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A Characterization of Equilibrium in Incomplete Markets with Real Assets*

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Abstract A closed-form solution of equilibrium outcomes is generally unavailable in a general equilibrium model with incomplete markets (GEI model). Competitive equilibrium may not be directly computable from the aggregate excess demand functions because they need not be continuous in GEI economies. This paper provides a new characterization of competitive equilibrium in the GEI model by putting into a new perspective the sudden shrinkage of risk-sharing opportunities at a spot price that makes some of the assets redundant and thus, may cause the discontinuity of demand functions. The new characterization is based on the notions of 'pre-GEI equilibrium' and 'test equilibrium'. Pre-GEI equilibrium outcomes yield a computational equivalent of competitive equilibrium while test equilibrium outcomes may provide existential information on equilibrium. Competitive equilibrium for the GEI model is generically computable as an outcome of pre-GEI equilibrium without resort to any element of the Grassmann manifold.

Keywords general equilibrium, incomplete markets, real assets, generic computability.

JEL Classification D52, C62, C63, C02

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1.INTRODUCTION

Economic decisions and transactions are sequentially made and frequently revised in the world with an incomplete system of contingent-claims markets as the state of nature is resolved over time. A general equilibrium model with incomplete markets (GEI model) differs from the classical Arrow-Debreu model in several respects. As demonstrated in Geanakoplos and Polemarchakis (1986), competitive markets may fail to attain efficiency even in a constrained sense in GEI economies.[1] The GEI model provides a rich environment for studying economic issues such as financial innovation, market crash, and welfare-improving economic policies that are hard to address in the classical complete-market framework. Moreover, Hart (1975) illustrates that competitive equilibrium with real assets may not exist under standard conditions such as the convexity and continuity of individual preferences.[2] The existence failure of competitive equilibrium is ascribed to the sudden shrinkage in marketed income spans at a price that makes some of the marketed assets redundant.

Radner (1972) is the first attempt to verify the existence of equilibrium with incomplete markets by imposing a lower bound on individual asset holdings.[3] Hart (1975) illustrates that equilibrium may fail to exist in unconstrained asset markets. The example of Hart (1975) has ignited research into the generic existence of equilibrium in incomplete markets. A full-fledged solution on this matter is found in the seminal work of Duffie and Shafer (1985) which leads to the flourish of the literature: to take a few, Geanakoplos and Shafer (1990), Hirsch et al. (1990), Husseini et al. (1990), Brown, DeMarzo and Eaves (1996), Bottazzi (1995, 2002), Zhou (1997), and Momi (2003). Duffie and Shafer (1985) develop a pseudo-equilibrium approach to the GEI model in which the full-rank span of the payoff matrix is identified as a point in the Grassmannian manifold.[4] When the asset payoffs are linear in spot prices, competitive equilibrium exists generically with respect to the initial endowments and the asset structure. Brown, DeMarzo and Eaves (1996) ingeniously reformulate the pseudo-equilibrium approach in a constructive way by developing a path-following algorithm for com-

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[1] Constrained efficiency is the notion of efficiency that is defined relative to the set of allocations attainable through a given set of asset markets.

[2] Assets are called real if the payoffs are linearly dependent on spot prices. Examples for real assets include stocks, commodity futures and forwards.

[3] As clarified in Hart (1975), lower bounds on asset holdings prevent the failure of the upper semicontinuity of budget correspondences.

[4] A manifold is a set which is locally diffeomorphic to a Euclidean space. For positive integers J and S, the Grassmannian $G_{J,S}$ is a manifold that consists of J-dimensional subspaces in $\mathbb{R}^S$.

puting competitive equilibrium in the GEI framework.

Immense need for computing equilibrium arises in the area of finance, macroeconomics or public finance that attempt to explain economic phenomena in the GEI framework which does not admit a closed-form solution for equilibrium outcomes in general. Competitive equilibrium may not be directly computable from the aggregate excess demand functions because they need not be continuous in GEI economies. This paper provides a new characterization of competitive equilibrium in the GEI model by putting into a new perspective the sudden shrinkage of risk-sharing opportunities at a spot price that makes some of the assets redundant. The new characterization is based on the notions of 'pre-GEI equilibrium' and 'test equilibrium'. Pre-GEI equilibrium outcomes yield a computational equivalent of competitive equilibrium while test equilibrium outcomes provide existential information on equilibrium. Competitive equilibrium for the GEI model is generically computable as an outcome of pre-GEI equilibrium without resort to any element of the Grassmann manifold. As described below, the properties of test equilibria carry critical information on the existence of equilibrium.

The discussion about pre-GEI and test equilibria involves building a sequence of test budget sets and pre-GEI budget sets. The test budget is built from replacing the payoff matrix by an artificial payoff matrix with constant rank at every spot price. To define a pre-GEI budget, we introduce a 'critical price domain' that consists of spot prices at which the rank of the payoff matrix drops, and a decreasing sequence of open neighborhoods whose intersection coincides with the critical price domain. For each neighborhood of the critical price domain, a pre-GEI budget set is built from smoothly pasting the original and test budget sets such that it offers the same set of income transfers in its neighborhood as the test budget set and elsewhere as the original budget set. Pre-GEI (test, resp.) equilibrium is defined as a collection of prices and individual optimal choices that satisfy the equilibrium conditions where each individual budget set is replaced by the pre-GEI (test, resp.) budget set. Both pre-GEI and test equilibria exist because the corresponding budgets are born to be continuous. It is worth noting that since the test and pre-GEI budget constraints can be differentiable up to any desired order without involving no element of the Grassmannian manifold, so can the corresponding demand functions in the whole price domain. This result is a distinct feature of the current approach compared to the literature that addresses the existence of GEI equilibrium based on the Grassmannian manifold.

If every test equilibrium price lies outside a neighborhood of the critical price domain, the economy has competitive equilibrium. In this case, equilibrium outcomes of the economy satisfy pre-GEI equilibrium conditions and thus, computing pre-GEI equilibrium amounts to computing GEI equilibrium. Another interesting result is that test equilibrium provides information about the nonexistence of equilibrium as well. The economy fails to have full-rank equilibrium when every sequence of test equilibrium prices converge to a price in the critical price domain. On the other hand, the result of Duffie and Shafer (1985) ensures that equilibrium exists generically with respect to the initial endowments and the asset structures. By combining the properties of pre-GEI equilibrium with the generic existence theorem of Duffie and Shafer (1985), we show that GEI equilibrium are generically computable as an outcome of pre-GEI equilibrium.

2. THE MODEL

A typical two-period GEI model is considered with finitely many agents, goods, and assets. There are a finite set of events $\mathcal{S} = \{1, \ldots, S\}$ in the second period. Agents consume L goods at date 0 and in each contingency of date 1. Let $\mathcal{I} = \{1, 2, \ldots, I\}$ denote the set of agents, $\mathcal{J} = \{1, 2, \ldots, J\}$ the set of financial assets, and $\mathcal{L} = \{1, 2, \ldots, L\}$ the set of consumption goods. Assets are traded in the first period (denoted by date 0) and make payoffs in the second period (denoted by date 1). The asset payoff may depend on the spot prices in each contingency of the second period. It is assumed that $J < S$, i.e., asset markets are incomplete. Let $\ell = S(L+1)$ and $\ell_1 = SL$. The set $\mathbb{R}^\ell$ indicates the space of state-contingent consumptions. The following notation is used.

$$P = \mathbb{R}^\ell_+, \, P^\circ = \mathbb{R}^\ell_{++}, \, P_1 = \mathbb{R}^{\ell_1}_+, \text{ and } P_1^\circ = \mathbb{R}^{\ell_1}_{++}.$$

Each $i \in \mathcal{I}$ is characterized by the consumption set P, the initial endowment $e_i \in P$ of goods, and the preferences represented by a utility function $u_i : P \to \mathbb{R}$. A vector $y \in \mathbb{R}^\ell$ is decomposed as $y = (y_0, y_1)$ where $y_0 = y(0) \in \mathbb{R}^L$ and $y_1 = (y(1), \ldots, y(S))$ is a collection of S vectors in $\mathbb{R}^L$. In particular, a consumption $x_i \in P$ indicates the collection of contingent consumptions $(x_{i0}, x_{i1}) = (x_i(0), x_i(1), \ldots, x_i(S))$. We set $e = (e_1, \ldots, e_I)$.

