal distribution functions, F_{XY}, F_X and F_Y. A copula $C(F_X(x), F_Y(y))$ is a multivariate distribution that decomposes a joint distribution function F_{XY} into marginal distributions F_X and F_Y which describe the marginal behavior of X and Y and the dependence structure between X and Y. That is, $F_{XY} = C(F_X(x), F_Y(y))$. Conversely, if $F_{XY}(x,y)$ is a joint distribution function with continuous marginal distributions F_X and F_Y, there exists a unique copula $C(F_X(x), F_Y(y))$. We can also denote this relationship in terms of a bivariate density function. That is, $f_{XY} = f_X(x) f_Y(y) c(F_X(x), F_Y(y))$. The density function can be derived to be used for maximum likelihood estimation provided that two marginal distributions F_X and F_Y are differentiable and F_{XY} and $C(F_X(x), F_Y(y))$ are twice differentiable.

$$\begin{aligned} f_{XY}(x,y) &= \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} = \frac{\partial F_X(x)}{\partial x} \frac{\partial F_Y(y)}{\partial y} \frac{\partial^2 C(F_X(x), F_Y(y))}{\partial u \partial v} \\ &= f_X(x) f_Y(y) c(u,v),\ \forall (x,y) \in R \times R \end{aligned} \tag{1}$$

where we use $F_{XY} = C(F_X(x), F_Y(y)) = C(u,v),\ u = F_X(x)$ and $v = F_Y(y)$. So, the maximum likelihood function is set up as $ln f_{XY}(x,y) = ln f_X(x) + ln f_Y(y) + ln\ c(F_X(x), F_Y(y))$. The U and V are the probability integral transforms of X and Y and are uniformly distributed marginal distributions on [0,1].

In order to construct the conditional distribution of asset returns where the temporal dependence is specified by the relevant copula, we need to identify the parametric copulas. Bouye et al. (2002) show that the Bayes theorem provides a simple way of deriving the lowest dimensional copula which captures the temporal association of the time series. A wide range of copula models are available in the extant literature. We choose the following copulas because they are frequently used in the literature. We do not provide the detailed specifications of all copula models we estimate to save space. The interested readers are referred to Joe (1997), Cherubini et al. (2004), Nelson (2006) and Zivot and Wang (2006).

The Gaussian copula with the correlation parameter $0 \le \delta \le 1$ can be denoted as:

$$C(u,v) = \Phi_\delta(\Phi^{-1}(u), \Phi^{-1}(v)), 0 \leq u,\ v \leq 1 \tag{2}$$

where $\Phi(\cdot)$ is the standard normal distribution function, and $\Phi_\delta(\cdot)$ is the bivariate standard normal distribution function. The Gaussian copula is symmetric and independent in the extreme tails of the distribution. The upper and lower tail dependence coefficients are the same as zero. Therefore, the Gaussian copula may be inappropriate if the bivariate return processes exhibit nonlinear asymmetric dependence and no tail dependence.

The Kimeldorf-Sampson (Clayton) copula is a class of Archimedean copulas and represented as $C(u,v;\theta) = [u^{-\theta} + v^{-\theta} - 1]^{-\frac{1}{\theta}}$, where $\theta > 0$ and has the lower tail dependence parameter $\tau^L = 2^{-\frac{1}{\theta}}$ and the upper tail dependence parameter $\tau^U = 0$. The parameter θ denotes the measure of the association between the two margins. Due to the presence of the asymmetric lower tail dependence, the Kimeldorf-Sampson copula is used when the joint negative extreme returns are clustered in the time series scatter plot.

