Journal of Economic Theory and Econometrics, Vol. 27, No. 1, Mar. 2016, 1–15

Asymmetric Information about Rivals' Types: Existence of Equilibrium in First-Price Auction

Jinwoo Kim*

Abstract This paper establishes the equilibrium existence in the first-price auction in the setup where bidders are partitioned into "knowledge groups" so that each bidder observes the types of rival bidders in the same group but not the others' types.

Keywords First-Price Auction; Knowledge of Rival Bidders' Types; Multidimensional Information; Equilibrium Existence

JEL Classification C70, D44, D82.

*Department of Economics, Seoul National University,1 Gwanak-ro Gwanak-gu, Seoul 151-742, South Korea, jikim72@snu.ac.kr. The author acknowledges the financial support from the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology through its grant of Global Research Network (2013S1A2A2035408).

Received January 21, 2016, Revised February 25, 2016, Accepted February 26, 2016

1. INTRODUCTION

This paper establishes the existence of Bayes Nash equilibrium for the first-price auction in the setup where some bidders know more about each other than about others. Specifically, bidders are partitioned into subsets called "knowledge groups" such that bidders in the same knowledge group are assumed to know each other's types (or values) precisely while they only know the value distribution of bidders in other groups. This information structure modifies the traditional setup with independent and private values (IPV)—where all bidders only know the value distribution of others—to capture situations where some bidders do *not* have uncertainty about each other's values due, for instance, to certain interactions prior to the auction. Some practical examples include a procurement auction contested by domestic firms and foreign firms in which domestic firms know much about each other's technological capabilities or revenue/cost structures, or auctions for government assets, such as mineral, timber harvesting, and spectrum rights, in which bidders often consist of incumbent firms with a long history of operation in the industry and relative newcomers in the area.

In this setup, Kim and Che (2004) study the standard auctions—first-price auction and second-price auction—to show that the second-price auction performs better than the first-price auction in terms of both efficiency and revenue. Their results are based on the existence of equilibrium in the first-price auction, which is not formally established in the paper, however. One may wonder if the existence follows simply from applying the standard equilibrium analysis to the current setup, since it can be seen as a special case of IPV setup with correlated values where some values are more correlated than others. This is not the case, however, since bidders in the current setup, in particular those in the knowledge group consisting of multiples bidders, hold *multidimensional* private information, that is, values of others in the same knowledge group as well as his own value. The standard analysis is not suited for handling the equilibrium bidding strategies that depend on such multidimensional information. In fact, the analysis is made more difficult due to the arbitrary partition structure assumed by the current paper, which renders the auction generally asymmetric since the group sizes may be different.

The current paper adapts the general existence result in Reny (1999) to establish that under the above setup, the first-price auction admits the existence of undominated Bayes Nash equilibrium (which may be mixed). Reny (1999) proves the existence of Nash equilibrium in games with discontinuous payoffs that typically arise in the auction games due to tie-braking rules. He provides mild conditions on payoff discontinuities that admit the existence of Nsah equilibrium and establishes, as an application, the existence of (Bayes) Nash equilibrium for the first-price auction—more precisely, discriminatory price auction—where bidders have multidimensional private information associated with their multi-unit demands. The current paper establishes the existence for the first-price auction in another novel setup with multidimensional private information that arises due to the asymmetry of bidders' knowledge about rivals' types. This paper also makes a contribution to the line of literatures that are devoted to proving the existence of equilibrium in the first-price auction, such as Maskin and Riley (2000) and Lebrun (1999).

2. MODEL

A seller has a single indivisible good to sell to $n \geq 2$ risk neutral bidders.[1] The seller is assumed to put no value on the good. Bidders have independent private values. Specifically, bidder i's valuation v_i is drawn from the interval $[0,1]$, following a common distribution function $F(\cdot)$ whose density function $f(\cdot)$ is bounded away from zero. F is assumed to be common knowledge. Letting N denote the set of all bidders, the profile of bidders' valuations, $v := (v_i)_{i \in N} \in V := [0,1]^n$, has the joint density $f_N(v) := \prod_{i \in N} f(v_i)$.[2]