For a price $p = (p_0, p_1)$ in P, asset $j \in \mathcal{J}$ yields the payoff $r^j_s(p_1)$ in state s of date 1. The asset payoffs at p are summarized into the $S \times J$ matrix $R(p_1)$ which has $r^j_s(p_1)$ as the (s, j) th element. Let $r_s(p_1)$ and $r^j(p_1)$ denote the s th row and the j th column of the payoff matrix $R(p_1)$, respectively. The column $r^j(p_1)$ indicates the payoffs of asset j across the S states of date 1 while the row $r_s(p_1)$ the payoffs of the J assets in state s. Let $R_1(p_1)$ denote the $J \times J$ submatrix of $R(p_1)$ which consists of the first J rows of $R(p_1)$ and $R_2(p_1)$ the $(S-J)\times J$ submatrix which constitutes the rest of $R(p_1)$. Then $R(p_1)$ is decomposed as

$$R(p_1)=\begin{bmatrix}R_1(p_1)\\ R_2(p_1)\end{bmatrix}$$

When a portfolio $\theta \in R^J$ is taken at an asset price $q \in \mathbb{R}^J$ at date 0, it costs $q\cdot\theta$ at date 0 and yields an income transfer $R(p_1)\cdot\theta \in \mathbb{R}^S$ at date 1.

We assume that the asset structure is in the class of (primary) real assets. A real asset j is a contract which promises to deliver in each $s \in \mathcal{S}$ a vector of commodities $a^j(s)=(a_1^j(s),\ \ldots,\ a_L^j(s)) \in R^L$. For each $s \in \mathcal{S}$, let $a(s)$ denote $L\times J$ matrix with jth columun $a^j(s)$. The asset structure $a=(a(1),\ldots,a(S))$ will be often treated as a point in $\mathbb{R}^{LSJ}$. Real asset j pays income $r_s^j(p_1)=p(s)\cdot a^j(s)$ in state s that linearly depends on the spot price $p(s)$. In this case, it holds that $r_s(p_1)=p(s)\cdot a(s)$ for each $s\in\mathcal{S}$. A forward contract is a simple example of real assets. When forward contracts are available for each commodity, the forward markets give $a(s)=I_L$ for each $s\in\mathcal{S}$ where I_L indicates the $L\times L$ identity matrix, and have the payoff matrix[5]

$$R(p_1)=\begin{bmatrix}p_1(1) & \cdots & p_L(1)\\ \vdots & \ddots & \vdots\\ p_1(S) & \cdots & p_L(S)\end{bmatrix}.$$

Let $\mathcal{E}(e)=\langle(P,u_i,e_i)_{i\in\mathcal{I}},R(\cdot)\rangle$ denote the economy described above. The following is a list of assumptions imposed on the economy $\mathcal{E}(e)$.

Assumption 1. Each u_i is continuous, strictly increasing, and quasiconcave in P.[6]
Assumption 2. Each e_i is in P°.
Assumption 3. The asset structure consists of real assets.

Assumption 1 is a standard condition in the general equilibrium literature. Assumption 2 imposes a strong survival condition on the initial endowments for analytical simplicity. Assumption 3 requires that the payoff of assets is linear in spot prices. Thus, it excludes derivative assets with nonlinear payoffs such as options.[7]

[5] For other interesting classes of real assets, see Magill and Shafer (1991).

[6] The function u_i is strictly increasing if for any x,x' in P with $x-x'\in P$ and $x\neq x'$, $u_i(x)>u_i(x')$.

[7] Ku and Polemarchakis (1990) illustrate that the generic existence of equilibrium fails in the presence of options.

The following notation is used in defining the budget set.

$$p \,\square\, (x_i - e_i) = \begin{bmatrix} p(0) \cdot (x_i(0) - e_i(0)) \\ p(1) \cdot (x_i(1) - e_i(1)) \\ \vdots \\ p(S) \cdot (x_i(S) - e_i(S)) \end{bmatrix},$$

$$p \,\square_1 (x_i - e_i) = \begin{bmatrix} p(1) \cdot (x_i(1) - e_i(1)) \\ \vdots \\ p(S) \cdot (x_i(S) - e_i(S)) \end{bmatrix}, \text{ and } W(p_1, q) = \begin{bmatrix} -q \\ R(p_1) \end{bmatrix}.$$

For a given pair $(p,q) \in P \times \mathbb{R}^J$, agent i has the budget constraint and demand correspondence in the economy $\mathcal{E}(e)$ defined by

$$\begin{aligned} \mathcal{B}_i(p,q,e_i) &= \left\{ (x_i, \theta_i) \in P \times \mathbb{R}^J : p \,\square\, (x_i - e_i) \leq W(p_1, q) \cdot \theta_i \right\}, \\ \xi_i(p,q,e_i) &= \Big\{ (x_i^*, \theta_i^*) \in P \times \mathbb{R}^J : \\ &\qquad (x_i^*, \theta_i^*) \in \arg\max \left\{ u_i(x); (x_i, \theta_i) \in \mathcal{B}_i(p,q,e_i) \right\} \Big\}. \end{aligned}$$

On the other hand, the agent i has the demand correspondence in a complete-market economy defined by

$$\chi_i(p, p \cdot e_i) = \{ x_i^* \in P : x_i^* \in \arg\max \{ u_i(x); p \cdot (x_i - e_i) \leq 0 \} \}.$$

Equilibrium of the economy $\mathcal{E}(e)$ is defined as follows.

Definition 1. A list $(p,q,x,\theta) \in P^\circ \times \mathbb{R}^J \times P^I \times \mathbb{R}^{IJ}$ is a *GEI equilibrium* of $\mathcal{E}(e)$ if it satisfies the conditions

(*i*) $(x_i, \theta_i) \in \xi_i(p,q,e_i)$ for every $i \in \mathcal{I}$,

(*ii*) $\sum_{i \in \mathcal{I}} (x_i - e_i) = 0$, and

(*iii*) $\sum_{i \in \mathcal{I}} \theta_i = 0$.

The list (p,q,x,θ) is a *full-rank GEI equilibrium* if $R(p_1)$ has rank J.

The budget set $\mathcal{B}_i$ may lose continuity at spot prices at which the payoff matrix R falls short of full rank. The critical price domain is the set of the rank-dropping spot prices $C = \{p_1 \in P_1 : \rho(R(p_1)) < J\}$ where $\rho(A)$ indicates the rank of the matrix A. The set C contains prices at which some assets become redundant, and has an alternative expression

$$C = \{p_1 \in P_1^\circ : |R(p_1)'R(p_1)| = 0\}$$

where B' indicates the transpose of the matrix B and $|A|$ the determinant of the square matrix A. Let $\mathbb{N}$ denote the set of positive integers $\{1, 2, \ldots, \infty\}$. We introduce a sequence of open neighborhoods $\{C_\tau, \tau \in \mathbb{N}\}$ of C such that for each τ,

$$C_\tau = \left\{ p_1 \in P_1^\circ \,\middle|\, |R(p_1)'R(p_1)| < \frac{1}{\tau^2} \right\}.$$

It holds that $C_1 \supset C_2 \supset \cdots$ and $C = \bigcap_{\tau=1}^{\infty} C_\tau$, i.e., C_τ coincides in the limit with C.

The payoff matrix $R(p_1)$ has a $J \times J$ submatrix with rank J at each $p_1 \in P_1^\circ \setminus C$. To exploit this fact, let $\mathcal{C}_J$ denote the collection of J-element subsets $\{s_1, \ldots, s_J\}$ of $\mathcal{S}$ with $1 \le s_1 < s_2 < \cdots < s_J \le S$. Let Σ denote the set of permutations on $\{1, 2 \ldots, S\}$. For each $\sigma \in \Sigma$, let σ^{-1} denote the inverse permutation of σ and P_σ denote the $S \times S$ permutation matrix corresponding to σ. For each $\tilde{s} = \{s_1, \ldots, s_J\} \in \mathcal{C}_J$, let $R(p_1; \tilde{s})$ denote the $J \times J$ submatrix of $R(p_1)$ whose jth row coincides with the s_jth row of $R(p_1)$ for each $j \in \mathcal{J}$ and $R(p_1; \tilde{s}^c)$ denote the $(S-J) \times J$ submatrix that consists of the $(S-J)$ rows of $R(p_1)$ in $\mathcal{S} \setminus \tilde{s}$. Then there exists a permutation matrix $P_{\sigma(\tilde{s})}$ such that

$$P_{\sigma(\tilde{s})} R(p_1) = \begin{bmatrix} R(p_1; \tilde{s}) \\ R(p_1; \tilde{s}^c) \end{bmatrix}$$