The Frank copula is a simple Archimedean copula and has parameters describing both positive (when $\theta \in (0,1)$) and negative (when $\theta > 1$) dependence structures:

$$C(u,v;\theta) = -\frac{1}{\theta} ln(1 + \frac{(exp(-\theta u) - 1)(exp(-\theta v) - 1)}{exp(-\theta) - 1}. \tag{3}$$

The Kimeldorf-Sampson and the Frank copula of the Archimedean copulas listed above have a single parameter θ which measures the dependence structure. The BB3 copula is also a type of Archimedean copula with lower (δ) and upper (θ) tail dependence parameters:

$$\begin{aligned} C_{\delta,\theta}(u,v;\theta) &= exp(1 - [\delta^{-1} ln(exp(\delta u^{-\theta}) + exp(\delta v^{-\theta}) - 1)]^{\frac{1}{\theta}}), \\ & where\ \theta \geq 1,\ \delta > 0 \end{aligned} \tag{4}$$

The Joe-Clayton (BB7) copula is a specific Archimedean copula with lower (δ) and upper (θ) tail dependence parameters:

$$\begin{aligned} C_{\delta,\theta}(u,v;\theta) &= 1 - (1 - [(1 - (1-u)^{\theta})^{-\delta} + (1 - (1-v)^{\theta})^{-\delta} - 1]^{-\frac{1}{\delta}})^{\frac{1}{\theta}}, \\ & where\ \theta \geq 1,\ \delta > 0 \end{aligned} \tag{5}$$

Although the correlation coefficient estimate of the Gaussian copula subsume the overall dependence structure information, the Joe-Clayton copula is convenient in identifying the tail area dependence measures. The parametric estimates of the lower (τ_L) and upper (τ_U) tail dependence measures are obtained by $\tau_L = 2^{-\frac{1}{\delta}}$ and $\tau_U = 2 - 2^{\frac{1}{\theta}}$ respectively. The lower (λ_L) and upper (λ_U) tail dependence measures are representative of the asymptotic behavior of the copula in the left and right tails.

For the Archimedean copulas, Bouye et al. (2002) develop a copula based empirical method to measure the nonlinear dynamic dependence structure between non-Gaussian bivariate random variables. They show that the measures of association, the Kendall's τ and the Spearman's ρ estimate the difference between the probabilities of concordance and discordance of the bivariate random variables. As the non-parametric measures of concordance of the bivariate series, the Kendall's concordance coefficient τ is

$$\tau = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1 = 4E(C(U,V)) - 1 \tag{6}$$

and the Spearman's concordance coefficient is

$$\begin{aligned}\rho &= 12\int_0^1\int_0^1 C(u,v)dudv - 3 = 12\int_0^1\int_0^1 uvdC(u,v) - 3\\ &= 12E(UV) - 3 = \frac{E(UV)-1/4}{1/12} = \frac{E(UV)-E(U)E(V)}{\sqrt{Var(U)}\sqrt{Var(V)}}\end{aligned} \tag{7}$$

These two measures are different from the Pearson's product-moment correlation coefficient which is a representation of linear dependence and is directly related to the generator of the families of Archimedean copulas. Thus, for these families of the Archimedean copulas, the parameter of the copulas, θ and the concordance coefficient, τ represent the same dependence structure. If this is the case, the parameter estimates of the copula can be interpreted as the measures of association.

As an alternative to a wide range of parametric copulas, we fit the Deheuvels' empirical copula to the data and measure the sample version of dependence in a nonparametric way. The empirical copula methodology has the advantage of estimating copula function parameters and other related measures of the copula without assuming any functional form for the copula and/or the margins. Once estimates of empirical copulas are obtained, other estimates expressed in terms of copulas can be empirically computed. According to Nelson (2006), for example, the sample version of dependence measures such as the Spearman's ρ and Kendall's τ are obtained from the empirical copula.

In this paper, we employ a semiparametric estimation procedure of the empirical marginal distribution function and parametric/nonparametric estimate of the parameters of the copula function. Our estimation strategy is to use the GPD as a marginal model coupled with different parametric/nonparametric copula distribution functions. This enables us to appropriately model marginal behaviors such as skewness and heavy tails on both extremes and symmetric or asymmetric positive and negative tail dependence. The aim of the paper is to characterize the transition distribution and conditional quantile such as the VaR using the marginal distribution and the copula dependence parameter.

Similar to our estimation method, Chen and Fan (2006) propose to separate out dependence structure as parametric copulas and nonparametric marginal distributions which can be estimated as the rescaled empirical distribution or the kernel smoothed distribution. The copula-based semiparametric model is then applied to calculating the conditional risk measures of portfolio.

