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Group Contests and Technologies: Weakest-Link and Best-Shot Group Contests*

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Abstract

We study the role of returns to scale of production function in contests between groups. Two types of group contests are considered, the weakest-link group contest and the best-shot group contest. In each group contest, we investigate the existence and characteristics of the Nash equilibrium of the game, given different returns to scale of the production function for the contest. We find that, in both group contests, the constant returns and decreasing returns to scale bring the same nature of equilibrium. On the other hand, the increasing returns to scale causes the nonconcavity of players' payoff functions, complicates the analysis of the game, and makes changes in equilibrium properties.

Keywords Group contests; Public good; Technology; Group impact functions

JEL Classification C72; D70; H41

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1. INTRODUCTION

Contests among groups are ubiquitous around us. Inter-party competition to win votes, rivalries between interest groups to secure group profits, technology competition between research groups to get a patent, sporting contests between teams to win a medal, and music contests between girl/boy groups to gain popularity, just to name a few. These situations where groups compete for the prize are referred to as *group contests* in economics, and the study on group contests has been growing.

The seminal work of Katz et al. (1990) considers the group contest in which players within a group are symmetric, each group's winning probability is stochastically determined by the efforts of all the groups in the contest through a lottery contest success function (Tullock, 1980), and each group's effort is defined as the sum of the group members' efforts, i.e., the perfect-substitute group impact function is used. The impact function is a function that translates the efforts of the group members within a group into the group effort (W*ä*rneryd, 1988). The group contest of Katz et al. (1990) is extended by many researchers in various ways. Among them, Baik (1993, 2008) extends it to the case in which players within each group are not symmetric, i.e., they may have different valuations on the prize, and have the constant returns to scale of production function for the contest. In this setting, he finds that, at equilibrium, the highest-valuation players in each group expend efforts and the others free ride. In the same setting of Baik (1993, 2008), Epstein and Mealem (2009) adopt the decreasing returns to scale of the production function for the contest, and obtain different results. At the equilibrium of Epstein and Mealem (2009), the free riding does not occur and the players make more efforts in proportion to their valuations. On the contrary to Epstein and Mealem (2009), Lee (2015) takes on the increasing returns to scale of the production function, and finds another type of equilibrium where the players' efforts decrease in proportion to their valuations.

Note that the abovementioned works study the perfect-substitute contests between groups. That is, they use the perfect-substitute group impact function, assuming that the efforts of the individual members within a group are perfectly interchangeable. On the other hand, many researchers employ different types of group impact functions and study various group contests: Lee (2012), Kolmar and Rommeswinkel (2013), Chowdhury et al. (2013), Barbieri et al. (2014), Lee and Song (2015), Chowdhury and Topolyan (2016), Chowdhury et al. (2016), Chowdhury and Topolyan (2016), and Lee and Lee (2016). Among them, the works of Lee (2012) and Chowdhury et al. (2013) are closely related to the current paper. Lee (2012) considers the group contest in which each member in a group has his own essential role in producing the group performance, i.e., group members' efforts within the group are perfect complements. For example, the competition among research teams each of which consists of different fields of experts applies to this case. He uses the weakest-link group impact function and assumes the constant returns to scale of the production function in the contest, which implies that players have the constant marginal costs for exerting their efforts in the contest. In such a setting, he finds a multiplicity of Nash equilibria in the weakest-link contest. On the other hand, Chowdhury et al. (2013) employs the best-shot impact function in the contest in which the performance of a group is determined as the maximum of the group members' performances within the group. The competition among Research Joint Ventures, each of which consists of several firms sharing the results of a research project, has a feature of the best-shot contest. If one of the firms in a research joint venture makes the great technological innovation, then it strengthens the overall competitiveness of that research joint venture while other innovations made by the other firms are diluted. Assuming the constant returns to scale of the production function in the contest, they shows that there may exist the equilibrium in which the highest-valuation players in a group free ride on the low-valuation player in the group as well as the typical equilibrium in which only highest-valuation players in the groups exert efforts and the others free ride. Note that both of Lee (2012) and Chowdhury (2013) assume the constant returns to scale of production function, which implies that the players face constant marginal costs for exerting efforts in the contest. Then the natural question to ask is what is the role of the other returns to scale of the production function (decreasing and increasing returns to scale) in the weakest-link and best-shot group contest? To put it differently, what would happen to the equilibrium in the weakest-link and best-shot contest if the experts in the research teams and the firms in the research joint ventures face increasing marginal costs (decreasing returns to scale of the production function) or decreasing marginal costs (increasing returns to scale of the production function) for investing efforts? This question has not yet been investigated, and we try to answer it in this paper. This is the motivation and the contribution of the paper. Table 1 summarizes the fit of the current study in the related literature.

Our main findings are as follows. Unlike the perfect-substitute group contest in which different types of equilibrium appears accordingly to the constant returns and decreasing returns to scale of the production function, each weakest-link and best-shot group contest has the same nature of equilibrium regardless of whether its production function exhibits constant or decreasing returns. That is, the equilibrium properties is invariant to the constant and decreasing returns.

Table 1: The fit of the current study

Impact function / *Returns to Scale*	Perfect-Substitute	Weakest-Link	Best-Shot
Constant Returns	Baik (2008)	Lee (2012)	Chowdhury et al. (2013)
Decreasing Returns	Epstein and Mealem (2009)	**Section 3.1.2** in current study	**Section 3.2.2** in current study
Increasing Returns	Lee (2015)	**Section 3.1.3** in current study	**Section 3.2.3** in current study

However, the increasing returns to scale of the production function makes a difference in both the weakest-link and the best-shot contest. First, the increasing returns to scale causes the nonconcavity of the players' payoff functions in the contest, and complexifies the payoff-maximization problems for players. Therefore, our close attention must be drawn when we try to figure out the equilibrium in the weakest-link and best-shot contests with the increasing returns to scale of the production function. Second, it makes changes in equilibrium properties. There always exist multiple pure-strategy Nash equilibria in the weakest-link contest with the constant and decreasing returns. However, a unique equilibrium may exist in case of the increasing returns to scale. And, in the best-shot contest with the constant and decreasing returns, there always exists the equilibrium in which only highest-valuation player in a group expends effort and the others in the group free rides on him, and there may exist the other equilibrium in which the highest-valuation player free rides on the others in the group. On the contrary, in case of the increasing returns to scale, the latter equilibrium may exist even in the case where the former does not exist at all. Besides, we find that there may not exist any pure-strategy Nash equilibrium. We believe that these findings add relevant knowledge/understanding to the literature on group contests.

The paper proceeds as follows. In Section 2, our model is presented and we analyze it in Section 3. Finally, Section 4 concludes.