For a positive integer τ, we define sets

$$\begin{aligned} C(\tilde{s}) &= \{p_1 \in P_1^\circ : |R(p_1; \tilde{s})| = 0\}, \\ C_\tau(\tilde{s}) &= \{p_1 \in P_1^\circ : -1/\tau < |R(p_1; \tilde{s})| < 1/\tau\}. \end{aligned}$$

To illustrate an intuition into the pre-GEI and test budget sets, we consider a special case that $R(p_1)$ is of the form (to be used in the examples of Section 3)

$$\begin{bmatrix} p_1(1) - p_2(1) \\ p_1(2) - p_2(2) \end{bmatrix}.$$

The matrix $R(p_1)$ has rank zero if $p_1(1) = p_2(1)$ and $p_1(2) = p_2(2)$, and otherwise, has rank 1. If $p_1(1) \neq p_2(1)$, $R(p_1)$ produces the same income span at date 1 as the following payoff matrix.

$$\begin{bmatrix} 1 \\ \frac{p_1(2) - p_2(2)}{p_1(1) - p_2(1)} \end{bmatrix}.$$

The new payoff matrix is not defined at p_1 with $p_1(1) = p_2(1)$. The dilemma does not occur to the following slight variant

$$\left[\begin{array}{c} 1 \\ f_\tau(p_1)(p_1(2) - p_2(2)) \end{array}\right],$$

where for each τ,

$$f_\tau(p_1) = \begin{cases} \tau^2(p_1(1) - p_2(1)), & \text{if } |p_1(1) - p_2(1)| < 1/\tau \\ \dfrac{1}{p_1(1) - p_2(1)}, & \text{if } |p_1(1) - p_2(1)| \geq 1/\tau \end{cases}.$$

It is worth noting that the τ-variant payoff matrix has full rank on P_1°. As τ increases, the price domain of disparity between $R(\cdot)$ and its τ-variant matrix is shrunk. The test budget set is built from replacing $R(\cdot)$ in $\mathcal{B}_i(\cdot)$ with the τ-variant payoff matrix.

To develop the above intuition in a full-fledged way, we introduce a function Φ_τ on P_1° such that

$$\Phi_\tau(p_1;\tilde{s}) = \begin{cases} \phi_\tau(|R(p_1;\tilde{s})|), & \text{if } p_1 \in C_\tau(\tilde{s}) \\ \dfrac{1}{|R(p_1;\tilde{s})|}, & \text{if } p_1 \in P_1^\circ \setminus C_\tau(\tilde{s}) \end{cases}$$

where for each $x \in \mathbb{R}$, $\phi_\tau(x)$ will indicate one of the three functions $\tau^2 x$, $\tau^2 x(2 - \tau^2 x^2)$, and $\tau^2 x(3 - 3\tau^2 x^2 + \tau^4 x^4)$, depending on the need for smoothness on the artificial budget sets. For each τ, the function

$$\psi_\tau(x) = \begin{cases} \phi_\tau(x), & \text{if } -1/\tau < x < 1/\tau \\ 1/x, & \text{otherwise} \end{cases}$$

is continuous, once or twice differentiable as $\phi_\tau(x)$ is chosen to be $\tau^2 x$, $\tau^2 x(2 - \tau^2 x^2)$, or $\tau^2 x(3 - 3\tau^2 x^2 + \tau^4 x^4)$, respectively.[8] Noting $\Phi_\tau(p_1;\tilde{s}) = \psi_\tau(|R(p_1;\tilde{s})|)$ for each τ, the smoothness of $\Phi_\tau(\cdot\,;\tilde{s})$ in P_1° depends on that of $\phi_\tau(\cdot)$ and $|R(\cdot\,;\tilde{s})|$. If $\phi_\tau(\cdot)$ and $|R(\cdot\,;\tilde{s})|$ are continuous, once or twice differentiable, so is $\Phi_\tau(\cdot\,;\tilde{s})$ accordingly. For each τ, we define two matrices

$$V_\tau^1(p_1;\tilde{s}) = \left[\begin{array}{c} I_J \\ \Phi_\tau(p_1;\tilde{s}) R(p_1;\tilde{s}^c) R^*(p_1;\tilde{s}) \end{array}\right]$$

[8] The complexity of ϕ_τ depends on the need for the smoothness of $\Phi_\tau(\cdot\,;\tilde{s})$. For example, if the continuity of $\Phi_\tau(\cdot\,;\tilde{s})$ serves well the goal at hand, then what we need is $\phi_\tau(x) = \tau^2 x$. When $\phi_\tau(x) = \tau^2 x(2 - \tau^2 x^2)$, $\Phi_\tau(\cdot\,;\tilde{s})$ is once differentiable.

and

$$V_\tau^2(p_1;\tilde{s}) = \begin{bmatrix} I_J \\ \phi_\tau(|R(p_1;\tilde{s})|)R(p_1;\tilde{s}^c)R^*(p_1;\tilde{s}) \end{bmatrix},$$

where I_J is the $J \times J$ identity matrix and A^* is the adjoint of the matrix A. It is worth noting that $V_\tau^1(\cdot;\tilde{s})$ and $V_\tau^2(\cdot;\tilde{s})$ coincide in $C_\tau(\tilde{s})$. The matrix $V_\tau^1(\cdot;\tilde{s})$ will become the payoff matrix for the pre-GEI budget set while $V_\tau^2(\cdot;\tilde{s})$ the payoff matrix for the test budget set.

For simplicity, we use the following short notation.

$$\bar{s} = \{1,2,\dots,J\}, C^J = C(\bar{s}), C_\tau^J = C_\tau(\bar{s}), \Phi_\tau(p_1) = \Phi_\tau(p_1;\bar{s}),$$
$$V_\tau^1(p_1) = V_\tau^1(p_1;\bar{s}), \text{ and } V_\tau^2(p_1) = V_\tau^2(p_1;\bar{s}).$$

Note that $\tilde{s}$ becomes $\bar{s}$ when $s_j = j$ for all $j = 1,\dots,J$. Recalling that $P_{\sigma(\bar{s})} = I_S$, $R_1(p_1) = R(p_1;\bar{s})$, and $R_2(p_1) = R(p_1;\bar{s}^c)$, we have

$$V_\tau^1(p_1) = \begin{bmatrix} I_J \\ \Phi_\tau(p_1)R_2(p_1)R_1^*(p_1) \end{bmatrix} \text{ and } V_\tau^2(p_1) = \begin{bmatrix} I_J \\ \phi_\tau(|R_1(p_1)|)R_2(p_1)R_1^*(p_1) \end{bmatrix}.$$

For each $(p,\tau,\tilde{s}) \in P \times \mathbb{N} \times \mathcal{C}_J$ and each $k = 1,2$, we introduce an artificial budget set $\mathcal{B}_{i,\tau}^k(p,e_i;\tilde{s})$ and demand correspondence $\xi_{i,\tau}^k(p,e_i)$ of agent i.

$$\mathcal{B}_{i,\tau}^k(p,e_i;\tilde{s}) = \Big\{(x_i,\theta_i) \in P \times \mathbb{R}^J \,:\, p \cdot (x_i - e_i) \le 0,$$
$$P_{\sigma(\tilde{s})} \cdot (p \,\square_1 (x_i - e_i)) = V_\tau^k(p_1;\tilde{s}) \cdot \theta\Big\}$$
$$\xi_{i,\tau}^k(p,e_i;\tilde{s}) = \Big\{x_i^* \in P \,:\, (x_i^*,\theta_i^*) \in \arg\max\{u_i(x);(x_i,\theta_i) \in \mathcal{B}_{i,\tau}^k(p,e_i;\tilde{s})\}\Big\}.$$

The set $\mathcal{B}_{i,\tau}^1(p,e_i;\bar{s})$ denotes the pre-GEI budget set and $\mathcal{B}_{i,\tau}^2(p,e_i;\bar{s})$ the test budget set of agent i. It is worth noting that they do not undergo the shrinkage of risk-sharing opportunities at prices in C. For each $k = 1,2$, the following short notation is used.

$$\mathcal{B}_{i,\tau}^k(p,e_i) = \mathcal{B}_{i,\tau}^k(p,e_i;\bar{s}) \quad \text{and} \quad \xi_{i,\tau}^k(p,e_i) = \xi_{i,\tau}^k(p,e_i;\bar{s}).$$

The above procedure involves transforming the budget set of the original GEI economy into the pre-GEI and test budget sets. In particular, the pre-GEI budget set is built to provide the same set of risk-sharing opportunities as the original budget set at prices outside C. Based on the characterization of the new budget sets, we provide the notions of pre-GEI equilibrium and test equilibrium for $\mathcal{E}(e)$.