2.3. COPULA ESTIMATION

The joint density function of X and Y is $f_{XY}(x,y) = f_X(x)f_Y(y)c(F_X(x),F_Y(y))$, where

$$c(F_X(x),F_Y(y)) = \frac{\partial^2 F_{XY}(x,y)}{\partial F_X(x)\partial F_Y(y)} \tag{8}$$

If the marginal distributions are specified, then the likelihood function is:

$$l(\zeta) = \sum_{t=1}^{T} lnc(F_X(x),F_Y(y)) + \sum_{t=1}^{T} lnf_X(x) + \sum_{t=1}^{T} lnf_Y(y) \tag{9}$$

where ζ is the parameters of both the marginal distributions and the copula.

Then, the exact maximum likelihood method is used to estimate the parameters of the copula and the marginal distributions simultaneously:

$$\hat{\zeta}_{MLE} = argmax_{\zeta} l(\zeta) \tag{10}$$

Joe and Xu (1996) suggest using the inference functions for the margins (IFM), a two-step procedure to resolve the computational intensity of the exact maximum likelihood method in a high dimensional multivariate distribution.

Firstly, we estimate the following maximization to obtain the parameters of the univariate probability density functions:

$$\hat{\zeta}_1 = argmax_{\zeta_1} \sum_{t=1}^{T} lnf_X(x;\zeta_1) + \sum_{t=1}^{T} lnf_Y(y;\zeta_1). \tag{11}$$

Secondly, given the estimated parameters of the marginals ζ_1, we estimate the parameters of the copula density function ζ_2:

$$\hat{\zeta}_2 = argmax_{\zeta_2} \sum_{t=1}^{T} lnc(F_X(x), F_Y(y); \zeta_2, \hat{\zeta}_1). \tag{12}$$

The property of asymptotic normality under regular conditions and the efficiency of the IFM estimator are presented in Joe (1997), Cherubini et al. (2004) and Patton (2006a).

The optimal copula which describes the appropriate measure of dependence can be selected using the information criteria. Vandenhende and Lambert (2000) suggest choosing an optimal copula function based on the AIC from the MLE of a variety of copula functions with the same marginal distribution function. Vandenhende and Lambert (2005) expand their previously cited paper by introducing a flexible semiparametric Archimedean copula using a local linear combination of quantile functions in describing the dependence measure.

The two stock returns processes prove to be non-normal with excess kurtosis and negative skewness. We estimate a wide range of parametric copulas along with the empirical copula. We estimate the 16 parametric copula models such as the Gaussian, the Gumbel, the Joe-Clayton (BB7), the BB3 and the Frank copula etc. reported in Tables 4, 5 and 6. The Gaussian copula is used for the benchmark model due to the fact that the bivariate Gaussian copula results in the tail area independence and can be used for comparison with the case where there is tail area dependence in the bivariate return processes. Combining these with the GPD estimation of the marginal models would result in the class of semiparametric copula-based calculation of the VaR.

In the first step of the procedure, we fit the 16 bivariate copula models to the Microsoft and Boeing returns with the IFM estimator. The log-likelihood value and the information criteria of the copula models are reported in Table 4. From the log-likelihood estimation results, the BB1 copula is evaluated to maximize the log-likelihood value and minimize the information criteria. Although the BB1 copula provides the best fit of the description of the dependence structure to the bivariate data, the Joe-Clayton (BB7) and the BB3 copulas are also fit to the data well. The dependence parameter estimates of the copula models are shown in Table 5. The entries in the table are the measure of association