We depart from standard IPV models (represented by Myerson (1981) and Riley and Samuelson (1981) by allowing some bidders to know about each other's valuation. To express this idea formally, we impose a partition structure $\mathscr{G}$ on the set N, where $\mathscr{G}$ is the set of disjoint groups (or disjoint sets) with each group consisting of bidders whose realized valuations are common knowledge among themselves. That is, for each group $G \in \mathscr{G}$, the profile of its members' valuations, $v_G := (v_i)_{i \in G}$, is common knowledge among those bidders.[3] Note that *every realization* of v_G is commonly known to bidders in G while only the *distribution* of v_G is known to bidders in the other groups. A bidder is referred to as a *group leader* if he has the highest valuation in the group. For later use, we let $v_{G \setminus i} := (v_j)_{j \in G \setminus i} \in V_{G \setminus i}$ and $v_{-G} := (v_j)_{j \in N \setminus G} \in V_{-G}$ denote the profile of valuations for group G except for bidder i and the valuation profile for all bidders except for group G, respectively. The joint density for the vector v_G is denoted by $f_G(v_G) := \prod_{i \in G} f(v_i)$, and the joint densities for the vector $v_{G \setminus i}$ and v_{-G} are similarly denoted by $f_{G \setminus i}$ and f_{-G}, respectively. Also, the partition structure is

[1]The description of the model here closely follows that in Kim and Che (2004).

[2]Throughout the paper, bold face letters are used to denote vectors.

[3]The partition structure is critical for the analysis in this paper. Whether the existence result holds in non-partitional structure is beyond the scope of the current paper.

assumed to be common knowledge and exogenous (i.e., fixed prior to the realization of v). It is assumed that the information structure described so far is common knowledge among bidders.

Throughout the paper, we focus on the first-price auction with zero reserve price. The assumption of zero reserve price is made purely for simplicity, and all subsequent results extend in the natural way (i.e., with no change in the comparisons) if a binding reserve price is introduced. In the first-price auction format, all bidders tender sealed bids simultaneously, and the highest bidder wins and pays his bid. A tie-breaking rule is important in guaranteeing existence of an equilibrium in this format. For instance, in a standard Bertrand game played by two firms with heterogeneous costs, a Nash equilibrium exists only when the ties are broken in favor of the lower-cost firm. For the same reason, we assume that (1) a tie is broken in favor of a bidder with a higher valuation if there are multiple highest bidders; and that (2) if there are multiple highest bidders with the same valuation, then the object is assigned randomly with equal probability among those bidders. While this tie-breaking rule is endogenous, it can be implemented by performing an auxiliary second-price auction with bidders who submitted the highest bid in first-price auction.[4] In both games, we look for a Nash equilibrium in weakly undominated strategies.

Our model includes as special cases two extreme partition structures. In the one case, every set in the partition is a singleton, so every bidder knows only his valuation. This is the standard assumption made in the auction literature. The other case has one grand set in the partition, which means that bidders know all other bidders' types. The resulting game is precisely the Bertrand game. We do not impose any restriction on the partition structure. Naturally, different groups may contain different numbers of bidders, so the partition may be asymmetric. For instance, if there are four bidders with $N = \{1,2,3,4\}$, our model encompasses three different possibilities than the two extreme ones: two groups of two (e.g., $\{\{1,2\},\{3,4\}\}$), one group of three and one group of one (e.g., $\{\{1,2,3\},\{4\}\}$), two groups of one and one group of two (e.g., $\{\{1\},\{2\},\{3,4\}\}$).

[4] See Maskin and Riley (2000) for a similar assumption about tie breaking. Further, our tie-breaking rule can be justified as producing a limiting equilibrium of a game in which bidders must bid in a discrete space and a random tie-breaking rule is used.

3. EXISTENCE OF EQUILIBRIUM FOR FIRST-PRICE AUCTION

We first establish several necessary conditions for an equilibrium, given any arbitrary knowledge partition. These conditions will constitute partial characterizations of equilibrium, which will be used for establishing existence of the equilibrium and for comparison with the second-price auction.

To this end, fix an equilibrium (whose existence will be shown later). Fix any bidder $i \in G$ and let $m(i) \in \arg\max_{j \in G \setminus i} v_j$ be the highest valuation bidder, excluding bidder i, in the same group, and let $v_{m(i)}$ be his valuation. Let $b_i(v_G)$ and $\underline{b}_i(v_G)$ respectively denote an arbitrary selection of bidder i's equilibrium bids and their infimum, given the valuation profile, v_G, of group G. Since we restrict the equilibrium strategies to be undominated, we must have $b_i(v_G) \leq v_i$. The following lemma shows that $b_i(v_G) \geq v_{m(i)}$ whenever $v_i > v_{m(i)}$.

Lemma 1. *If $v_i > v_{m(i)}$, then, in any equilibrium of the first-price auction, $\underline{b}_i(v_G) \geq v_{m(i)}$ and i beats all bidders in G.*

Proof. See the Appendix. ▌

This lemma implies that the allocation is efficient within each group in any equilibrium. It also implies that competition is effectively among the group leaders, so attention can be restricted to group leaders when we search for an equilibrium. The next lemma shows that the equilibrium distributions of their bids have no mass points.[5] It refers to $y_G(b)$, which denotes the probability of outbidding all bidders outside a group G with a bid of b in an arbitrary equilibrium.