2. THE MODEL

We set up our model on the basis of the model of Lee (2015), while using the same terminology, terms, notations, assumptions, etc. n groups contest a group-specific public-good prize, where $n \geq 2$. Each group i consists of m_i risk-neutral players who exert effort to win the prize, where $m_i \geq 2$. Player k in group i puts his valuation v_{ik} on the prize and the valuations of players are assumed as follows.

Assumption 1. $v_{i1} \geq v_{i2} \geq \cdots \geq v_{im_i} > 0 \; \forall \; i = 1, \ldots, n.$

Player k in group i exerts effort and his non-negative effort level is denoted by e_{ik}. Through an *production function for the contest* $f(\cdot)$, the effort of player k in group i is transformed into his individual performance in the contest. Denoting the performance of player k in group i by x_{ik}, we define it as follows:

$$x_{ik} := f(e_{ik}) = e_{ik}^{\alpha} \tag{1}$$

where $\alpha > 0$ and it represents returns to scale for the production function. When $\alpha = 1$, the production function exhibits constant returns to scale (CRS). α less than 1 exhibits decreasing returns to scale (DRS), while α greater than 1 exhibits increasing returns to scale (IRS).

All the players have a common linear effort-cost function: $c(e_{ik}) = e_{ik}$.

Individual performances of the players within each group are mapped onto the performance of that group through a group impact function $F(\cdot)$. Denoting the performance of group i by X_i, we define it as follows:

$$X_i := F(x_{i1}, \ldots, x_{im_i}) = \min\{x_{i1}, \ldots, x_{im_i}\} \;\text{ or }\; \max\{x_{i1}, \ldots, x_{im_i}\}. \tag{2}$$

If $F(\cdot) = \min\{x_{i1}, \ldots, x_{im_i}\}$, our group contest becomes the weakest-link group contest studied in Lee (2012). And, if $F(\cdot) = \max\{x_{i1}, \ldots, x_{im_i}\}$, it becomes the best-shot group contest studied in Chowdhury et al. (2013).[1]

Group i wins the prize with probability p_i. The winning probability of each group depends on the performance of that group and the other groups' performances as well:

$$p_i(X_1, \ldots, X_n), \tag{3}$$

which satisfies the following regularity conditions for the contest success function.

Assumption 2. $0 \leq p_i \leq 1$, $\sum_{i=1}^{n} p_i = 1$, $p_i(0, \ldots, 0) = \frac{1}{n}$, $\frac{\partial p_i}{\partial X_i} \geq 0$, $\frac{\partial^2 p_i}{\partial X_i^2} \leq 0$, $\frac{\partial p_i}{\partial X_j} \leq 0$, $\frac{\partial^2 p_i}{\partial X_j^2} \geq 0$, $\frac{\partial p_i}{\partial X_i} > 0$ and $\frac{\partial^2 p_i}{\partial X_i^2} < 0$ for some $X_j > 0$, $\frac{\partial p_i}{\partial X_j} < 0$ and $\frac{\partial^2 p_i}{\partial X_j^2} > 0$ for $X_i > 0$, where $i \neq j$.

[1] If $F(\cdot) = \sum_{l=1}^{m_i} x_{il}$, the contest becomes the perfect-substitute group contest.

Assumption 2 indicates that the win probability of group i increases in the performance of that group and decreases in performances of the other groups at a decreasing rate. It also means that the group that makes the greatest performance in the contest does not surely win the prize as long as there exist at least two groups making positive group performances. The group with the greatest performance just has the highest probability of winning. That is, the continuous contest success function defined by Assumption 2 is imperfectly discriminating (Hillman and Riley, 1989). This type of contest success function has been widely employed in various fields such as economics of advertising (Schmalensee, 1978), rent-seeking (Tullock, 1980; Nizan, 1994), political campaigns (Baron, 1994), litigation (Hirshleifer and Osborne, 2001; Robson and Skaperdas, 2008), sports economics (Szymanski, 2003), and contests in general (Konrad, 2009), to name a few. The reason is because it satisfies set of intuitive axioms and properties necessary for the probabilistic choice function in contests (Skaperdas, 1996) and also provides analytical convenience. Especially, the ratio-form (Tullock-form) contest success function, one of the continuous imperfectly discriminating contest success functions, is the most widely used in sporting contests. Using data on the 4 major American sports leagues, Peeters (2011) estimated the ratio-form contest success function and found that the data fit well with it. On the other hand, the perfectly discriminating contest success function follows the all-pay auction rule, i.e., the group making the largest group performance in the contest wins the prize with probability one (Moldovanu and Sela, 2001). This type of contest success function is extremely sensitive to the group performances and it could be considered as the the extreme (limit) case for the imperfectly discriminating contest success function (Hilman and Riley, 1989).

The expected payoff for player k in group i is denoted by π_{ik}. Then the payoff function for player k in group i is defined as follows:

$$\pi_{ik} = v_{ik} p_i(X_1, ..., X_n) - e_{ik}. \tag{4}$$

All the players in the contest choose their effort levels independently and simultaneously. All of the above is common knowledge among the players, and we employ Nash equilibrium as our solution concept.

3. ANALYSIS OF THE MODEL

Depending on the types of the impact function and the production function, i.e., whether $F(\cdot) =$ min (weakest-link) or max (best-shot) and whether $\alpha = 1$

(CRS), $\alpha < 1$ (DRS), or $\alpha > 1$ (IRS), there appear 6 different types of group contests. We analyze each of them in the following subsections.

3.1. $F(\cdot) = $ MIN: THE WEAKEST-LINK GROUP CONTEST

3.1.1 $\alpha = 1$: the CRS production function

When $\alpha = 1$, group i's performance X_i is defined as $\min\{e_{i1}, \ldots, e_{im_i}\}$. Player k in group i chooses e_{ik} that maximizes his expected payoff

$$\pi_{ik} = v_{ik} p_i(X_i, X_{-i}) - e_{ik}, \tag{5}$$

where $X_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$.