Definition 2. A pair $(p,x) \in P^\circ \times P^I$ is a *pre-GEI equilibrium* (*test equilibrium*, resp.) of $\mathcal{E}(e)$ for a pair $(\tau,\tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$ if it satisfies the conditions

(i) $x_i \in \xi^1_{i,\tau}(p, e_i; \tilde{s})$ ($x_i \in \xi^2_{i,\tau}(p, e_i; \tilde{s})$, resp.) for every $i \in \mathcal{I}$, and

(ii) $\sum_{i \in \mathcal{I}}(x_i - e_i) = 0$.

Pre-GEI equilibrium arises when agents face the pre-GEI budget set in making choices. Both GEI and pre-GEI equilibria turns out to be equivalent in terms of risk-sharing opportunities when the payoff matrix has full rank at GEI equilibrium prices. Since the pre-GEI budget set is continuous on the price set, the computation of pre-GEI equilibrium does not suffer the discontinuity problem that can be faced in computing GEI equilibrium. The notions of equilibrium in Definition 2 are equivalent to the following definitions which turn out to be useful later in applying the Cass trick to verify the existence of pre-GEI and test equilibria.

Definition 2′. A pair $(p, x) \in P^\circ \times P^I$ is a *pre-GEI equilibrium* (*test equilibrium*, resp.) of $\mathcal{E}(e)$ for a pair $(\tau, \tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$ if it satisfies the conditions

(i) $x_1 \in \chi_1(p, p \cdot e_i)$,

(ii) $x_i \in \xi^1_{i,\tau}(p, e_i; \tilde{s})$ ($x_i \in \xi^2_{i,\tau}(p, e_i; \tilde{s})$, resp.) for every $i \neq 1$, and

(iii) $\sum_{i \in \mathcal{I}}(x_i - e_i) = 0$.

A pre-GEI equilibrium (p, x) for some $(\tau, \tilde{s})$ turns out to be a GEI equilibrium when p lies outside $C_\tau(\tilde{s})$. This fact is exploited in finding out GEI equilibrium from computing pre-GEI equilibrium. The notion of test equilibrium provides a preliminary information on the existence of GEI equilibrium in the economy. The following section illustrates how to exploit both equilibrium concepts in computing GEI equilibrium.

3. MAIN RESULTS

This section presents the main results of the paper by articulating the relationship between GEI, pre-GEI and test equilibria. First, we provide the existence theorem for pre-GEI and test equilibria. Then Comparison Theorem is presented to characterize full-rank GEI equilibrium in terms of pre-GEI equilibrium. Generic computability of GEI equilibrium is established by combining Comparison Theorem with the generic existence of Duffie and Shafer (1985).

3.1. COMPARISON THEOREM

It is shown below that the economy has a full-rank equilibrium if there exists a pair $(\tau,\tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$ for which $\mathcal{E}(e)$ has a pre-GEI equilibrium (p,x) in which p_1 lies outside $C_\tau(\tilde{s})$. The existence failure may occur when every sequence of pre-GEI equilibrium prices converges to a point in C. These results are built on the following existence theorem for pre-GEI and test equilibria.

Theorem 1. For each pair $(\tau,\tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$, the economy $\mathcal{E}(e)$ has pre-GEI equilibrium as well as test equilibrium.

PROOF : See Appendix.

Theorem 1 states that pre-GEI and test equilibria always exist under Assumptions 1 – 3. The result is in contrast to the generic existence of GEI equilibrium established in Duffie and Shafer (1985). The reason lies in the fact that the payoff matrix $V^1_\tau(p_1;\tilde{s})$ has the constant rank at each $(p_1,\tau,\tilde{s})$, which makes the budget set $\mathcal{B}^1_{i,\tau}(p,e_i;\tilde{s})$ change continuously in respond to price changes.

It is shown below that for some pair $(\tau,\tilde{s})$, pre-GEI and GEI equilibrium outcomes coincide in goods markets when pre-GEI equilibrium prices are outside the set $C_\tau(\tilde{s})$. The result provides a theoretical foothold for computing GEI equilibria.

Theorem 2. (Comparison Theorem) A point (p,q,x,θ) in $P^\circ \times \mathbb{R}^J \times P^I \times \mathbb{R}^{IJ}$ with $p_1 \in P^\circ_1 \setminus C$ is a full-rank GEI equilibrium of $\mathcal{E}(e)$ with $1_S = (1,1,\ldots,1) \in \mathbb{R}^S$ as equilibrium state prices of agent 1 if and only if for some $(\tau,\tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$, (p,x) is a pre-GEI equilibrium of $\mathcal{E}(e)$ with $p_1 \in P^\circ_1 \setminus C_\tau(\tilde{s})$.

PROOF : Let (p,q,x,θ) is a GEI equilibrium of $\mathcal{E}(e)$. Then for each $i \in \mathcal{I}$, (x_i,θ_i) satisfies the relation

$$p(0)\cdot(x_i(0)-e_i(0)) + q\cdot\theta_i = 0 \text{ and } p \,\square_1 (x_i - e_i) = R(p_1)\cdot\theta_i. \tag{1}$$

To show that $x_1 \in \chi_1(p, p\cdot e_1)$, we define sets

$$\langle W(p_1,q)\rangle = \{m \in \mathbb{R}^{S+1} : m = W(p_1,q)\cdot\gamma' \text{ for some } \gamma' \in \mathbb{R}^J\}$$
$$M(x_1) = \{m \in \mathbb{R}^{S+1} : m = p \,\square\, (x'_1 - e_1) \text{ for some } x'_1 \in P \text{ with } u_1(x'_1) > u_1(x_1)\}.$$

Since $p_1 \in P^\circ_1 \setminus C$, $\langle W(p_1,q)\rangle$ is the J-dimensional subspace in $\mathbb{R}^{S+1}$ spanned by the columns of $W(p_1,q)$. By Assumption 1, $M(x_1)$ is convex and open in

$\mathbb{R}^{S+1}$. The fact that $(x_1,\theta_1) \in \xi_1(p,q,e_1)$ implies $M(x_1) \cap \langle W(p_1,q)\rangle = \emptyset$. By the separating hyperplane theorem (Theorem 11.2 of Rockafellar (1970)), there exists a nonzero $\lambda = (\lambda_0,\lambda_1,\ldots,\lambda_S) \in \mathbb{R}^{S+1}$ such that for all $\gamma' \in \mathbb{R}^J$ and all $x_1' \in P$ with $u_1(x_1') > u_1(x_1)$,

$$\lambda \cdot (p \,\Box\, (x_1' - e_1)) > \lambda \cdot (W(p_1,q)\cdot\gamma'). \tag{2}$$

Since $p \,\Box\, (x_1 - e_1) = W(p_1,q)\cdot\theta_1 \in \langle W(p_1,q)\rangle$, it holds that $\lambda\cdot(p \,\Box\, (x_1'-e_1)) > \lambda(p \,\Box\, (x_1 - e_1))$ for all $x_1' \in P$ with $u_1(x_1') > u_1(x_1)$. Especially, the strict monotonicity of u_1 implies $\lambda_s > 0$ for all $s = 0,1,\ldots,S$. Moreover, (2) gives $\lambda\cdot(W(p_1,q)\cdot\gamma') = 0$ for all $\gamma' \in \mathbb{R}^J$, which yields $\lambda\cdot W(p_1,q) = 0$. It follows that $\lambda\cdot(p \,\Box\, (x_1'-e_1)) > 0$ for all $x_1' \in P$ with $u_1(x_1') > u_1(x_1)$. Since q and $p(s)$ can be rescaled into $\lambda_0 q$ and $\lambda_s p(s)$ for each $s \in \mathcal{S}$, without loss of generality, it can be assumed that $\lambda = (1,1_S)$. In this case, it holds that $p\cdot x_1' > p\cdot e_1$ for all $x_1' \in P$ with $u_1(x_1') > u_1(x_1)$. This implies that $x_1 \in \chi_1(p,p\cdot e_1)$.