Lemma 2. *In any equilibrium of a first-price auction, $y_G(b)$ is continuous in b for every $G \in \mathscr{G}$.*

Proof. See the Appendix. ▌

We next show that any equilibrium in a first-price auction is essentially pure.

[5]While this result would be standard in the auction literature, our informational structure makes it nontrivial and warrants a separate proof. The standard proof is based on the argument along the following line: if a bidder puts mass on a bid b, then there exists an interval $(b-\varepsilon, b)$ to which everyone else assigns probability 0, which then provides a profitable deviation from b (see Fudenberg and Tirole (1991 pp.223-225). This argument does not work in our model since, by Lemma 1, a group leader is constrained by the second-highest valuation in his group, so bids will be placed on every interval with some probability. Our proof basically amounts to showing that if a positive mass is put on b by a group leader, the opponent group leaders will submit bids between $(b-\varepsilon, b)$ with such a small probability that it pays the former leader to move the mass point.

Lemma 3. *For each $G \in \mathscr{G}$, the equilibrium bid, $b_i(v_G)$, is unique for almost every v_G such that $v_i > v_{m(i)}$.*

Proof. See the Appendix. ▌

We are now in a position to address the existence issue. Two features of the model make the standard existence result inapplicable in our setting. First, each bidder observes the entire profile of valuations of his group members, so this creates a multi-dimensional signal problem. Second, since we assume an arbitrary knowledge partition, the environment is generally asymmetric. Our proof builds on the existence result of Reny (1999). A sketch of the proof is presented below, with its detailed version contained in the appendix.

Proposition 4. *There exists a pure-strategy equilibrium in which each bidder, say i, employs a bidding strategy, $b_i(v_G)$ that is nondecreasing and takes a value in $[v_{m(i)}, v_i]$ when $v_i \geq v_{m(i)}$, or else $b_i(v_G) = v_i$.*

Proof Sketch. (A detailed proof is collected in the Appendix.) We consider a hypothetical first-price auction game in which there is one player representing each group. Specifically, for each group $I \in \mathscr{G}$ in the original game, we assign a single player with a signal, $v_I = (v_i)_{i \in I} \in [0,1]^{|I|}$. That player realizes the highest valuation of v_I, upon winning the game (and zero when losing) and chooses a bid $\beta_I(v_I) \in [v_I^2, v_I^1]$, where v_I^r denotes the r^{th} order statistic of vector v_I, and $v_I^2 := 0$ if $|I| = 1$. We then follow the arguments of Reny (1999) to prove that this hypothetical game has a pure-strategy equilibrium, β^*, in which each (hypothetical) player is choosing a nondecreasing bid function $\beta_I^*(v_I) \in [v_I^2, v_I^1]$.

Given the equilibrium, β^*, of the hypothetical game, we can construct the equilibrium of the original game as follows: Each bidder $i \in I$ bids $\min\{v_i, \beta_I^*(v_I)\}$ for a value realization v_I of the group I. To see that this forms an equilibrium of the original game, suppose first that i is not a group leader. Then, given our endogenous tie-breaking rule and $\beta_I^*(v_I) \geq v_i$, bidder i has no incentive to deviate. If i is a group leader, his equilibrium bid is $\beta_I^*(v_I) \leq v_i$. Given the behavior of other bidders in his group, bidder i cannot benefit from bidding below v_I^2. Given this constraint, bidding $\beta_I^*(v_I)$ is optimal since β_I^* is an equilibrium strategy in the hypothetical game. ▌

Given the private value specification, a group leader's equilibrium bid is likely to depend only on his valuation and the second-highest valuation in the group. In fact, it is likely that there exists an unconstrained bidding strategy, $B_I(v)$, for group I such that a leader of group I bids $\max\{B_I(v_I^1), v_I^2\}$, given the first and second-highest valuations, v_I^1 and v_I^2, respectively. The unconditional bid $B_I(v_I^1)$ is thus the optimal bid that a group leader with value v_I^1 would have submitted if there were no competition from the group members. When $B_I(v_I^1)$ is smaller than the second highest value, the leader raises his bid up to v_I^2, i.e. just enough to beat the highest group rival. Notice here the role of the endogenous tie-breaking rule that awards the object to the group leader when his bid is tied with the highest group rival. The following example illustrates the existence of (pure strategy) Bayes Nash equilibrium that involves the unconditional bidding strategy.