To derive the best response of player k in group i to the other players' effort levels, we use the "imaginary" best response of the player. Let e^b_{ik} denote the imaginary best response of player k in group i, which means the best response of player k in group i when he is a unique player within the group. Namely, e^b_{ik} is the non-negative effort level that maximizes

$$\pi^b_{ik} = v_{ik} p_i(e_{ik}, X_{-i}) - e_{ik}, \tag{6}$$

given X_{-i}. Thus, the interior maximizer e^b_{ik} satisfies the following first-order condition for maximizing π^b_{ik}:

$$v_{ik} \frac{\partial p_i(e_{ik}, X_{-i})}{\partial e_{ik}} - 1 = 0. \tag{7}$$

The second-order condition for maximizing is satisfied from Assumption 2. By Assumption 1, the first-order condition (7) means that

$$e^b_{i1}(X_{-i}) \geq e^b_{i2}(X_{-i}) \geq \cdots \geq e^b_{im_i}(X_{-i}) \text{ for all } X_{-i}. \tag{8}$$

Let e^B_{ik} denote the best response of player k in group i to effort levels of all the other players in the contest. By using the imaginary best response e^b_{ik} and the nature of the weakest-link impact (minimum) function, we obtain the following best response of player k in group i:

$$e^B_{ik}(e_{-ik}, X_{-i}) = \begin{cases} e^b_{ik}(X_{-i}) & \text{for} \quad e^b_{ik}(X_{-i}) < \min\{e_{il}\}^{m_i}_{l=1(\neq k)} \\ \min\{e_{il}\}^{m_i}_{l=1(\neq k)} & \text{for} \quad e^b_{ik}(X_{-i}) \geq \min\{e_{il}\}^{m_i}_{l=1(\neq k)}, \end{cases} \tag{9}$$

where $e_{-ik} = (e_{i1}, \ldots, e_{ik-1}, e_{ik+1}, \ldots, e_{im_i})$.

Finally, from the best responses of the players in (9), we obtain the pure-strategy Nash equilibrium of the game. Let a $(\sum_{j=1}^{n} m_j)-$tuple vector of effort levels $(e_{11}^*, \ldots, e_{1m_1}^*, \cdots, e_{n1}^*, \ldots, e_{nm_n}^*)$ represent the Nash equilibrium of the game. The proposition below describes the equilibrium, which is found by Lee (2012).

Proposition in Lee (2012) *The pure-strategy Nash equilibrium in the weakest-link group contest with the CRS production function has the following properties:*

(a) The players in each group play the same strategy: $e_{i1}^* = e_{i2}^* = \cdots = e_{im_i}^* (\equiv e_i^*)$.

(b) There exist multiple pure-strategy Nash equilibria: e_i^* *can be any value that satisfies* $0 \le e_i^* \le e_{im_i}^b(X_{-i}^*)\ \forall\ i = 1, \ldots, n,$ *where* $X_{-i}^* = (X_1^*, \ldots, X_{i-1}^*, X_{i+1}^*, \ldots, X_n^*)$ *and* $X_i^* = F({e_{i1}^*}^{\alpha}, \ldots, {e_{im_i}^*}^{\alpha})$.

(c) Among the multiple equilibria, there is a unique coalition-proof Nash equilibrium. At that equilibrium, the players in each group play the strategy: $e_i^* = e_{im_i}^b(X_{-i}^*)$.

The proposition says that all the players within each group choose an equal effort level in equilibrium, and the lowest-valuation player in each group plays a decisive role in determining the Nash equilibria of the game, because, given the players' efforts in the other groups, his willingness to exert effort is the smallest within the group. There is no free riding problem in equilibrium.

3.1.2 $\alpha < 1$: the DRS production function

Group i's performance X_i is now defined as $\min\{e_{i1}^{\alpha}, \ldots, e_{im_i}^{\alpha}\}$. Player k in group i chooses e_{ik} that maximizes his expected payoff

$$\pi_{ik} = v_{ik} p_i(X_i, X_{-i}) - e_{ik}. \tag{10}$$

As in the previous section, we use the imaginary best response e_{ik}^b in order to derive the best response of player k in group i. e_{ik}^b is the effort level that maximizes

$$\pi_{ik}^b = v_{ik} p_i(e_{ik}^{\alpha}, X_{-i}) - e_{ik}. \tag{11}$$

Thus, the interior e_{ik}^b satisfies the following first-order condition for maximizing π_{ik}^b:

$$v_{ik} \frac{\partial p_i}{\partial (e_{ik}^{\alpha})} \frac{\alpha}{e_{ik}^{1-\alpha}} - 1 = 0. \tag{12}$$

The second-order condition for the maximization is as follows:

$$v_{ik}\alpha\left(\frac{\partial^2 p_i}{\partial(e_{ik}^{\alpha})^2}\frac{\alpha}{e_{ik}^{2(1-\alpha)}}-(1-\alpha)\frac{\partial p_i}{\partial(e_{ik}^{\alpha})}\frac{1}{e_{ik}^{2-\alpha}}\right)<0, \tag{13}$$

which is satisfied for any arbitrary value of $e_{ik}>0$ due to Assumption 2 and α less than 1. And, by Assumption 1, the first-order condition (12) implies that

$$e_{i1}^{b}(X_{-i})\geq e_{i2}^{b}(X_{-i})\geq\cdots\geq e_{im_i}^{b}(X_{-i}) \text{ for all } X_{-i}. \tag{14}$$

By using the imaginary best response e_{ik}^{b} and the nature of the weakest-link impact function, the best response of player k in group i is given as follows:

$$e_{ik}^{B}(e_{-ik},X_{-i})=\begin{cases} e_{ik}^{b}(X_{-i}) & \text{for} \quad e_{ik}^{b}(X_{-i})<\min\{e_{il}\}_{l=1(\neq k)}^{m_i} \\ \min\{e_{il}\}_{l=1(\neq k)}^{m_i} & \text{for} \quad e_{ik}^{b}(X_{-i})\geq\min\{e_{il}\}_{l=1(\neq k)}^{m_i}. \end{cases} \tag{15}$$

Finally, from the best responses of the players, we obtain the pure-strategy Nash equilibrium of the game. Let a $(\sum_{j=1}^{n}m_j)-$tuple vector of effort levels $(e_{11}^{*},\ldots,e_{1m_1}^{*},\cdots,e_{n1}^{*},\ldots,e_{nm_n}^{*})$ represent a Nash equilibrium. Proposition 1 describes the equilibrium.

Proposition 1. *The pure-strategy Nash equilibrium in the weakest-link group contest with the DRS production function has the following properties:*

(a) The players in each group play the same strategy: $e_{i1}^{*}=e_{i2}^{*}=\cdots=e_{im_i}^{*}(\equiv e_i^{*})$.

(b) There exist multiple pure-strategy Nash equilibria: e_i^{*} *can be any value that satisfies* $0\leq e_i^{*}\leq e_{im_i}^{b}(X_{-i}^{*})\ \forall\ i=1,\ldots,n$.

(c) Among the multiple equilibria, there is a unique coalition-proof Nash equilibrium. At that equilibrium, the players in each group play the strategy: $e_i^{*}=e_{im_i}^{b}(X_{-i}^{*})$.

(d) The equilibrium structure and characteristics are the same as those in the weakest-link group contest with the CRS production function.

The decreasing returns to scale of the production function gives us the same structure of imaginary best responses, and consequently the same structure of best responses of the players as those obtained in case of the constant returns to scale. As a result, the decreasing returns to scale of the production function has no effect on changing the nature of the Nash equilibrium of the weakest-link group contest with the constant returns to scale.