Noting that $|R(p_1)'R(p_1)| > 0$, we can pick a positive integer τ' such that $|R(p_1)'R(p_1)| > (1/\tau')^2$. By the Binet-Cauchy formula,[9] it holds that

$$|R(p_1)'R(p_1)| = \textstyle\sum_{\tilde{s}\in\mathcal{C}_J} \left|R(p_1;\tilde{s})'R(p_1;\tilde{s})\right| > (1/\tau')^2.$$

This implies that there exists a positive integer τ and $\tilde{s}^* = (s_1^*,\ldots,s_J^*)$ such that

$$|R(p_1;\tilde{s}^*)'R(p_1;\tilde{s}^*)| > 1/\tau^2.$$

The states in $\mathcal{S}$ can be rotated into a new ordered scheme $\mathcal{S}'$ such that $s_j = s_j^*$ for each $j = 1,\ldots,J$ and elements in $\mathcal{S}'\setminus\tilde{s}^*$ match those in $\mathcal{S}\setminus\tilde{s}^*$ in the order-preserving way. Thus, without loss of generality, we can assume that $\tilde{s}^* = \bar{s} = (1,2,\ldots,J)$. Then p_1 is in $P_1^\circ \setminus C_\tau$. Now we set $\tilde{\theta}_i = R_1(p_1)\cdot\theta_i$. It holds that $(x_i,\tilde{\theta}_i)$ is in the pre-GEI budget set $\mathcal{B}^1_{i,\tau}(p_1,e_i)$ for each $i \neq 1$ because

$$\begin{aligned} R(p_1)\cdot\theta_i &= \begin{bmatrix} I_J \\ R_2(p_1)R_1^{-1}(p_1) \end{bmatrix}\cdot\tilde{\theta}_i \\ &= \begin{bmatrix} I_J \\ \frac{1}{|R_1(p_1)|}R_2(p_1)R_1^*(p_1) \end{bmatrix}\cdot\tilde{\theta}_i = V_\tau^1(p_1)\cdot\tilde{\theta}_i \end{aligned} \tag{3}$$

where $R_1^{-1}(p_1)$ indicates the inverse of $R_1(p_1)$. We claim that $x_i \in \xi^1_{i,\tau}(p,e_i)$. Suppose otherwise. Then there exists $(x_i',\theta_i') \in \mathcal{B}^1_{i,\tau}(p_1,e_i)$ with $u_i(x_i') > u_i(x_i)$.

[9] Brualdi and Cvetkovic (2009) is one reference on the Binet-Cauchy formula.

Since $q = 1_S \cdot R(p_1) = 1_S \cdot V_\tau^1(p_1)R_1(p_1)$, by setting $\gamma_i = R_1^{-1}(p_1) \cdot \theta_i'$, we see that

$$p(0) \cdot (x_i'(0) - e_i(0)) + q \cdot \gamma_i \leq 0 \text{ and } p \,\square_1 (x_i' - e_i) = R(p_1) \cdot \gamma_i. \tag{4}$$

This implies that $(x_i', \gamma_i) \in \mathcal{B}_i(p,q,e_i)$, which contradicts the optimality of (x_i, θ_i) in $\mathcal{B}_i(p,q,e_i)$.

Now we show the converse. For some $(\tau, \tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$, let (p,x) be a pre-GEI equilibrium of $\mathcal{E}(e)$ with $p_1 \in P_1^\circ \setminus C_\tau(\tilde{s})$. Without loss of generality, we will assume that $\tilde{s} = \bar{s}$. For each $i \geq 2$, let $\tilde{\theta}_i$ be a point in $\mathbb{R}^J$ which satisfies $p \,\square_1 (x_i - e_i) = V_\tau^1(p_1) \cdot \tilde{\theta}_i$. Setting $\theta_i = R^{-1}(p_1) \cdot \tilde{\theta}_i$ and reversing the previous arguments, we can show that $(x_i, \theta_i) \in \xi_i(p,q,e_i)$ for each $i \geq 2$ where $q = 1_S \cdot R(p_1)$. Now we set $\theta_1 = -\sum_{i \geq 2} \theta_i$. Since $p \,\square\, (x_1 - e_1) = -\sum_{i \geq 2}(p \,\square\, (x_i - e_i)) = W(p_1,q) \cdot \theta_1$, we also have $(x_1, \theta_1) \in \xi_1(p,q,e_1)$. □

The result of the comparison theorem provides a conceptual framework for computing GEI equilibria. Whenever $\mathcal{E}(e)$ has full-rank GEI equilibrium, by the comparison theorem it can be computed as a pre-GEI equilibrium of the economy $\mathcal{E}(e)$. In this case, pre-GEI equilibrium is a computational equivalent of full-rank equilibrium. The consequence of Theorem 2 leads to the following characterization of the existential failure for GEI equilibrium.

Corollary 1. Let $\{(p^\tau, x^\tau)\}$ be a sequence of pre-GEI equilibria for any given $\tilde{s} \in \mathcal{C}_J$. If every sequence $\{p^\tau\}$ converges to a point in C such that $p_1^\tau \in C_\tau(\tilde{s})$ for all τ, then the economy $\mathcal{E}(e)$ has no full-rank GEI equilibrium.

PROOF : Suppose that $\mathcal{E}(e)$ has a full-rank GEI equilibrium (p,q,x,θ). Since $p_1 \in P^\circ \setminus C$, by Theorem 2, there exist $\tilde{\tilde{s}} \in \mathcal{C}_J$ and $\tilde{\tau} \in \mathbb{N}$ such that (p,x) with $p_1 \in P_1^\circ \setminus C_{\tilde{\tau}}(\tilde{\tilde{s}})$ is a pre-GEI equilibrium of $\mathcal{E}(e)$. This contradicts the fact that $p_1 \in C_\tau(\tilde{s})$ for each $(\tau, \tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$. □

The consequences of Theorems 1 and 2 lead to the following criterion for the economy to possess full-rank GEI equilibrium by exploiting the properties of pre-GEI and test equilibria. For a pair $(\tau, \tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$, let $E^1(\tau, \tilde{s})$ denote the set of pre-GEI equilibria and $E^2(\tau, \tilde{s})$ the set of test equilibria. By Theorem 1, both $E^1(\tau, \tilde{s})$ and $E^2(\tau, \tilde{s})$ are not empty.

Theorem 3. If every (p', x') in $E^2(\tau, \tilde{s})$ satisfies $p_1' \in P_1^\circ \setminus C_\tau(\tilde{s})$ for some $(\tau, \tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$, then $\mathcal{E}(e)$ has a full-rank GEI equilibrium (p,q,x,θ) with $p_1 \in$

$P_1^\circ \setminus C_\tau(\tilde{s})$. More generally, full-rank GEI equilibria exist if $E^1(\tau,\tilde{s}) \setminus E^2(\tau,\tilde{s}) \neq \emptyset$ for some $(\tau,\tilde{s})$.

PROOF : By Theorem 1, $E^1(\tau,\tilde{s}) \neq \emptyset$ for each $(\tau,\tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$. Suppose that the economy has no full-rank GEI equilibrium (p,q,x,θ) which satisfies $p_1 \in P_1^\circ \setminus C_\tau(\tilde{s})$. Then by Theorem 2, every $(p,x) \in E^1(\tau,\tilde{s})$ must satisfy $p_1 \in C_\tau(\tilde{s})$. Since $V_\tau^1(\cdot;\tilde{s})$ and $V_\tau^2(\cdot;\tilde{s})$ coincide in $C_\tau(\tilde{s})$, every (p,x) is also a test equilibrium with $p_1 \in C_\tau(\tilde{s})$, which is impossible. Thus, $\mathcal{E}(e)$ has a full-rank GEI equilibrium. Let (p,x) be a point in $E^1(\tau,\tilde{s}) \setminus E^2(\tau,\tilde{s}) \neq \emptyset$. Since $p_1 \in P_1^\circ \setminus C_\tau(\tilde{s})$, the result of Theorem 2 ensures that (p,x) is a full-rank GEI equilibrium. $\square$

Theorem 3 shows that the property of test equilibrium can provide a sufficient condition for the existence of full-rank GEI equilibrium. Theorem 1 found an application in Example 2.

3.2. GENERIC COMPUTABILITY OF GEI EQUILIBRIUM

Comparison Theorem discussed above enables us to show that GEI equilibrium can be generically computed as an outcome of pre-GEI equilibrium. This result is built on the generic existence result of Duffie and Shafer (1985). To verify the generic computability of GEI equilibrium, the GEI economy is parameterized in terms of the initial allocation $e \in P^I$ and the real asset structure $a \in \mathbb{R}^{LSJ}$. For a pair $(e,a) \in P^I \times \mathbb{R}^{LSJ}$, we denote the underlying economy by $\mathcal{E}(e,a)$. To discuss the generic computability of GEI equilibrium, we need to impose a certain degree of differentiability on preferences.

Assumption 1a. Each $u_i : P^\circ \to \mathbb{R}$ is twice continuously differentiable (C^2) and satisfies the strict concavity, i.e., $v(D^2u_i(x_i))v < 0$ for all $v \neq 0$ in $\mathbb{R}^\ell$ and $x_i \in P^\circ$.