Example 5. *Suppose that there are four bidders in k equal-sized groups: $\mathscr{G} = \{\{1,2\},\{3,4\}\}$. Suppose also that each bidder draws his valuation uniformly from $[0,1]$. It is then a (symmetric) equilibrium for each bidder to bid*

$$\min\{v_i, \max\{\frac{2}{3}v_i, v_{m(i)}\}\}$$

when his valuation is v_i and that of the other bidder in the group is $v_{m(i)}$. In this equilibrium, a group leader with valuation v_i adopts an unconstrained bid $B(v_i) = \frac{2}{3}v_i$, unless it is less than the valuation of the other bidder in his group.[6] *That is, a leader behaves much as in a standard first-price auction (i.e., singleton partitions), except that he behaves like a Bertrand player against his within-group rivals. Note also that the unconstrained bid, $B(v_i) = \frac{2}{3}v_i$, adopted by the group leader is the same as the equilibrium bid that would be employed in a standard first-price auction game (i.e., singleton knowledge partitions) with* ***only three*** *players. Essentially, the group leader acts as if he faces no competition from the lower valuation bidder in his group as long as $\frac{2}{3}v_i \geq v_{m(i)}$. Clearly this will reduce the competition, all else equal. On the other hand, whenever $\frac{2}{3}v_i < v_{m(i)}$, the second highest valuation acts as a constraint, thus raising the intensity of competition. How these two conflicting effects will affect the revenue will be the subject of the next section.*

[6] That this is an equilibrium can be seen as follows. Since the two groups are symmetric,

$$\begin{aligned} y_G(b) &= \text{Prob}\{v_G|\ \max\{B(v_i), v_{m(i)}\} \leq b, \text{ where } i = \arg\max_{j\in G} v_j\} \\ &= \text{Prob}\{v_G|\ v_i \leq B^{-1}(b) \text{ and } v_{m(i)} \leq b, \text{ where } i = \arg\max_{j\in G} v_j\} \\ &= 2F(3b/2)F(b) - F^2(b) = 2b\max\{1, 3b/2\} - b^2. \end{aligned}$$

A group leader with valuation v_i maximizes $y_G(b)(v_i - b)$ subject to the constraint $b \geq v_{m(i)}$. It can be easily verified that with $y_G(b)$ given above, $y_G(b)(v_i - b)$ is increasing with b for $b < \frac{2}{3}v_i$ and decreasing for $b > \frac{2}{3}v_i$, which implies $\max\{\frac{2}{3}v_i, v_{m(i)}\}$ is indeed an equilibrium.

APPENDIX

Proof of Lemma 1. Bidder i must receive a strictly positive (expected) payoff in equilibrium (since he can bid slightly higher than $v_{m(i)}$, which will win with positive probability). If $\underline{b}_i(v_G) < v_{m(i)}$, then bidder $m(i)$ must also earn a strictly positive payoff in equilibrium. For both bidders to earn positive payoffs, their infimum must coincide and each must put a mass point there. But then it pays either one of them to raise the mass point slightly, which will increase the probability of winning discontinuously while lowering his payoff conditional on winning only slightly. Hence, we have a contradiction, so we must have $\underline{b}_i(v_G) \geq v_{m(i)}$. The last statement follows directly from the first statement and our tie-breaking rule. ▌

Proof of Lemma 2. Suppose to the contrary that $y_G(b)$ jumps up at b for some group G. This can only occur if the leader of another group, say $\tilde{G} \neq G$, puts mass on b. We must have only one such group since, otherwise, one of leaders of those groups would want to bid slightly above b to increase the winning probability discontinuously. Therefore, $y_{\tilde{G}}$ is continuous at b. Since the leader of $\tilde{G}$ bids b for a positive measure of type profiles of $\tilde{G}$ bidders, there must exist such a profile, $v_{\tilde{G}}$, with $v_{\tilde{G}}^2 < b \leq v_{\tilde{G}}^1$, where $v_{\tilde{G}}^1$ and $v_{\tilde{G}}^2$ are the first and second order statistics of $v_{\tilde{G}}$. The leader of $\tilde{G}$ can ensure himself a positive surplus given $v_{\tilde{G}}$, so we must also have $y_{\tilde{G}}(b) > 0$.