3.1.3 $\alpha > 1$: the IRS production function

Group i's performance X_i is defined as $\min\left\{e_{i1}^{\alpha},\ldots,e_{im_i}^{\alpha}\right\}$. Player k in group i chooses e_{ik} that maximizes his expected payoff

$$\pi_{ik} = v_{ik}p_i(X_i, X_{-i}) - e_{ik}. \tag{16}$$

As in the previous sections, we first try using the imaginary best-responses of the players in each group to derive the best responses of them and find the equilibrium of the game. Given X_{-i}, the imaginary best response of player k in group i, e_{ik}^b, is the effort level that maximizes

$$\pi_{ik}^b = v_{ik}p_i(e_{ik}^{\alpha}, X_{-i}) - e_{ik}. \tag{17}$$

Then the interior maximizer e_{ik}^b satisfies the following first-order condition for maximizing π_{ik}^b:

$$v_{ik}\frac{\partial p_i}{\partial(e_{ik}^{\alpha})}\alpha e_{ik}^{\alpha-1} - 1 = 0. \tag{18}$$

The second-order condition for the maximization is as follows:

$$v_{ik}\alpha\left(\frac{\partial^2 p_i}{\partial(e_{ik}^{\alpha})^2}\alpha e_{ik}^{2(\alpha-1)} + (\alpha-1)\frac{\partial p_i}{\partial(e_{ik}^{\alpha})}e_{ik}^{\alpha-2}\right) < 0, \tag{19}$$

which is not necessarily satisfied for any arbitrary value of $e_{ik} > 0$, because the sign of the first term within the bracket is negative, while the sign of the second one is positive. This implies that e_{ik}^b satisfying both the first-order condition (18) and the second-order condition (19) does not guarantee player k in group i the global maximum of π_{ik}^b. I.e., the solution e_{ik}^b for the first-order condition is surely the local maximizer, but not necessarily the global one. Therefore, we need to check if the interior maximizer e_{ik}^b obtained from the first-order and second-order conditions indeed gives the player the highest expected payoff or not, by comparing the payoff given by the interior maximizer e_{ik}^b and the payoff given by the boundary solution, 0. For this reason, when the production function exhibits increasing returns to scale ($\alpha > 1$), the imaginary best responses of the players derived from the first-order and second-order conditions may lead us to an erroneous way of constructing the best responses of the players and finding the Nash equilibrium of the game. In order to understand this more specifically and see the existence and structure of the Nash equilibrium in case of the IRS of the production function, we consider the following simple weakest-link group contest with $\alpha > 1$ as an example.

Two symmetric groups compete against each other, each group consists of two members, and the members within each group have the valuations on the prize, $k \geq 1$ and 1, respectively. I.e., $n = 2$, $m_1 = m_2 = 2$, $v_{11} = v_{21} = k$ (≥ 1), and $v_{12} = v_{22} = 1$. We use the Tullock-form contest success function (CSF) for our analysis:

$$p_i(X_1, X_2) = \frac{X_i}{X_1 + X_2} = \frac{\min\left\{e_{i1}^{\alpha}, e_{i2}^{\alpha}\right\}}{\min\left\{e_{11}^{\alpha}, e_{12}^{\alpha}\right\} + \min\left\{e_{21}^{\alpha}, e_{22}^{\alpha}\right\}} \text{ for } i = 1, 2.$$

In the weakest-link group contest, the players within each group choose an equal level of efforts in equilibrium, because the performance of each group is defined as the minimum of individual players' performances in the group. Namely, $e_{11} = e_{12}$ and $e_{21} = e_{22}$ hold at equilibrium. Using these necessary conditions for equilibrium, we examine the pure-strategy Nash equilibrium in the following numerical examples.

Example i. A symmetric 2-group-2-member case in which multiple pure-strategy Nash equilibria exist: $\alpha = 1.5$, $n = 2$, $m_1 = m_2 = 2$, $k = 1.2$.

Given these parameters, we first check if the $4-$tuple vector of effort levels $(e_{11}, e_{12}^K, e_{21}, e_{22}^K)$ constitutes a Nash equilibrium of the game or not, where the superscript K denotes the "key" players in each group and the key player in each group means the player within the group who plays a decisive role in determining the Nash equilibrium of the game. The strategy profile $(e_{11}, e_{12}^K, e_{21}, e_{22}^K)$ then implies that the low-valuation players in each group are the key players.

Since the key players in the contest are the low-valuation players in each group, we first compute e_{12}^K and e_{22}^K that satisfy the first-order and second-order conditions for maximizing the payoffs of the low-valuation players in each group, i.e., π_{12}^b and π_{22}^b. We then have $e_{12}^K = e_{22}^K = 0.375$.[2] And, from the necessary conditions for equilibrium, we have $e_{11} = e_{12}^K$ and $e_{21} = e_{22}^K$. We then have the following vector of effort levels as an equilibrium candidate:

$$(e_{11} = 0.375, e_{12}^K = 0.375, e_{21} = 0.375, e_{22}^K = 0.375).$$

For this strategy profile to be a Nash equilibrium, each player shouldn't have an incentive to change his effort level, given the other players' effort levels in this profile. We have checked the existence of deviation incentives of all the players (including the key players' deviation incentives, because, when $\alpha > 1$, each e_{12}^K

[2]However, note that when $\alpha > 1$ each solution e_{12}^K and e_{22}^K guarantees the local maximum, but not necessarily the global one. We need to check if each e_{12}^K and e_{22}^K is indeed the global maximizer, given the other players' effort levels.

and e_{22}^K satisfying the first-order and second-order conditions for maximizing guarantees the local maximum but not necessarily the global one) by plotting each player's expected payoff for given the other players' effort levels. We find that there is no such an incentive for the players. So, the above strategy profile constitutes a Nash equilibrium of the game.

In the same way, we propose the following two vectors of effort levels as our equilibrium candidates. The first one indicates that the high-valuation players in each group are the key players. The second one indicates that the high-valuation player is the key player in a group, while, in the other group, the low-valuation player is the key player:

$$(e_{11}^K = 0.45, e_{12} = 0.45, e_{21}^K = 0.45, e_{22} = 0.45),$$

$$(e_{11}^K = 0.441689619, e_{12} = 0.441689619, e_{21} = 0.368074683, e_{22}^K = 0.368074683).$$

We, for each of these strategy profiles, have checked whether each player has an incentive to change his effort level in the profile. We find that the low-valuation player (who is not the key player) in the group has the incentive to decrease his effort level, given the other players' effort levels, in both strategy profiles. Hence, the two strategy profiles do not constitute the Nash equilibrium of the game. This is intuitively understandable if we note that in a group the low-valuation player's willingness to exert effort is less than the high-valuation player's, given effort levels of the players in the other group. In other words, it means that in equilibrium the key player in each group should be the low-valuation player in that group. To sum up, we find that there exists the following pure-strategy Nash equilibrium which is the coalition-proof Nash equilibrium as well:

$$(e_{11}^* = 0.375, {e_{12}^*}^K = 0.375, e_{21}^* = 0.375, {e_{22}^*}^K = 0.375).$$

Along with this equilibrium, due to the perfect complementarity among the performances of the individual players within the group, there also exist many pure-strategy Nash equilibria as follows:

$$\left\{(e_{11} = e_1^*, e_{12} = e_1^*, e_{21} = e_2^*, e_{22} = e_2^*) \mid 0 \leq e_1^* \leq e_{12}^b(e_2^*) \text{ and } 0 \leq e_2^* \leq e_{22}^b(e_1^*)\right\}.$$

Note that the strategy profile $(e_{11}^* = 0, e_{12}^* = 0, e_{21}^* = 0, e_{22}^* = 0)$ constitutes a Nash equilibrium of the game.