Table 4: Log-likelihood and information criteria

Copula	Log-likelihood	AIC	BIC	HQ
Gaussian	298.55	-595.09	-588.29	-592.77
Gumbel	288.35	-574.70	-567.99	-572.38
Normal mixture	298.55	-591.09	-570.97	-584.11
BB1	**365.72**	**-727.43**	**-714.01**	**-722.77**
BB2	305.78	-607.55	-594.13	-602.89
BB3	**364.21**	**-724.41**	**-711.00**	**-719.76**
BB4	347.06	-690.11	-676.69	-685.46
BB5	288.35	-572.70	-559.29	-568.05
BB6	193.19	-382.38	-368.96	-377.72
BB7 (Joe-Clayton)	**362.53**	**-721.05**	**-707.63**	**-716.39**
Galambos	257.70	-513.40	-506.69	-511.07
Husler Reiss	233.26	-464.52	-457.81	-462.19
Tawn	315.81	-625.62	-605.49	-618.63
Frank	295.58	-589.15	-582.44	-586.82
Kimeldorf-Sampson	305.81	-609.62	-602.91	-607.30
Joe	193.19	-384.38	-377.67	-382.05

Note: The table reports the log-likelihood value and the information criteria of the sixteen bivariate copula models to the Microsoft and Boeing returns with the IFM estimator. The BB1 copula minimizes the information criteria and provides the best fit of the description of the dependence structure of the bivariate data.

for the bivariate log daily returns of the Microsoft and Boeing. The IFM estimator of δ of the Gaussian copula model is 0.31 and statistically significant with an asymptotic standard error of 0.01. The dependence parameter estimates of the copula models are used to calculate the tail area dependence measures. For example, the Joe-Clayton copula is convenient in identifying the tail area dependence measures from the dependence parameter estimates in Table 5. The parametric estimates of the lower (τ_L) and upper (τ_U) tail dependence measures of the Joe-Clayton copula model are obtained by $\tau_L = 2^{-\frac{1}{\delta}}$ and $\tau_U = 2 - 2^{-\frac{1}{\theta}}$, respectively. The lower ($\tau_L = 2^{-\frac{1}{\delta}}$) and upper ($\tau_U = 2 - 2^{-\frac{1}{\theta}}$) tail dependence measures are representative of the asymptotic behavior of the copula in the left and right tails. From the bivariate Joe-Clayton copula function estimates, the lower (λ_L) and upper (λ_U) tail dependence measures are calculated as 0.13 and 0.16, respectively.

Table 5: Dependence parameter estimate of the copula

Copula	θ	δ
Gaussian		0.31 (0.01)
Gumbel		1.23 (0.01)
Normal mixture		
BB1	0.26 (0.02)	1.11 (0.01)
BB2		
BB3	1.12 (0.01)	0.22(0.02)
BB4	0.28 (0.02)	0.31(0.02)
BB5		
BB6	1.26 (0.02)	1.00 (0.01)
BB7 (Joe-Clayton)	1.14 (0.02)	0.34 (0.02)
Galambos		0.47 (0.01)
Husler Reiss		0.79 (0.02)
Tawn		
Frank	1.98 (0.08)	
Kimeldorf-Sampson		0.42 (0.02)
Joe	1.26 (0.02)	

Note: The table reports the estimation results of the sixteen bivariate copula models to the Microsoft and Boeing returns with the IFM estimator. The benchmark model is the Gaussian copula. The tail area independence in the Gaussian copula is used for comparison with the case where there is tail area dependence in the bivariate return processes. The entries in parentheses are the standard deviations.

Table 6: Measures of concordance: Kendall's τ and Spearman's ρ

Copula	Kendall's τ	Spearman's ρ
Gaussian	**0.20**	**0.29**
Gumbel	**0.19**	**0.27**
Normal mixture		
BB1	0.21	-0.42
BB2	0.17	2.05
BB3	0.20	-0.42
BB4	0.20	-0.90
BB5	**0.19**	**0.27**
BB6	0.13	4.95
BB7 (Joe-Clayton)	0.20	-0.43
Galambos	0.18	0.26
Husler Reiss	0.16	0.24
Tawn	**0.19**	**0.28**
Frank	**0.21**	**0.31**
Kimeldorf-Sampson	0.17	2.05
Joe	0.13	4.95

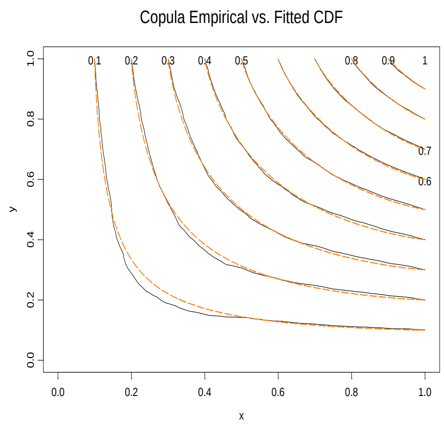
Note: The measures of concordance are stable with respect to the specifications of the copulas. The boldface entries are close to the Kendall's τ and the Spearman's ρ of 0.21 and 0.30 computed from the empirical copula. The measures of concordance indicate overall moderate positive dependence between the Microsoft and Boeing returns.