In equilibrium, the group $\tilde{G}$ leader should have no incentive to deviate by bidding below b given $v_{\tilde{G}}$, which requires that, for $\varepsilon > 0$,

$$[v_{\tilde{G}}^1 - (b-\varepsilon)]y_{\tilde{G}}(b-\varepsilon) \leq [v_{\tilde{G}}^1 - b]y_{\tilde{G}}(b),$$

or

$$[v_{\tilde{G}}^1 - b]\frac{y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\varepsilon)}{\varepsilon} \geq y_{\tilde{G}}(b-\varepsilon). \tag{A.1}$$

For sufficiently small $\varepsilon > 0$, $y_{\tilde{G}}(b-\varepsilon)$ must be strictly positive since $y_{\tilde{G}}(\cdot)$ is continuous at b and $y_{\tilde{G}}(b) > 0$. Hence, to prove that such a deviation is profitable, it suffices to show

$$\limsup_{\varepsilon \downarrow 0} \frac{y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\varepsilon)}{\varepsilon} \leq 0. \tag{A.2}$$

We prove this below. Consider again any group $G \neq \tilde{G}$. If G consists of a single bidder, then for small enough ε, he would assign probability 0 to the interval $[b-\varepsilon, b)$. Hence, no single-bidder group can contribute to $y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\varepsilon)$, for a sufficiently small ε, and we are done if all groups other than $\tilde{G}$ have single bidders. Assume therefore that there exists a group $G \neq \tilde{G}$, which contains more than one bidder. We show below that even such a group chooses almost zero probability in the interval $[b-\varepsilon, b)$ for a small $\varepsilon > 0$.

To prove this, we find an upper bound for $y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\varepsilon)$ for a small ε, which is accomplished by identifying a set of v_G for which a leader of G should not make a bid between b and $b-\varepsilon$. To begin, note that since y_G jumps up at b as mentioned above, we have $\overline{p} := \lim_{b' \uparrow b} y_G(b') < y_G(b)$. Let $r := \frac{y_G(b)}{\hat{p}} > 1$, and take any $K_1 > \frac{1}{r-1}$. Then, for any $\varepsilon > 0$, a leader of G with $v_G^1 > b + K_1\varepsilon$ strictly prefers b to any $\tilde{b} \in [b-\varepsilon, b)$ since

$$\frac{[v_G^1 - b] y_G(b)}{[v_G^1 - \tilde{b}] y_G(\tilde{b})} \geq \left(\frac{v_G^1 - b}{v_G^1 - \tilde{b}} \right) r \geq \left(\frac{v_G^1 - b}{v_G^1 - b + \varepsilon} \right) r > 1,$$

where the numerator and denominator are the payoffs from the bidding b and $\tilde{b}$, respectively, the first inequality follows from $y_G(\tilde{b}) \leq \hat{p}$, the second inequality follows from $\tilde{b} \geq b - \varepsilon$, and the last inequality follows from $v_G^1 > b + K_1\varepsilon$ and from $K_1 > \frac{1}{r-1}$. It follows that a bid $\tilde{b} \in [b-\varepsilon, b)$ can only be made by the group G leader if $v_G^1 \in [b-\varepsilon, b+K_1\varepsilon]$.

Next, set $K_2 := K_1 + 3$ for K_1 chosen above and assume that the group G leader has $v_G^1 \in [b-\varepsilon, b+K_1\varepsilon]$ — the only possibility that causes the leader to bid in $[b-\varepsilon, b)$. Suppose that the second-highest rival in the group has $v_G^2 \leq b - K_2\varepsilon$. Then, the group G leader will face no within-group challenge by bidding $b - K_2 < b - \varepsilon$. In fact, for a sufficiently small $\varepsilon > 0$, a group G leader with $v_G^1 \in [b-\varepsilon, b+K_1\varepsilon]$ strictly prefers $b - K_2$ to any bid $\tilde{b} \in [b-\varepsilon, b)$, since

$$\frac{[v_G^1 - \tilde{b}] y_G(\tilde{b})}{[v_G^1 - (b - K_2\varepsilon)] y_G(b - K_2\varepsilon)} \leq \frac{(K_1+1) y_G(\tilde{b})}{(K_1+2) y_G(b - K_2\varepsilon)} < 1,$$

where the numerator and the denominator represent the payoffs from bidding $\tilde{b} \in [b-\varepsilon, b)$ and $b - K_2\varepsilon$, respectively, and the first inequality holds (for $\varepsilon < 1$ say) since $v_G^1 \in [b-\varepsilon, b+K_1\varepsilon]$ and $\tilde{b} \geq b - \varepsilon$, and the second inequality holds since $y_G(\cdot)$ is continuous at b. It follows that, for a sufficiently small $\varepsilon > 0$, the group G leader will never bid in $[b-\varepsilon, b)$ if $v_G^2 \leq b - K_2\varepsilon$.