Example ii. A symmetric 2-group-2-member case in which a unique pure-strategy Nash equilibrium exists: $\alpha = 2.5$, $n = 2$, $m_1 = m_2 = 2$, $k = 1.2$.

As in Example i, we check if the 4−tuple vector of effort levels $(e_{11}, e_{12}^K, e_{21}, e_{22}^K)$ constitutes a Nash equilibrium of the game. Solving the first-order conditions for maximizing the payoffs for the low-valuation players in each group, we have the following vector of effort levels as an equilibrium candidate:

$$(e_{11} = 0.625, e_{12}^K = 0.625, e_{21} = 0.625, e_{22}^K = 0.625).$$

For this strategy profile to be a Nash equilibrium, each player shouldn't have an incentive to change his effort level, given the other players' effort levels in the profile. We have checked the existence of the low-valuation player's deviation incentive by plotting his expected payoff for given the other players' effort levels, and find that there is indeed such an incentive to decrease his effort level to zero. This implies that the solution e_{12}^K and e_{22}^K satisfying the first-order and second-order conditions for maximizing the low-valuation players' payoffs are the local maximizers, not the global ones. Thus, the above strategy profile does not constitute a Nash equilibrium of the game.

Although there is no pure-strategy Nash equilibrium which is derived from the first-order and second-order conditions for maximizing the payoffs of the key players in each group, due to the perfect complementarity among the performances of the individual players within the group, there exists the following pure-strategy Nash equilibrium and it is a unique pure-strategy Nash equilibrium of the game:

$$(e_{11}^* = 0, e_{12}^* = 0, e_{21}^* = 0, e_{22}^* = 0).$$

So far we have examined two numerical examples for the symmetric two-group-two-member weakest-link contest. Through this exercise, we heuristically have gained a little understanding of the existence and the structure of the Nash equilibrium of the weakest-link group contest with the IRS production function. Based on our understanding and the finding of Pérez-Castrillo and Verdier (1992), we extend the symmetric two-group-two-member contest to the symmetric n-group-m_i-member case. I.e., we examine the case in which $n \geq 2$, $m_i \geq 2$, $v_{1m_1} = v_{2m_2} = \cdots = v_{nm_n} (\equiv v)$, and $p_i = \frac{X_i}{\sum_{j=1}^n X_j}$. The following lemma summarizes our results on this case. All the proofs in the paper are presented in the appendix.

Lemma 2. *The pure-strategy Nash equilibrium in the symmetric weakest-link contest has the following properties:*

(a) When $\alpha < 2$, there exist multiple pure-strategy Nash equilibria.

(b) When $\alpha \geq 2$, there exists a unique pure-strategy Nash equilibrium, $(0, \ldots, 0, \cdots, 0, \ldots, 0)$.

Lemma 2 says that there may exist multiple pure-strategy Nash equilibria or a unique Nash equilibrium, depending on the size of α, when the key players (the lowest-valuation players) in each group are symmetric. Then, what if the key players are not symmetric? To answer this, we consider an asymmetric two-group-two member case: $n = 2$, $m_1 = m_2 = 2$, $v_{12} = k > 1$, and $v_{22} = 1$.[3] Our results on this case are summarized in the following lemma. Lemma 3 shows that there may exist multiple or a unique pure-strategy Nash equilibrium, depending on the relative size of the key players' valuations (the structure of the valuations) as well as the size of α.

Lemma 3. *The pure-strategy Nash equilibrium in the asymmetric 2-group weakest-link contest has the following properties:*

(a) When $\alpha < 2$ and $k < \frac{1}{(\alpha-1)^{1/\alpha}}$, there exist multiple pure-strategy Nash equilibria.

(b) When $\alpha < 2$ and $k \geq \frac{1}{(\alpha-1)^{1/\alpha}}$ or $\alpha \geq 2$, there exists a unique pure-strategy Nash equilibrium, $(0,0,0,0)$.

From Lemma 2 and Lemma 3 we see that the increasing returns to scale of the production function in the weakest-link contest may result in different characteristics of the pure-strategy Nash equilibrium of the game in terms of the multiplicity of the equilibrium, comparing with the constant and decreasing returns to scale of production functions. The following proposition summarizes this.

Proposition 4. *The pure-strategy Nash equilibrium in the weakest-link group contest with the IRS production function has the following properties:*

(a) The players in each group play the same strategy: $e^*_{i1} = e^*_{i2} = \cdots = e^*_{im_i} (\equiv e^*_i)$.

(b) Depending on the value of α and the valuation structure of the key (lowest-valuation) players in each group, there may exist multiple pure-strategy Nash equilibria as in the CRS and DRS cases.

(c) However, unlike the CRS and DRS cases, there may exist a unique pure-strategy Nash equilibrium in which all the players choose zero effort levels, i.e., $e^*_i = 0\ \forall i$.

[3] We have tried to analyze the asymmetric $n > 2$ group case. However, we could not complete it even in the case of $n = 3$, due to the intricateness in solving the payoff-maximization problems for the key players whose valuations are different each other. We ask for the readers' and the reviewer's understanding on this.