Table 6 reports the Kendall's τ and the Spearman's ρ for the two stocks with the copulas we estimate. The measures of concordance are close to each other, meaning that the estimates are stable with respect to the specifications of the copulas. The boldface entries in the table are close to the Kendall's τ and the Spearman's ρ of 0.21 and 0.30 computed from the empirical copula, which indicates an overall moderate positive dependence between the two returns series.

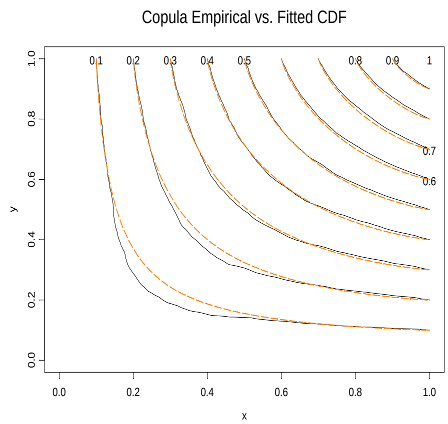
To evaluate the performance of the fit of the parametric copulas, we estimate the empirical copula for the Microsoft and Boeing stock returns. The iso-probability contours of the estimated bivariate copula densities with the estimated GPD margins are compared with the contours of the empirical copula and given in Figure 6. The BB1, BB3 and the BB7 copulas provide clearly better fits to the bivariate returns than the Guassian, Gumbel, and the Kimeldorf-Sampson copulas. From the estimation results, the BB1, BB3 and the BB7 copulas fit well to the bivariate returns conditional on using the semiparametric GPD models for the marginal models.

Figure 6: Contour plots of the nonparametric copula vs. the estimated parametric copulas

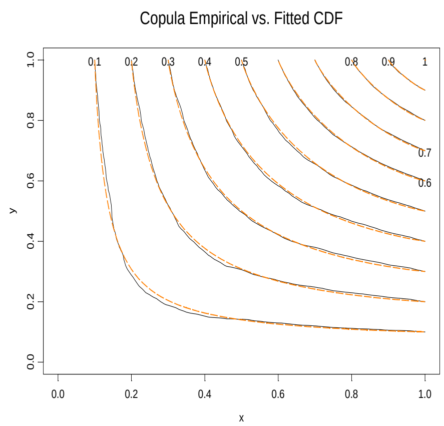
<Gaussian>



<Gumbel>



<BB1>



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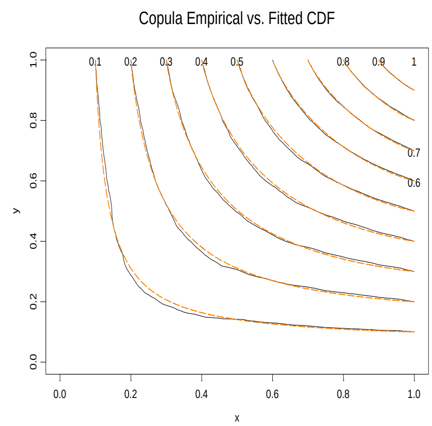