From now on, Assumption 1 is replaced by Assumption 1a. The following is a restatement of the main result of Duffie and Shafer (1985).

Duffie-Shafer Theorem: There exists an open set $\Omega \subset P^I \times \mathbb{R}^{LSJ}$ of Lebesgue-measure-zero complement such that a finite number of full-rank equilibria exist for each $(e,a) \in \Omega$.

To take advantage of Duffie and Shafer (1985) from the viewpoint of generic computatility, we provide a slight variation of Duffie-Shafer Theorem.

Theorem 4. There exists an open set $\Omega(\bar{s}) \subset P^I \times \mathbb{R}^{LSJ}$ with Lebesgue-measure-zero complement such that for each $(e,a) \in \Omega$, $\mathcal{E}(e,a)$ has finitely many full-rank equilibria (p,q,x,q)'s with $\rho(R_1(p_1)) = J$.

PROOF : The result is verified by slightly changing the arguments in step (d) of Proof of Theorem 2 in Duffie and Shafer (1985) in the following way. To make things clear, the asset structure is explicitly expressed in the payoff matrix. For a price $p \in P$ and an asset structure $a \in \mathbb{R}^{LSJ}$, let $R(p_1,a)$ denote the payoff matrix. Let $a_1 = (a(1),\dots,a(J))$ be a substructure of a in $\mathbb{R}^{J^2L}$ and $R_1(p_1,a_1)$ denote the $J \times J$ matrix whose sth row coincides with the sth row of $R(p_1,a)$ for each $s = 1,\dots,J$. Define a set

$$A_1 = \{(p,a) \in P^I \times \mathbb{R}^{LSJ} : \rho(R_1(p_1,a_1)) = J\}.$$

The matrix R_1 can be viewed as a map $R_1 : P^I \times \mathbb{R}^{LSJ} \to \mathbb{R}^{J^2}$. Let M denote the set of all $J \times J$ matrices of rank J in $\mathbb{R}^{J^2}$. It holds that $A_1 = R_1^{-1}(M)$ and $\partial R_1(p_1,a_1)/\partial a_1$ has maximum rank J^2. This implies that $A_1 = R_1^{-1}(M)$ is open in $P^I \times \mathbb{R}^{LSJ}$ with Lebesgue-measure-zero complement. The result of the theorem is verified by following just the arguments made in the rest of the proof of Duffie and Shafer (1985). $\square$

Theorem 4 gives a little stronger result for the income spanning of the asset structure in the first J states than Duffie-Shafer Theorem in that for every $(e,a) \in \Omega(\bar{s})$, $\mathcal{E}(e,a)$ has finitely many equilibria in which the submatrix $R_1(p_1,a_1)$ has rank J. It is worth noting that $\Omega(\bar{s}) \subset \Omega$ but the difference $\Omega \setminus \Omega(\bar{s})$ is a closed set of measure zero in $P^I \times \mathbb{R}^{LSJ}$. The result of Theorem 4 is built on the special choice $\bar{s}$ in $\mathcal{C}_J$. The same generic existence of Theorem 4 holds for any $\tilde{s}$ in $\mathcal{C}_J$. Let $\Omega(\tilde{s})$ denote the full-measure open set of economies with finitely many full-rank equilibria. By construction, we see that

$$\Omega = \bigcup\nolimits_{\tilde{s} \in \mathcal{C}_J} \Omega(\tilde{s}).$$

Theorems 2 and 4 lead to the generic computability of GEI equilibrium in terms of pre-GEI equilibrium.

Theorem 5. For each $(e,a) \in \Omega(\bar{s})$, full-rank GEI equilibria of the economy $\mathcal{E}(e,a)$ are obtained as an outcome of pre-GEI equilibrium.

We need only any single choice $\tilde{s}$ in $\mathcal{C}_J$ to show the generic computability of GEI equilibrium via pre-GEI equilibrium because $\Omega(\tilde{s})$ is an open set of full measure in $P^I \times \mathbb{R}^{LSJ}$ and thus, $\Omega \setminus \Omega(\tilde{s})$ is of measure zero. Here the choice is $\bar{s}$. By

Theorem 5, it is a non-generic event that a successful procedure for computing pre-GEI equilibria fails to find out a full-rank GEI equilibrium.

4. EXAMPLES: EXISTENTIAL INFORMATION FROM TEST EQUILIBRIA

This section gives two examples where information on the existence of GEI equilibrium is inferred from test equilibria. In the first example, a closed-form solution for test equilibria is available. There exists no GEI equilibrium when the test economies produce a sequence of equilibrium prices that converge to a price in the critical price domain C. Test equilibria in the second example display dependence on the choice of τ. As τ gets sufficiently large, test equilibrium reveals useful information on the existence of full-rank equilibrium. The examples at hand deal with a GEI economy with the characteristics $I=2, L=2, S=2$, and $J=1$ described as following.

$$\begin{array}{ll}
(a) & \text{Utility functions:} \\
& u_1(x) = \frac{1}{4}\log x_1(0) + \frac{1}{2}\log x_2(0) + \frac{1}{2}\left(\log x_1(1) + \log x_2(1)\right) \\
& \qquad + \frac{1}{2}\left(\frac{1}{2}\log x_1(2) + \log x_2(2)\right) \\
& u_2(x) = \frac{1}{4}\log x_1(0) + \frac{1}{2}\log x_2(0) + \frac{1}{2}\left(\log x_1(1) + 2\log x_2(1)\right) \\
& \qquad + \frac{1}{2}\left(\log x_1(2) + 2\log x_2(2)\right) \\
(b) & \text{Payoffs:} \\
& R(p_1) = \begin{bmatrix} p_1(1) - p_2(1) \\ p_1(2) - p_2(2) \end{bmatrix}
\end{array}$$

Example 1. It is assumed in the example that for a point $\varepsilon > -17/11$, agents have the initial endowments

$$e_1 = \begin{bmatrix} e_{11}(0) & e_{11}(1) & e_{11}(2) \\ e_{12}(0) & e_{12}(1) & e_{12}(2) \end{bmatrix} = \begin{bmatrix} 1 & 17/11+\varepsilon & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} e_{21}(0) & e_{21}(1) & e_{21}(2) \\ e_{22}(0) & e_{22}(1) & e_{22}(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}.$$

Note that $e_{11}(1)$ is a ε deviation from 17/11. It is shown below that no GEI equilibrium exists at $\varepsilon = 0$ because test equilibrium prices fall in the critical price domain C. When $\varepsilon \neq 0$, test equilibrium prices lie outside C. This information will confirm the existence of full rank GEI equilibrium at $\varepsilon \neq 0$. That is, full-rank equilibrium exists if pre-GEI equilibrium prices lie outside C. The next section demonstrates that the result holds in general.

In this example, we see that

$$P = \mathbb{R}^6_+, \quad P_1^\circ = \mathbb{R}^4_{++} \quad \text{and} \quad C = \{p_1 \in P_1^\circ : p_1(1) = p_2(1), p_1(2) = p_2(2)\}.$$

The payoff matrix has rank 0 in the set C. The example also has

$$\begin{aligned}
&C^1 = \{p_1 \in P_1^\circ : p_1(1) = p_2(1)\},\\
&C^1_\tau = \{p_1 \in P_1^\circ : |p_1(1) - p_2(1)| < 1/\tau\} \text{ for each } \tau,\\
&V^1_\tau(p_1) = \begin{bmatrix} 1 \\ \Phi_\tau(p_1)(p_1(2) - p_2(2)) \end{bmatrix},\\
&V^2_\tau(p_1) = \begin{bmatrix} 1 \\ \phi_\tau(|R_1(p_1)|)(p_1(2) - p_2(2)) \end{bmatrix},
\end{aligned}$$

where $\phi_\tau(|R_1(p_1)|) = \tau^2(p_1(1) - p_2(1))$ and

$$\Phi_\tau(p_1) = \begin{cases} \tau^2(p_1(1) - p_2(1)), & \text{if } p \in C^1_\tau \\ \dfrac{1}{p_1(1) - p_2(1)}, & \text{if } p \in P_1^\circ \setminus C^1_\tau \end{cases}.$$

(Since the continuity of $\phi_\tau(x)$ serves well in the example, $\phi_\tau(x)$ is set to be $\tau^2 x$.) For each τ, the test budget set is built from replacing the payoff $R(\cdot)$ in the original budget set with $V^2_\tau(\cdot)$.