3.2. $F(\cdot) = $ MAX: THE BEST-SHOT GROUP CONTEST

3.2.1 $\alpha = 1$: the CRS production function

Group i's performance X_i is now defined as the maximum function $\max\{e_{i1},\ldots, e_{im_i}\}$. Player k in group i seeks to maximize his expected payoff by choosing e_{ik}

$$\pi_{ik} = v_{ik} p_i(X_i, X_{-i}) - e_{ik}. \tag{20}$$

For getting the best responses of the players in the contest, we use the "imaginary" best responses of the players as we did in Section 3.1.1, and have the following fact that is the same as the one in (8):

$$e^b_{i1}(X_{-i}) \geq e^b_{i2}(X_{-i}) \geq \cdots \geq e^b_{im_i}(X_{-i}) \text{ for all } X_{-i}. \tag{21}$$

Using the imaginary best responses of the players, the property in (21), and the characteristics of the best-shot group impact function (maximum function), i.e., the performance of a group is determined by the maximum of the individual players' performances within that group, we obtain the best responses of the players in each group i:

$$\begin{aligned} &e^B_{ik}(e_{-ik}, X_{-i}) \qquad (22)\\ &= \begin{cases} 0 & \text{for } e^b_{ik}(X_{-i}) \leq \max\{e_{il}\}^{m_i}_{l=1(\neq k)} \\ \\ e^b_{ik}(X_{-i}) & \text{for } e^b_{ik}(X_{-i}) > \max\{e_{il}\}^{m_i}_{l=1(\neq k)} \text{ and} \\ & \pi_{ik}(e^b_{ik}(X_{-i}), e_{-ik}, X_{-i}) > \pi_{ik}(0, e_{-ik}, X_{-i}) \\ 0 & \text{for } e^b_{ik}(X_{-i}) > \max\{e_{il}\}^{m_i}_{l=1(\neq k)} \text{ and} \\ & \pi_{ik}(e^b_{ik}(X_{-i}), e_{-ik}, X_{-i}) \leq \pi_{ik}(0, e_{-ik}, X_{-i}). \end{cases} \end{aligned}$$

From the best responses of the players in (22), we obtain the pure-strategy Nash equilibrium of the game. Let a $(\sum_{j=1}^n m_j)-$tuple vector of effort levels $(e^*_{11},\ldots,e^*_{1m_1},\ldots,e^*_{n1},\ldots,e^*_{nm_n})$ represent a Nash equilibrium of the game. The proposition describes the equilibrium, which is examined in Chowdhury et al. (2013).

Proposition in Chowdhury et al. (2013) *The pure-strategy Nash equilibrium in the best-shot group contest with the CRS production function has the following properties:*

- *There exists an equilibrium in which the highest-valuation player in each group exerts effort and the others in that group free ride:*

(a) *For group i with $v_{i1} > v_{i2}$, the players play the strategies: $e^*_{i1} = e^b_{i1}(X^*_{-i})$ and $e^*_{il} = 0$ for $l = 2, \dots, m_i$.*

(b) *For group j with $v_{j1} = v_{jt} > v_{jt+1}$ for some t, the players play the strategies: $e^*_{jo} = e^b_{j1}(X^*_{-j})$ where $o = 1$ or 2 or $3 \cdots$ or t and $e^*_{jl} = 0$ for all $l \neq o$.*

- *There may exist an equilibrium in which the k_ith-highest-valuation player in each group exerts effort and the others in that group free ride:*

  (a) *For group i with $v_{i1} > v_{i2}$, the players play the strategies: $e^*_{ik_i} = e^b_{ik_i}(X^*_{-i})$ where $k_i \neq 1$ and $e^*_{il} = 0$ for all $l \neq k_i$.*

  (b) *For group j with $v_{j1} = v_{jt} > v_{jt+1}$ for some t, the players play the strategies: $e^*_{jk_j} = e^b_{jk_j}(X^*_{-j})$ where $k_j \neq 1, 2, \dots, t$ and $e^*_{jl} = 0$ for all $l \neq k_j$.*

The proposition implies that, in equilibrium, only a player in each group is active. The active player within the group is the highest-valuation player or may be one of the others. In other words, the equilibrium found in Baik (2008) always exists in the best-shot group contest with the CRS production function. In addition to this equilibrium, depending on the parameters within the model, there may also exist different type of equilibria in which the highest-valuation players in the group free ride.

3.2.2 $\alpha < 1$: the DRS production function

Group i's performance X_i is now defined as the maximum function $\max\left\{e^{\alpha}_{i1}, \dots, e^{\alpha}_{im_i}\right\}$. Player k in group i chooses e_{ik} that maximizes his expected payoff

$$\pi_{ik} = v_{ik} p_i(X_i, X_{-i}) - e_{ik}. \tag{23}$$

As in the previous section, we use the “imaginary” best responses in order to get the best-responses of the players in each group. e^b_{ik} is the effort level that maximizes

$$\pi^b_{ik} = v_{ik} p_i(e^{\alpha}_{ik}, X_{-i}) - e_{ik}. \tag{24}$$

Thus, e^b_{ik} satisfies the following first-order condition:

$$v_{ik} \frac{\partial p_i}{\partial (e^{\alpha}_{ik})} \frac{\alpha}{e^{1-\alpha}_{ik}} - 1 = 0^4. \tag{25}$$

Since α is less than 1, the first-order condition means that

$$e^b_{i1}(X_{-i}) \geq e^b_{i2}(X_{-i}) \geq \cdots \geq e^b_{im_i}(X_{-i}) \text{ for all } X_{-i}. \tag{26}$$

Using the imaginary best responses of the players, the property in (26), and the fact that the performance of a group is determined by the maximum of the individual players' performances within that group, we obtain the following the best responses of the players in group i:

$$e^B_{ik}(e_{-ik}, X_{-i}) \tag{27}$$

$$= \begin{cases} 0 & \text{for } e^b_{ik}(X_{-i}) \leq \max\{e_{il}\}^{m_i}_{l=1(\neq k)} \\ e^b_{ik}(X_{-i}) & \text{for } e^b_{ik}(X_{-i}) > \max\{e_{il}\}^{m_i}_{l=1(\neq k)} \text{ and} \\ & \pi_{ik}(e^b_{ik}(X_{-i}), e_{-ik}, X_{-i}) > \pi_{ik}(0, e_{-ik}, X_{-i}) \\ 0 & \text{for } e^b_{ik}(X_{-i}) > \max\{e_{il}\}^{m_i}_{l=1(\neq k)} \text{ and} \\ & \pi_{ik}(e^b_{ik}(X_{-i}), e_{-ik}, X_{-i}) \leq \pi_{ik}(0, e_{-ik}, X_{-i}). \end{cases}$$

From the best responses of the players in (27), we obtain the pure-strategy Nash equilibrium of the game. Let a $(\sum_{j=1}^n m_j)-$tuple vector of effort levels $(e^*_{11}, \ldots, e^*_{1m_1}, \ldots, e^*_{n1}, \ldots, e^*_{nm_n})$ represent a Nash equilibrium. Proposition 5 describes the equilibrium.