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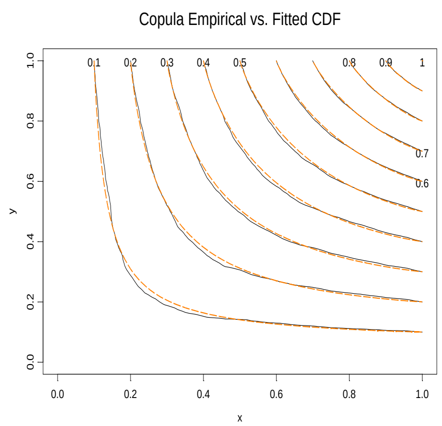
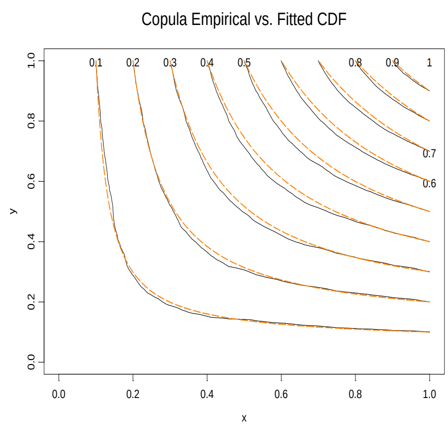


<Kimeldorf-Sampson>



3. COPULA-BASED RISK MEASURES

Chen and Fan (2006) develop a method of combining parametric copulas with unspecified marginal distributions to obtain a univariate copula-based semiparametric stationary time series models and estimate the value-at-risk (VaR) of portfolio of assets. They propose to separate out the dependence structure as parametric copulas from nonparametric marginal distributions which can be estimated as the rescaled empirical distribution or the kernel smoothed distribution. The copula-based semiparametric model is then applied to calculating the conditional VaR of portfolio. Chiou and Tsay (2008) combine the GARCH marginal models and two copula models, the Plackett copula and the Frank copula to price financial derivatives and assess the risk of a portfolio. Embrechts et al. (2003a) construct optimal bounds for the VaR under various dependence structures based on the copula theory and show that the nonadditivity of the risk measure is attributed to the different dependence structures between the component random variables.

This paper employs an alternative procedure for calculating the VaR. Firstly, the semi-parametric estimates of the tails of the empirical marginal models are obtained using the non-parametric estimate of the random proportion of the data and the GPD function of estimating the tails of the marginal distributions. Armed with the parameter estimates of the empirical marginal distributions, we use the IFM method for estimating the parameters of the copula function. We then use these results to calculate the copula-based semiparametric VaR and the expected shortfall (ES).

Given any $q \in (0,1)$, the daily $(1-q)\times 100\%$ VaR on the daily losses of the stock portfolio is the $100\times q\%$ quantile of the portfolio return distribution function F:

$$VaR_q = F^{-1}(q). \tag{13}$$

The level of the loss is used as the threshold, and the estimate of the threshold is the extreme quantile estimate. If the loss on the stock portfolio is exceeded with the $(1-q)\times 100\%$ probability, then the level of the loss is the $100\times q\%$ quantile or the VaR_q. If we assume the losses with the cumulative distribution function F are distributed to normal, the $100\times q\%$ quantile of the distribution function F of the daily losses can also be calculated. If the distribution of the daily losses has fatter tails than the normal distribution then the $100\times q\%$ quantile of the standard normal distribution could be misleading and underestimate the extreme quantile estimation. Estimates of the extreme quantile based on the GPD procedure are more accurate.

We also compute the ES which can be calculated as the conditional expectation of the daily loss -R given that -R is greater than the VaR_q. That is, the $(1-q) \times 100\%$ ES is computed as follows:

$$ES_q = E[-R| -R > VaR_q]. \tag{14}$$

If the distribution of the daily losses is a fat-tailed distribution, ES_q computed from the standard normal distribution could be misleading and underestimate the expected shortfall. We compute the ES_q as the conditional expectation of the threshold excesses $F_{VaR_q}(-R)$ given that -R is greater than the VaR_q. The GPD approximation to $F_{VaR_q}(-R)$ has the shape parameter ξ and the scale parameter $\beta(u) + \xi(VaR_q - u)$. Consequently, the GPD approximation to ES_q can be computed from the parameter estimates of the GPD model. Since the portfolio return distribution function F is generally unknown, analytic expressions for the VaR_q and ES_q are generally not available.