Combining the two arguments, we conclude that a group G bidder will bid in $[b-\varepsilon, b)$ only if $v_G^1 \in [b-\varepsilon, b+K_1\varepsilon]$ and $v_G^2 \in (b - K_2\varepsilon, b)$. The probability of the joint event is no greater than $\binom{|G|}{2} \left(F(b+K_1\varepsilon) - F(b-K_2\varepsilon)\right)^2$. Therefore,

$$y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\varepsilon) \leq \sum_{\substack{G \neq \tilde{G} \\ |G| \geq 2}} \binom{|G|}{2} \left(F(b+K_1\varepsilon) - F(b-K_2\varepsilon)\right)^2. \qquad \text{(A.3)}$$

Hence, we obtain

$$\begin{aligned}
& \limsup_{\varepsilon\downarrow 0} \frac{y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\varepsilon)}{\varepsilon} \\
\leq\ & \lim_{\varepsilon\downarrow 0} \sum_{\substack{G\neq\tilde{G} \\ |G|\geq 2}} \binom{|G|}{2} \frac{(F(b+K_1\varepsilon) - F(b-K_2\varepsilon))^2}{\varepsilon} \\
=\ & \lim_{\varepsilon\downarrow 0} \sum_{\substack{G\neq\tilde{G} \\ |G|\geq 2}} 2\binom{|G|}{2} (F(b+K_1\varepsilon) - F(b-K_2\varepsilon))(K_1 f(b+K_1\varepsilon) + K_2 f(b-K_2\varepsilon)) \\
=\ & 0,
\end{aligned}$$

where the inequality follows from (A.3), and the first equality follows from the L'hopital's rule.

The last string of inequalities implies that it pays the group $\tilde{G}$ leader to move the mass point down, which yields a contradiction to the conjectured equilibrium. Hence, we conclude that $y_G(\cdot)$ is continuous for all $G \in \mathscr{G}$. ▌

Proof of Lemma 3. Let $b_G(v_G)$ be an arbitrary selection from the support of a group G leader's (possibly mixed) equilibrium bids when the valuation profile of group G members is v_G. Consider two valuation profiles v_G and $\overline{v}_G$ with $v_G \leq \overline{v}_G$ and $v_G^1 < \overline{v}_G^1$, where v_G^1 (resp. $\overline{v}_G^1$) refers to the group G leader's valuation, given profile v_G (resp. $\overline{v}_G$). Let $b = b_G(v_G)$ and $\overline{b} = b_G(\overline{v}_G)$, and then we show that $b \leq \overline{b}$; i.e., an arbitrary equilibrium bidding strategy is nondecreasing. Since $\overline{b} \geq \overline{v}_{m(i)} \geq v_{m(i)}$, we are done if $b \leq \overline{v}_{m(i)}$. Hence, assume that $b > \overline{v}_{m(i)}$. This means that, given the profile of $\overline{v}_G$, the group G leader could beat all of his within group rivals by bidding b, so his winning probability would be simply that of outbidding other group leaders, $y_G(b)$. Likewise, given the profile of v_G, the group G leader would face the winning probability of $y_G(\overline{b})$ when bidding $\overline{b}$. Then, incentive compatibility requires $y_G(b)[v_G^1 - b] \geq y_G(\overline{b})[v_G^1 - \overline{b}]$ and we have $y_G(\overline{b})[\overline{v}_G^1 - \overline{b}] \geq y_G(b)[\overline{v}_G^1 - b]$. Combining last two inequalities gives

$$(\overline{v}_G^1 - v_G^1)(y_G(\overline{b}) - y_G(b)) \geq 0. \tag{A.4}$$

Suppose, to the contrary, that $b > \overline{b}$. Then, since $y_G(\cdot)$ is nondecreasing, we must have $y_G(\overline{b}) = y_G(b) > 0$. But this cannot hold since the group G leader would strictly prefer to bid $\overline{b}$ when v_G is realized. We therefore conclude that an arbitrary selection from the equilibrium strategies must be non-decreasing. Because there can be only countably many jumps in a non-decreasing and bounded correspondence, the equilibrium bidding strategy of i is almost pure when he is a leader. ▌