Proposition 5. *The pure-strategy Nash equilibrium in the best-shot group contest with the DRS production function has the following properties:*

- *There exists an equilibrium in which the highest-valuation player in each group exerts effort and the others in that group free ride:*
  - *(a) For group i with $v_{i1} > v_{i2}$, the players play the strategies: $e^*_{i1} = e^b_{i1}(X^*_{-i})$ and $e^*_{il} = 0$ for $l = 2, \ldots, m_i$.*
  - *(b) For group j with $v_{j1} = v_{jt} > v_{jt+1}$ for some t, the players play the strategies: $e^*_{jo} = e^b_{j1}(X^*_{-j})$ where $o = 1$ or 2 or $3 \cdots$ or t and $e^*_{jl} = 0$ for all $l \neq o$.*
- *There may exist an equilibrium in which the k_ith-highest-valuation player in each group exerts effort and the others in that group free ride:*

[4]The second-order condition for the maximum is satisfied as in Section 3.1.2.

(a) *For group i with $v_{i1} > v_{i2}$, the players play the strategies:* $e^*_{ik_i} = e^b_{ik_i}(X^*_{-i})$ *where* $k_i \neq 1$ *and* $e^*_{il} = 0$ *for all* $l \neq k_i$.

(b) *For group j with $v_{j1} = v_{jt} > v_{jt+1}$ for some t, the players play the strategies:* $e^*_{jk_j} = e^b_{jk_j}(X^*_{-j})$ *where* $k_j \neq 1, 2, \ldots, t$ *and* $e^*_{jl} = 0$ *for all* $l \neq k_j$.

- *The equilibrium structure and characteristics are the same as those in the the best-shot group contest with the CRS production function.*

As in the weakest-link group contest, the decreasing returns to scale of the production function gives us the same structure of "imaginary" best responses, and consequently the same structure of best responses of the players as those obtained in case of the constant returns to scale. Therefore, the decreasing returns to scale of the production function has no effect on changing the nature of the Nash equilibrium of the best-shot group contest with the constant returns to scale.

3.2.3 $\alpha > 1$: the IRS production function

Group i's performance X_i is defined as $\max\left\{e^\alpha_{i1}, \ldots, e^\alpha_{im_i}\right\}$. Player k in group i chooses e_{ik} that maximizes his expected payoff

$$\pi_{ik} = v_{ik} p_i(X_i, X_{-i}) - e_{ik}. \tag{28}$$

As before, we first try using the "imaginary" best responses of the players in each group to derive the best responses of them and find the equilibrium of the game. The imaginary best response of player k in group i, e^b_{ik}, is the effort level that maximizes

$$\pi^b_{ik} = v_{ik} p_i(e^\alpha_{ik}, X_{-i}) - e_{ik}. \tag{29}$$

Then the interior maximizer e^b_{ik} satisfies the following first-order condition for maximizing π^b_{ik}:

$$v_{ik} \frac{\partial p_i}{\partial (e^\alpha_{ik})} \alpha e^{\alpha-1}_{ik} - 1 = 0. \tag{30}$$

The second-order condition for the maximization is as follows:

$$v_{ik} \alpha \left(\frac{\partial^2 p_i}{\partial (e^\alpha_{ik})^2} \alpha e^{2(\alpha-1)}_{ik} + (\alpha - 1) \frac{\partial p_i}{\partial (e^\alpha_{ik})} e^{\alpha-2}_{ik} \right) < 0. \tag{31}$$

However, as in Section 3.1.3, the second-order condition is not necessarily satisfied for any arbitrary value of $e_{ik} > 0$, because the first term within the bracket has a negative sign but the second one has a positive sign. This implies that e_{ik}^b satisfying both the first-order condition (30) and the second-order condition (31) does not guarantee player k in group i the global maximum of π_{ik}^b. The solution e_{ik}^b for the first-order condition is surely the local maximizer, but not necessarily the global one. Therefore, we need to check if the interior maximizer e_{ik}^b obtained from the first-order and second-order conditions indeed gives the player the highest expected payoff. Therefore, when the production function exhibits increasing returns to scale ($\alpha > 1$), the imaginary best responses of the players derived from the first-order and second-order conditions may lead us to an erroneous way of finding the Nash equilibrium of the game. To understand this more specifically, we consider the following simple best-shot group contest with $\alpha > 1$ as an example.

Two symmetric groups compete against each other, each group consists of two members, and the members within each group have the valuations on the prize, $k \geq 1$ and 1, respectively. I.e., $n = 2$, $m_1 = m_2 = 2$, $v_{11} = v_{21} = k$ (≥ 1), and $v_{12} = v_{22} = 1$. We use the Tullock-form contest success function for our analysis:

$$p_i(X_1, X_2) = \frac{X_i}{X_1 + X_2} = \frac{\max\left\{e_{i1}^\alpha, e_{i2}^\alpha\right\}}{\max\left\{e_{11}^\alpha, e_{12}^\alpha\right\} + \max\left\{e_{21}^\alpha, e_{22}^\alpha\right\}} \quad \text{for} \quad i = 1, 2.$$

In the best-shot group contest, since each group's performance is determined by the maximum of the individual members' performances, only a player in each group would be active, i.e., choose a positive effort level, in equilibrium. So, we have the necessary conditions for equilibrium: $e_{i1}^* > 0$ and $e_{i2}^* = 0$ *or* $e_{i1}^* = 0$ and $e_{i2}^* > 0$ for $i = 1, 2$. Considering these conditions, we examine all the possible pure-strategy Nash equilibria in the following numerical examples.

Example iii. A symmetric 2-group-2-member case in which there exist all kinds of pure-strategy Nash equilibria: $\alpha = 1.5$, $n = 2$, $m_1 = m_2 = 2$, $k = 1.5$.

With these parameters, we first check if the $4-$tuple vector of effort levels $(e_{11}^A, e_{12}, e_{21}^A, e_{22})$ constitutes a Nash equilibrium of the game, where the superscript A denotes the "Active" players in each group and the active player(s) in each group means the player(s) within the group who chooses a strictly positive effort level. The strategy profile $(e_{11}^A, e_{12}, e_{21}^A, e_{22})$ then implies that the high-valuation players in each group are active in the contest.

Since the active players in the contest are the high-valuation players in each group, we first compute e_{11}^A and e_{21}^A that satisfy the first-order and second-order conditions for maximizing the payoffs of the high-valuation players in each group, i.e., π_{11}^b and π_{21}^b. We then have $e_{11}^A = e_{21}^A = 0.5625$.[5] And, from the necessary conditions for equilibrium, we have $e_{12} = e_{22} = 0$. We then have the following vector of effort levels as an equilibrium candidate:

$$(e_{11}^A = 0.5625, e_{12} = 0, e_{21}^A = 0.5625, e_{22} = 0).$$

For this strategy profile to be a Nash equilibrium, each player shouldn't have an incentive to change his effort level, given the other players' effort levels in this profile. We have checked the existence of deviation incentives of all the players (including the active players' deviation incentives) by plotting each player's expected payoff for given the other players' effort levels. We find that there is no such an incentive for the players. So, the above strategy profile constitutes a Nash equilibrium of the game.