We use the copula-based model to calculate the VaR_q and ES_q. In applying the copula method to the computation of the conditional VaR from the bivariate distribution of the daily log returns on the portfolio which is composed of the Microsoft and Being, we need to specify the models for the marginal distributions of each return and the model for the conditional copula. The appropriate conditional distribution for the portfolio should reflect heavy-tailed behavior of each return process and the dependence structure between the two processes. In specifying the model for the bivariate distribution, we use the fitted GPD marginal models and the optimal copula to generate bivariate returns of the Microsoft and Boeing. We generate 1,000,000 sample paths of bivariate returns and calculate the 5 % and 1 % VaR. For comparison, we calculate the 5 % and 1 % VaR under the assumption of dependence structure implied by the Gaussian copula model as the reference model. In the simulation, we assume a portfolio weight on the Microsoft and Boeing as 0.5 and 0.5, respectively.

Based on the Sklar's theorem, we generate the joint density of the bivariate process via two marginal distributions and a copula function, $f_{XY}(x,y) = f_X(x) f_Y(y) c(F_X(x), F_Y(y))$.

For the benchmark model, the margins are assumed as $X \sim N(0.0009, 0.0239)$ and $Y \sim N(0.0004, 0.0190)$. We employ the Gaussian copula which describes the concordance between the two margins at different levels of dependence measure. The estimate of the dependence measure of the Gaussian copula from the actual bivariate returns is 0.31, however, we employ a variety of levels of dependence for comparison. Due to the complexity of the bivariate distribution, we use a numerical approximation method of random simulations for evaluating the risk measures. From the simulated returns of the portfolio which invests 50 percent of the wealth into Microsoft and 50 percent into Boeing, we compute the daily $(1-q) \times 100\%$ VaR as the $100 \times q\%$ empirical quantile of the portfolio return distribution function F, and the ES which can be calculated as the conditional expectation of the daily loss -R given that -R is greater than the VaR_q.

Table 7 reports the computation results of the VaR_q and ES_q by simulation based on the GPD approximation. According to the estimation results in Table 7, the daily loss on the portfolio of the Microsoft and Boeing could be as low as -2.77% with the 5% probability when the parametric copula is the Kimeldorf-Sampson copula. And, under the condition that the daily negative return is less than -2.77%, the average loss on the portfolio return is -4.36%. Under the condition that the daily negative returns are less than -5.18%, the expected loss on the portfolio return is, on average, -7.32% when the parametric copula is the Kimeldorf-Sampson copula. If the distribution of the daily losses is a fat-tailed distribution, the VaR_q and ES_q computed from the standard normal distribution could be misleading and underestimate the true VaR_q and ES_q. To confirm this assertion, we compute the VaR_q and ES_q for q=0.95, 0.99 from the benchmark model. The results are presented in Table 8. When we assume low dependece measure of δ, the VaR_q and ES_q from the benchmark model generally underestimate the levels of the risk. These results are more conspicuous for the cases with $VaR_{0.99}$ than those with $VaR_{0.95}$.

Table 7: VaR_q and ES_q computation results by simulation with the GPD margins and the parametric copulas

Copula	$VaR_{0.95}$	$VaR_{0.99}$	$ES_{0.95}$	$ES_{0.99}$
Gaussian	2.7397	4.7897	4.0635	6.4225
Gumbel	2.6434	4.5528	3.8694	6.0411
BB1	2.7570	5.0668	4.2504	6.9952
BB3	2.7504	5.0504	4.2418	7.0171
BB5	2.6452	4.5509	3.8631	6.0246
BB7 (Joe-Clayton)	2.7644	5.0660	4.2675	7.0665
Frank	2.7282	4.5921	3.9255	6.0100
Kimeldorf-Sampson	2.7748	5.1799	4.3594	7.3187
Joe	2.5302	4.2983	3.6560	5.7738

Note: The table reports the computation results of the VaR_q and ES_q by simulation based on the GPD margins and a variety of parametric copulas. We can read off from the second column that the daily loss on the portfolio of the Microsoft and Boeing could be as low as -2.77% with the 5% probability when the parametric copula is the Kimeldorf-Sampson copula. If the daily negative return is less than -2.77%, the expected loss on the portfolio is -4.36% from the fourth column.