Proof of Proposition 4. Fix an arbitrary partition structure and suppose that there are K groups in that partition structure. As outlined in the text, we first consider a *hypothetical game* in which there are only K players, one for each group. In this hypothetical game, player $I \in \mathscr{G}$ observes as private information $v_I = (v_i)_{i\in I}$, the valuation profile of bidders in group I in the original game, and bids

$$\beta_I(v_I) \in [v_I^2, v_I^1], \tag{A.5}$$

where v_I^r denotes the r^{th} order statistic of the vector v_I, and $v_I^2 = 0$ if $|I| = 1$. All bidders bid simultaneously, and the good is allocated according to the first-price auction rule. Ties are broken according to our endogenous sharing rule. Formally, given the profile of bids submitted, $b = (b_1, \cdots, b_K)$, let $W(b,v) = \arg\max_J \{v_J^1 | J \in \arg\max_{H\in\mathscr{G}} b_H\}$ denote the set of highest-valuation bidders (in the hypothetical game) who submitted the highest bid. Then, the payoff of player I is described as:

$$U_I(b;v) := \begin{cases} \frac{1}{|W(b,v)|}(v_I^1 - b_I) & \text{if } I \in W(b,v) \\ 0 & \text{otherwise,} \end{cases} \tag{A.6}$$

when the players observed v and bid b. Notice that each player only realizes the highest valuation of his group. If the bidders play a strategy profile $\beta = (\beta_I)_{I\in\mathscr{G}}$, then bidder I receives payoff: $u_I(\beta) := \int U_I(\beta(v); v) f_N(v) dv$.

Given this description of hypothetical game, we turn to existence of the Nash equilibrium in this game. Reny (1999) provides us with conditions for the existence of a mixed strategy equilibrium (see Corollary 5.2 of Reny (1999)). First of all, as Reny (1999) did in the case of multi-unit pay-your-bid auction, we study a *restricted* version of this hypothetical game where bid functions are restricted to be nondecreasing. This latter restriction ensures that the strategy space is compact if endowed with the pointwise convergence topology, thereby making the set of mixed strategies compact with the weak* topology. An equilibrium of the *restricted hypothetical game* will be shown to be an equilibrium of the hypothetical game later when we show that there exists a best response satisfying the monotonicity constraint when all other bidders play the restricted equilibrium strategies.

Given the compactness, *better-reply security* (as defined in Reny (1999)) is sufficient to establish the existence of a mixed strategy equilibrium of the restricted hypothetical game. We prove that its sufficient conditions, reciprocal upper semicontinuity and payoff security, hold for the restricted hypothetical game.[7]

[7] A standard first-price auction does not satisfy reciprocal upper semicontinuity, given a random

Step 1: *The payoffs of the players the restricted hypothetical game satisfy reciprocal upper semicontinuity in mixed strategies.*

Proof. We prove reciprocal upper semicontinuity in the players' pure strategies, which is sufficient for reciprocal uppersemicontinuity in the mixed strategies. The former is in turn proven by showing that $u(\beta) = \sum_I u_I(\beta)$ is upper semicontinuous in β. To this end, we first show that $U(b;v) := \sum_I U_I(b;v)$ is upper semicontinuous in b for every v. For a v, pick an arbitrary b and a sequence $b^t = (b^t_I)_{I\in\mathscr{G}}$ converging to b. It suffices to show that for any given $\varepsilon > 0$, there exists T such that $U(b;v) + \varepsilon \geq U(b^t;v)$ for all $t \geq T$. Let $b = b_I$ and $v = v^1_I$ for $I \in W(b,v)$. Then, $U(b;v) = v - b$. For a sufficiently large t, we must have (i) $W(b^t,v) \subset \arg\max_{J\in\mathscr{G}} b_J$, and that (ii) $b^t_I \geq b - \varepsilon$ for every I. It then follows that

$$\begin{aligned} U(b^t;v) &= \sum\nolimits_{I\in W(b^t,v)} \frac{1}{|W(b^t,v)|}(v^1_I - b^t_I) \\ &\leq \sum\nolimits_{I\in W(b^t,v)} \frac{1}{|W(b^t,v)|}(v - b^t_I) \\ &\leq v - b + \varepsilon \\ &= U(b;v) + \varepsilon, \end{aligned}$$

where the first inequality follows from (i) since $v = \max_{J'}\{v^1_{J'} \mid J' \in \arg\max_{J\in\mathscr{G}} b_J\} \geq v^1_I$ for any $I \in W(b^t,v) \subset \arg\max_{J\in\mathscr{G}} b_J$, and the second inequality follows from (ii).

To show the upper semicontinuity of u, consider a sequence β^t converging to β pointwise. Then,

$$\begin{aligned} \limsup\nolimits_{t\to\infty} u(\beta^t) &= \limsup\nolimits_{t\to\infty} \int U(\beta^t(v),v) f_N(v)dv \\ &\leq \int \limsup\nolimits_{t\to\infty} U(\beta^t(v),v) f_N(v)dv \\ &\leq \int U(\beta(v),v) f_N(v)dv = u(\beta), \end{aligned}$$

where the first inequality follows from Fatou's Lemma (see Ash 1972, p.295, for instance) and the second inequality from the upper semicontinuity of U. ▌

Step 2: *The restricted hypothetical game is payoff secure in mixed strategies.*

---

tie-breaking rule (see Reny (1999), p.1040). Reciprocal upper semicontinuity holds here because of our endogenous sharing rule, which allocates the good efficiently among tying bidders.