For each τ, there exists a unique test equilibrium given by

$$\begin{aligned}
p^* &= \left(1, 1, 1, 1 + \frac{11\varepsilon}{30}, 1, 1\right)\\
x_1^* &= \left(\frac{8(45 + 22\varepsilon)}{420 + 165\varepsilon}, \frac{720 + 352\varepsilon}{420 + 165\varepsilon}, \frac{12}{11} + \frac{8\varepsilon}{15}, \frac{8(45 + 22\varepsilon)}{330 + 121\varepsilon}, \frac{2}{3}, \frac{4}{3}\right),\\
x_2^* &= \left(\frac{480 + 154\varepsilon}{420 + 165\varepsilon}, \frac{960 + 308\varepsilon}{420 + 165\varepsilon}, \frac{16}{11} + \frac{7\varepsilon}{15}, \frac{960 + 308\varepsilon}{330 + 121\varepsilon}, \frac{4}{3}, \frac{8}{3}\right).
\end{aligned}$$

The consumption choices x_1^* and x_2^* in test equilibrium are supported by the portfolio choices $\theta_1^* = -(40 + 33\varepsilon)/110$ and $\theta_2^* = (40 + 33\varepsilon)/110$ at $q^* = 44/(28 + 11\varepsilon)$, respectively. Note that the test equilibrium is independent of τ.[10] The price $p_2^*(1) = 1 + 11\varepsilon/30$ equals 1 at $\varepsilon = 0$. In this case, test equilibrium becomes pre-GEI equilibrium for each τ because $V^1_\tau(p_1^*) = V^2_\tau(p_1^*)$. This

[10] Such independence is related to the availability of a closed-form solution for test equilibrium. When test equilibrium depends on τ, it is hard to find out a closed-form solution of test equilibrium even in this simple example. The ε-perturbation of the initial endowments is intentionally chosen for test equilibrium to be independent of τ.

is the case that p_1^* falls in C_τ^1 for all τ, and, as shown below, no GEI equilibrium exists at $\varepsilon = 0$. For $\varepsilon \neq 0$, the test equilibrium price p_1^* lies outside C^1. In this case, GEI equilibrium exists.

Noting that $p_1^*(1) = p_1^*(2) = 1$, the economy has (p^*, x^*) as the unique pre-GEI equilibrium. Since $p_2^*(1) = 1 + 11\varepsilon/30$ is outside C_τ^1 for sufficiently large τ, the economy has GEI equilibrium (p^*, q, x^*, θ) where

$$q = -\frac{242\varepsilon}{420 + 165\varepsilon}, \quad \theta_1 = \frac{3(40 + 33\varepsilon)}{121\varepsilon}, \quad \text{and } \theta_2 = -\frac{3(40 + 33\varepsilon)}{121\varepsilon}.$$

(This result is formally stated in Theorem 2.) Note that as $\varepsilon \to 0$, $p_2^*(1) \to 1$ and θ_1 goes to infinity, and θ_1 is not defined at $\varepsilon = 0$. We claim that the economy $\mathcal{E}(e)$ has no GEI equilibrium at $\varepsilon = 0$. Suppose that it has an equilibrium (p', q', x', θ'). If $p_2'(1) \neq p_1'(1)$, it would hold that $p_1' \in P_1^\circ \setminus C_\tau^1$ for some τ, and $\langle V_\tau(p_1') \rangle = \langle R(p_1') \rangle$ where $\langle A \rangle$ denotes the income span by the matrix A. By Theorem 2, (p', x') is a pre-GEI equilibrium pair of price and consumption allocation. This contradicts the fact that the two goods in state 1 has the same price at the pre-GEI equilibrium price p^*. When $p_2'(2) \neq p_1'(2)$, we rotate state 1 and state 2 and apply the same arguments to get a contradiction. When $p_2'(1) = p_1'(1)$ and $p_2'(2) = p_1'(2)$, market demand for goods in state 1 does not satisfy the market clearing condition for the economy $\mathcal{E}(e)$. Consequently, $\mathcal{E}(e)$ has no GEI equilibrium at $\varepsilon = 0$.

The existence failure at $\varepsilon = 0$ is alluded from the behavior of θ_1 with $\varepsilon \to 0$. As ε moves across zero, the optimal portfolio of agents undergoes drastic change both in sign and size. Especially, the optimal portfolios are not defined at $\varepsilon = 0$. Eventually, no GEI equilibrium exists at $\varepsilon = 0$. This is also the case that the test equilibrium price p_1^* falls in C^1.

Example 2. In this example, the initial endowments are slightly changed to make test equilibrium depend on the choice of τ. To see the behavior of test and pre-GEI equilibria induced by tiny changes in $e_{11}(0)$ and $e_{22}(2)$, the initial endowments of agents are slightly changed as following.

$$e_1 = \begin{bmatrix} e_{11}(0) & e_{11}(1) & e_{11}(2) \\ e_{12}(0) & e_{12}(1) & e_{12}(2) \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{20} & \frac{17}{11} & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} e_{21}(0) & e_{21}(1) & e_{21}(2) \\ e_{22}(0) & e_{22}(1) & e_{22}(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 + \frac{1}{40} \end{bmatrix}.$$

It occurs that ε is set to 0, and $e_{11}(0)$ and $e_{22}(2)$ are increased by $1/20$ and $1/40$, respectively, compared to the initial endowments in Example 1. Interestingly, such small changes in the endowments do not allow a closed-form solution for equilibrium outcomes. Thus, $\tau = 10, 100, 1000$ are picked up to computationally figure out test equilibrium prices (p', q').

i) For the pair $(\tau, \tilde{s}) = (10, \{1\})$, there exists a unique test equilibrium price

$$(p', q') = ((1, 41/40, 1, 0.998961, 1, 160/161), 1.611726).$$

Since $|p'_1(1) - p'_2(1)| = |1 - 0.998961| < 1/10$, p'_1 is in C^1_{10}.

ii) For the pair $(\tau, \tilde{s}) = (100, \{1\})$, there exist three test equilibrium prices

$$\begin{aligned}(p', q') \in \{&((1, 41/40, 1, 0.996566, 1, 160/161), 1.944716),\\ &((1, 41/40, 1, 0.991567, 1, 160/161), 2.433877),\\ &((1, 41/40, 1, 0.989585, 1, 160/161), 2.629373)\}.\end{aligned}$$

Noting that $|p'_1(1) - p'_2(1)| < 1/100$ for the three p'_1's, they are in C^1_{100}.

iii) For the pair $(\tau, \tilde{s}) = (1000, \{1\})$, there exists a unique test equilibrium price

$$(p', q') = ((1, 41/40, 1, 0.959752, 1, 160/161), 404.0073).$$

Since it satisfies $|p'_1(1) - p'_2(1)| = 0.04 > 1/1000$, p'_1 is in $P^\circ_1 \setminus C^1_{1000}$.

Test equilibrium spot prices reside in the neighborhoods of the critical price domain at $\tau = 10, 100$. Existential information comes out at $\tau = 1000$ at which the unique test equilibrium price falls outside the narrower neighborhood C_{1000} of C. By Theorem 3, the result $p'_1 \in P^\circ_1 \setminus C^1_{1000}$ in step iii) implies that the economy has full-rank equilibrium.

Example 3. We attempt to compute GEI equilibrium in the economy of Example 2 which is shown to have GEI equilibrium. Since utility functions consists of log functions, we can find a system of polynomial equations that determine GEI equilibrium. First, we use normalized prices where the price of the first good is put to 1 in each state. Eventually, all the equations including the first-order conditions for utility maximization and the market clearing conditions for consumption goods and asset holdings are reduced to the five equations that determine the five equilibrium variables; the price $p^* = (p^*_0, p^*_1, p^*_2) \in \mathbb{R}^3_{++}$ of the second good in each state, the asset price $q^* \in \mathbb{R}$ and the asset holding $\theta^*_1 \in \mathbb{R}$ of agent 1 in equilibrium. The five equations are a polynomial function of the five variables.[11]

We use an algorithm based on the Gröbner basis to compute GEI equilibria that satisfy the five polynomial equations.[12] The algorithm yields a unique GEI equilibrium with $\theta_1^* = -27.3498$ and full-rank equilibrium price

$$(p_0^*, p_1^*, p_2^*, q^*) = (1.025, 0.989996, 0.993789, 0.025890).$$

5. CONCLUSION

The paper have attempted to characterize equilibrium of the GEI model in terms of pre-GEI equilibrium. Pre-GEI equilibrium always exists because the pre-GEI budget sets are built to change continuously or smoothly in response to price changes. Thus pre-GEI equilibrium does not suffer the non-generic existence failure that may occur to GEI equilibrium. Theorem 2 (Comparison Theorem) provides a foothold for computing GEI equilibrium as an outcome of pre-GEI equilibrium. Theorem 5 shows that GEI equilibrium is generically computable as an outcome of pre-GEI equilibrium. No attempt is made here to study an algorithmic implementation for computing pre-GEI equilibrium. It is left as a future research to develop a 'good' computational procedure for GEI equilibrium which takes an advantage of the characteristics of pre-GEI and test equilibria stated in Theorems 2 and 3. It is also an interesting and important task to extend the two-period computational procedure to stochastic OLG models, multi-period GEI models, and infinite-horizon GEI models, where prices of long-lived securities naturally enter the payoff matrix.