Table 8: VaR_q and ES_q computation results by simulation with the normal margins and the normal copulas

q	VaR_q	ES_q
Panel A	$\delta = 0.31$	
0.95	2.8028	3.5300
0.99	3.9871	4.5743
Panel B	$\delta = 0.10$	
0.95	2.5640	3.2315
0.99	3.6488	4.1900
Panel C	$\delta = 0.90$	
0.95	3.3825	4.2581
0.99	4.8066	5.5223

Note: The table reports the computation results of the VaR_q and ES_q by simulation based on the Gaussian margins and the Gaussian copulas. The $VaR_{0.95}$ and $ES_{0.95}$ from the benchmark model with $\delta = 0.10$ are -2.56% and -3.23%, respectively. These underestimate the levels of the risk, for example, compared to those with the $VaR_{0.95}$ and $ES_{0.95}$ from the GPD margins and the Kimeldorf-Sampson copula.

4. CONCLUSION

This paper aims to quantify the tail area dependence and use this information to evaluate the tail related risk measures from the semi-parametric estimates of the marginal distributions and the copula function. To those ends, firstly, we utilize the copula function methodology to separate out the components which describe the marginal behavior of the individual returns and the dependence structure between the random variables from the joint density of stationary time series. In applying the copula method to the modeling of the conditional bivariate distribution of the daily log returns on the Microsoft and Being, we specify the models for the marginal distributions of each return and the model for the conditional copula. The appropriate marginal distributions for the stock returns should reflect heavy-tailed behavior of the process. However, we also consider that the degrees of freedom parameters of the two marginal distributions and the copula should be the same to use the Student's t margins and the bivariate Student's t copula, which we regard the conditions as too restrictive. Also, in specifying the joint density of the returns, it seems to be inappropriate to use the bivariate normal distribution when the two margins deviate from the Gaussian normality. In order to reflect the non-ellipticity of the joint distribution and heavy tails in the extreme quantile of the marginal distributions of the asset returns, we use the generalized Pareto distribution (GPD) as the margins and a wide range of parametric copula functions along with a nonparametric copula function in the analysis. We select the optimal copulas from a variety of non-nested copulas based on the model selection criteria.

Extreme value theory based approach offers the fully parametric methods for estimating the tails of the marginal distribution. We fit the generalized Pareto distribution (GPD) to the threshold excesses. In theory, it is best to fit the GPD to the data solely pertained to the tail of the distribution and not included in the center of the distribution. At the same time, however, we want to reduce the variance of the parameter estimates by keeping the number of observations included in the tail shape and scale parameter estimation large enough to have a sufficient number of exceedances of the returns over a high threshold. We minimize the bias of the GPD parameter estimates by choosing a high threshold. In this paper, we follow the mean excess methodology for selecting the optimal threshold.

The two stock return processes prove to be non-normal with excess kurtosis and negative skewness. We estimate a variety of parametric copulas along with the empirical copula. We use the Gaussian, the Joe-Clayton (BB7), the BB3 and the Frank copula. The Gaussian copula is used for the benchmark due to the fact that the bivariate Gaussian copula results in the tail area independence and can be used for comparison with the case where there is tail area dependence in the bivariate time series. Combining these with the GPD estimation of the marginal distributions would result in the class of semiparametric copula-based calculation of the VaR. In the first step of the procedure, we fit the bivariate copulas to the Microsoft and Boeing returns with the IFM estimator.

Secondly, we proceed to evaluating the market risk using the VaR and the expected shortfall (ES). We use a semi-parametric method for estimating the market risk from an estimated copula functions. We combine a parametric assessment of the VaR and the ES with the fitted tail distribution to form a semi-parametric evaluation of the risk measures. By sampling from the tail of the distribution, the level of statistical precision is elevated in evaluating the risk measures. In calculating the risk measures, we assume that the returns are jointly distributed to the parametric copulas as well as to the empirical copula. We then compare the results with that from the bivariate normal distribution. The results show that the VaR and the ES calculated from the copula function which takes the complicated and possibly nonlinear dependence structure into account performs better than the one based on the linear correlation-based normality assumption.

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