Proof. Let m_I denote the bidder I's mixed strategy and $m = (m_I)_{I\in\mathscr{G}}$ its profile for all players. Note that m_I is a mixing over non-decreasing pure strategy bid functions satisfying (A.5). Then, our game is payoff secure if for every m and every $\varepsilon > 0$, each player i has a strategy $\overline{m}_I$ such that $u_I(\overline{m}_I, m'_{-I}) \geq u_I(m) - \varepsilon$ for all m'_{-I} in some open neighborhood of m_{-I}. This part of the proof follows precisely the same argument as in Reny (1999). The key step is to observe that, given m_{-I} and ε, a player I can achieve a payoff within $\varepsilon/2$ of his supremum payoff by adopting a bidding strategy that is strictly increasing in v_I^1.[8] Since the latter strategy does not put any mass on a single bid, $u_I(\overline{m}_I, \cdot)$ is continuous in m_{-I}. Thus, we can take a neighborhood of m_{-I} where $u_I(\overline{m}_I, \cdot)$ is at least $u_I(m) - \varepsilon$. ▌

Given that the two conditions are met, Corollary 5.2 of Reny (1999) implies that there exists a mixed strategy equilibrium, denoted m^*, whose support consists of non-decreasing bid functions. Now, we complete the proof by showing that I has a best response which is non-decreasing, which implies that m_I^* must be a best response overall.

Step 3: *When all other players play their equilibrium strategies of the restricted hypothetical game, player I has a best response strategy which is non-decreasing.*

Proof. As before, let $y_I(b)$ denote player I's probability of winning when bidding b. By Lemma 2,[9] the best response set $M_I(v_I) := \arg\max_{v_I^2 \leq b \leq v_I^1} y_I(b)[v_I^1 - b]$ is nonempty. Further, since $y_I(\cdot)$ nondecreasing, the objective function satisfies the single crossing property in (b, v_I). By Theorem 4 of Milgrom and Shannon (1994), then one can select a best response function, $\beta_I(v_I)$ that is non-decreasing in v_I. ▌

This last step implies that m^* is an equilibrium of the (unrestricted) hypothetical game. Furthermore, Lemma 3 guarantees that m^* is almost pure. Hence, there exists a pure strategy equilibrium β^*. Given the equilibrium, β^*, of the

[8] To see this point, suppose hypothetically that player I wins the auction whenever he makes the highest bid *even if a tie occurs* at that bid. Given this presumption, I's payoff is upper semi-continuous in his bid, so the maximum is well defined and is attained by a bidding function which is nondecreasing in v_I^i. The resulting maximum must constitute an upper bound for I's payoff (since he will not always win at a tie in the true game). This payoff can be arbitrarily closely approximated by modifying the bid function slightly to raise the bid at a tie and to avoid constant bids.

[9] While Lemma 2 establishes the continuity for the original game, the same proof applies to the restricted hypothetical game.

hypothetical game, one can construct the equilibrium strategies for the original game, as described in the main text. ▌

REFERENCES

Ash, R.B. (1972). Real Analysis and Probability. Academic Press, San Diego, CA.

Fudenberg, G., and J. Tirole (1991). Game Theory. MIT Press, Cambridge, MA.

Kim, J., and Y.-K. Che (2004). Asymmetric Information about Rivals' Types in Standard Auctions. *Games and Economic Behavior*, Vol. 46 (2), pp383-397.

Lebrun, B. (1999). First Price Auctions in the Asymmetric N Bidder Case. *International Economic Review*, Vol. 40 (1), 125-142.

Maskin, E., and J. Riley (2000). Equilibrium in Sealed High Bid Auctions. *Review of Economic Studies*, Vol.67 (3), 413-438.

Milgrom, P., and C. Shannon (1994). Monotone Comparative Statics. *Econometrica*, Vol. 62 (1), 157-180.

Myerson, R.B. (1981). Optimal Auction Design. *Mathematics of Operations Research*, Vol. 6 (1), 58-73.

Reny, P.J. (1999). On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games. *Econometrica*, Vol. 67 (5), 1029-1056.

Riley, J.G., and W. Samuelson (1981). Optimal Auctions. *American Economic Review*, Vol. 71 (3), 381-392.