APPENDIX

The appendix provides the proof of Theorem 1. The existence result of Theorem 1 is based on the following lemma that verifies the continuity of test and pre-GEI demand correspondences.

[11]The original five equations are a rational function. They are transformed into a multivariate polynomial equation by multiplying both sides of the rational functions with the denominators. They are omitted from the paper to save space. It will be available upon request.

[12]We use the Mathematica command 'GroebnerBasis' to find a Gröbner basis for the multivariate system of polynomial equations. The Gröbner basis produces a simple system of polynomial equations that has the same set of solutions as the original polynomial system. For extensive discussions on the Gröbner basis approach , see Kubler and Schmedders (2010).

Lemma A. For each $i \in \mathcal{I}$ and each pair $(\tau, \tilde{s}) \in \mathbb{N} \times \mathcal{C}_J$, the correspondences $\xi^1_{i,\tau}(p, e_i; \tilde{s})$ and $\xi^2_{i,\tau}(p, e_i; \tilde{s})$ are nonempty, compact, convex-valued, and upper semicontinuous at each $p \in P^\circ$.

PROOF : Since the stated properties of $\xi^1_{i,\tau}(\cdot\,; \tilde{s})$ and $\xi^2_{i,\tau}(\cdot\,; \tilde{s})$ are proved in the same way for each $\tilde{s} \in \mathcal{C}_J$, we proceed only with $\xi^1_{i,\tau}(\cdot) = \xi^1_{i,\tau}(\cdot\,; \bar{s})$. We pick $\varepsilon > 0$ such that $p_l(s) > \varepsilon$ for all $l = 1, \ldots, L$ and $s = 0, 1, \ldots, S$. Then it is easy to see that $\mathcal{B}^1_{i,\tau}(p, e_i)$ is compact and thus, $\xi^1_{i,\tau}(p, e_i)$ is not empty. The set $\xi^1_{i,\tau}(p, e_i)$ is convex because u_h is quasiconcave. Let p^n be a sequence in P° which converges to p. For each n, we pick $x^n_i \in \xi^1_{i,\tau}(p^n, e_i)$. Then there exists θ^n_i such that $(x^n_i, \theta^n_i) \in \mathcal{B}^1_{i,\tau}(p^n, e_i)$. For sufficiently large n, we have $p^n_l(s) > \varepsilon$ for all l and s. Since $p^n \cdot x^n_i \leq p^n \cdot e_i$ for all n, this implies that $\{x^n_i\}$ is bounded. Since $V^1_\tau(\cdot)$ has rank J over P°, $\{\theta^n_i\}$ is bounded as well. Without loss of generality, we may assume that $(x^n_i, \theta^n_i) \to (x_i, \theta_i)$.

Clearly, (x_i, θ_i) is in $\mathcal{B}^1_{i,\tau}(p, e_i)$. We claim that $x_i \in \xi^1_{i,\tau}(p, e_i)$. Otherwise, we could choose $x'_i \in \xi^1_{i,\tau}(p, e_i)$. Let θ'_i be a point in $\mathbb{R}^J$ such that $(x'_i, \theta'_i) \in \mathcal{B}^1_{i,\tau}(p, e_i)$. Noting that $u_i(x'_i) > u_i(x_i)$, we choose $\lambda \in (0,1)$ such that $u_i(x^\lambda_i) > u_i(x_i)$ where $x^\lambda_i \equiv \lambda x'_i + (1-\lambda)e_i$. Since e_i is in P°, so is x^λ_i. For each τ, we define a function $\Psi_\tau : P^\circ \times P^\circ \times \mathbb{R}^J \to \mathbb{R}^{S+1}$ such that for each $(r, y, \gamma) \in P^\circ \times P^\circ \times \mathbb{R}^J$,

$$\Psi^1_\tau(r, y, \gamma) = \begin{bmatrix} r \cdot (y - e_i) \\ r \,\square_1 (y - e_i) - V^1_\tau(r_1)\gamma \end{bmatrix}$$

It holds that

$$\Psi^1_\tau(p, x^\lambda_i, \lambda\theta'_i) = 0. \tag{5}$$

Let $x_{(1)}$ denote the vector $(x_1(0), x_1(1), \ldots, x_1(S))$ where $x_1(s)$ indicates consumption of the first good in state $s = 0, 1, \ldots, S$. The matrix

$$\frac{\partial \Psi^1_\tau(p, x^\lambda_i, \lambda\theta'_i)}{\partial x_{(1)}} = \begin{bmatrix} p(0) & p(1) & \cdots & p(S) \\ 0 & p(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(S) \end{bmatrix}.$$

has rank $S+1$. Recalling that $x^\lambda_i \in P^\circ$, by the implicit function theorem, we pick $(y^n, \gamma^n) \in P^\circ \times \mathbb{R}^J$ such that $(y^n, \gamma^n) \to (x^\lambda_i, \lambda\theta'_i)$ and $\Psi^1_\tau(p^n, y^n, \gamma^n) = 0$. Knowing that $u_i(x^\lambda_i) > u_i(x_i)$, we have $u_i(y^n) > u_i(x^n_i)$ for sufficiently large n.

This contradicts the optimality of (x_i^n, θ_i^n) in $\mathcal{B}^1_{i,\tau}(p^n, e_i)$. Therefore, we conclude that $x_i \in \xi^1_{i,\tau}(p, e_i)$ and thus, $\xi^1_{i,\tau}(\cdot, e_i)$ is upper semi-continuous. $\square$

To show the existence of pre-GEI and test equilibria, we introduce normalized prices.

$$\Delta = \{p \in P : p \cdot e_1 = 1\} \quad \text{and} \quad \Delta^\circ = \{p \in P^\circ : p \cdot e_1 = 1\}.$$

For each pair $(\tau, \tilde{s})$ and each $k = 1, 2$, we define the excess demand correspondence $Z^k_\tau(\cdot\,; \tilde{s}) : P^\circ \times (P^\circ)^I \to \mathbb{R}^{I\ell}$ by

$$Z^k_\tau(p, e; \tilde{s}) = \chi_1(p, 1) - e_1 + \sum\nolimits_{i \geq 2} \left(\xi^k_{i,\tau}(p, e_i; \tilde{s}) - e_i \right).$$

It is clear that $Z^k_\tau(p, e; \tilde{s}) = 0$ if only and only if $p \in \Delta^\circ$ and

$$\chi_1(p, p \cdot e_1) - e_1 + \sum\nolimits_{i \geq 2} \left(\xi^k_{i,\tau}(p, e_i; \tilde{s}) - e_i \right) = 0.$$

Moreover, $Z^k_\tau(\cdot\,; \tilde{s})$ has the following properties.

i) $Z^k_\tau(p, e; \tilde{s})$ is compact, convex-valued, and upper semicontinuous at each $p \in \Delta^\circ$.

ii) For each $p \in \Delta^\circ$, it holds that $p \cdot Z^k_\tau(p, e; \tilde{s}) = 0$.

iii) Let $\{p^n\}$ be a sequence in Δ° such that $p^n \to \bar{p}$ for some $\bar{p}$ in the boundary of Δ. If $z^n \in Z^k_\tau(p^n, e; \tilde{s})$ for each n, then $\|z^n\| \to \infty$.

The first property comes from Lemma A. The property ii) is obvious because $p \cdot \chi_1(p, 1) = p \cdot e_1 = 1$ and $p \cdot \xi^k_{i,\tau}(p, e_i; \tilde{s}) = p \cdot e_i$ for each $i \geq 2$. The strict monotonicity of preferences and the fact that χ_1 is a complete-market demand correspondence lead to the property iii). By applying Theorem 8 of Debreu (1982) to the excess demand correspondence $Z^k_\tau(\cdot\,; \tilde{s})$, we conclude that the economy $\mathcal{E}(e)$ has pre-GEI (test, resp.) equilibrium.

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