In the same way, we propose the following two vectors of effort levels as our equilibrium candidates and check them. The first one indicates that the low-valuation players in each group are active and the other one indicates that the high-valuation player is active in group 1, while in group 2 the low-valuation player is active:

$$(e_{11} = 0, e_{12}^A = 0.375, e_{21} = 0, e_{22}^A = 0.375),$$

$$(e_{11}^A = 0.513528818, e_{12} = 0, e_{21} = 0, e_{22}^A = 0.342352545).$$

We, for each of these strategy profiles, have checked whether each player has an incentive to change his effort level in the profile, and have found that there is no such an incentive for the players. Therefore, the above strategy profiles constitute the Nash equilibrium. This is intuitively understandable if we note that the valuation of player 1 in each group is not too high relative to the one of player 2. So, although in each group the high-valuation player's willingness to exert effort is greater than the low-valuation player's, the difference between them is not big enough and this makes it possible that the high-valuation player in a group free ride on the low-valuation player in the group. We predict that this free riding of the high-valuation player will disappear as the valuation difference becomes larger. We show this in the following examples.

In sum, there exist the following pure-strategy Nash equilibria:

$$(e_{11}^{*A} = 0.5625, e_{12}^* = 0, e_{21}^{*A} = 0.5625, e_{22}^* = 0),$$

[5] However, note that when $\alpha > 1$ each solution e_{11}^A and e_{21}^A guarantees the local maximum, but not necessarily the global one. We need to check if each e_{11}^A and e_{21}^A is indeed the global maximizer, given the other players' effort levels.

$$(e_{11}^* = 0, e_{12}^{*\,A} = 0.375, e_{21}^* = 0, e_{22}^{*\,A} = 0.375),$$

$$(e_{11}^{*\,A} = 0.513528818, e_{12}^* = 0, e_{21}^* = 0, e_{22}^{*\,A} = 0.342352545).$$

Example iv. A symmetric 2-group-2-member case in which there exist pure-strategy Nash equilibria where the active players are the high-valuation players or the low-valuation players in each group: $\alpha = 1.5$, $n = 2$, $m_1 = m_2 = 2$, $k = 2.5$.

As we did in the previous example, given these parameters, we have proposed all the possible strategy profiles as our equilibrium candidates and have checked if each of them constitutes a Nash equilibrium of the game. We find that, in this example, there exist the following two symmetric equilibria exist:

$$(e_{11}^{*\,A} = 0.9375, e_{12}^* = 0, e_{21}^{*\,A} = 0.9375, e_{22}^* = 0)$$

and

$$(e_{11}^* = 0, e_{12}^{*\,A} = 0.375, e_{21}^* = 0, e_{22}^{*\,A} = 0.375).$$

Note that, comparing to the equilibria existing in Example iii, the asymmetric equilibrium, in which the active players in the contest are the high-valuation player in a group and the low-valuation player in the other group, does not exist in this example. This is because the valuation difference increased: $k = 1.5$ in Example iii to $k = 2.5$ now.

Example v. A symmetric 2-group-2-member case in which there exists a unique pure-strategy Nash equilibrium where the active players are the high-valuation players in each group: $\alpha = 1.5$, $n = 2$, $m_1 = m_2 = 2$, $k = 3.5$.

With these parameters, we find that there exists only the following symmetric Nash equilibrium:

$$(e_{11}^{*\,A} = 1.3125, e_{12}^* = 0, e_{21}^{*\,A} = 1.3125, e_{22}^* = 0).$$

Note that the valuation difference increased: $k = 2.5$ in Example iv to $k = 3.5$ in current example. Since the valuation difference is large enough, the equilibrium in which the high-valuation player in a group free rides on the low-valuation player does not exist.

Example vi. A symmetric 2-group-2-member case in which there is no pure-strategy Nash equilibrium: $\alpha = 2.5$, $n = 2$, $m_1 = m_2 = 2$, $k = 1.1$.

Given these parameters, we first check if the 4−tuple vector of effort levels $(e^A_{11}, e_{12}, e^A_{21}, e_{22})$ constitutes a Nash equilibrium of the game. Solving the first-order conditions for maximizing the payoffs for the high-valuation players, we have the following vector of effort levels as an equilibrium candidate:

$$(e^A_{11} = 0.6875, e_{12} = 0, e^A_{21} = 0.6875, e_{22} = 0).$$

For this strategy profile to be a Nash equilibrium, each player shouldn't have an incentive to change his effort level, given the other players' effort levels in this profile. We first have checked the existence of the high-valuation player's deviation incentive by plotting his expected payoff for given the other players' effort levels, and find that there is indeed such an incentive to decrease his effort level to zero. This implies that the solution e^A_{11} and e^A_{21} satisfying the first-order and second-order conditions for maximizing the high-valuation players' payoffs are the local maximizers, not the global ones. Thus, the above strategy profile does not constitute a Nash equilibrium of the game. In the same way, we propose the other possible vectors of effort levels as our equilibrium candidates. We, for each of these strategy profiles, have checked whether each player has an incentive to deviate from the proposed profiles. We find that these do not constitute the Nash equilibrium, either. So, we conclude that there is no Nash equilibrium, given these parameters.

So far we have examined several numerical examples of the symmetric two-group-two-member best-shot contest. Through this exercise, we heuristically have gained a little understanding of the existence and the structure of the Nash equilibrium of the best-shot group contest with the IRS production function. Based on this and the finding of Pérez-Castrillo and Verdier (1992), we extend the symmetric two-group-two-member contest to the symmetric n-group-m_i-member one: $n \geq 2$, $m_i \geq 2$, $v_{11} = v_{21} = \cdots = v_{n1} (\equiv v)$, and $p_i = \frac{X_i}{\sum_{j=1}^n X_j}$. Our results on this case are summarized in the following lemma.

Lemma 6. *The pure-strategy Nash equilibrium in the symmetric best-shot contest has the following properties:*

(a) When $\alpha < 2$, there exists an equilibrium in which only the highest-valuation players in groups exert effort and the others free ride. And there may also exist an equilibrium in which the highest-valuation players in the groups free ride on.

(b) When $\alpha \geq 2$, there is no pure-strategy Nash equilibrium.

Lemma 6 says that, depending on the size of α, there may or may not exist the pure-strategy Nash equilibrium in which the highest-valuation players in groups are the active ones. Note that that equilibrium always exists in the CRS and DRS cases. Then what if the highest-valuation players in each group are not symmetric? To answer this, we consider an asymmetric two-group-two member case: $n = 2$, $m_1 = m_2 = 2$, $v_{11} = k > 1$, and $v_{21} = 1$. Our results on this case are summarized in the following lemma.

Lemma 7. *The pure-strategy Nash equilibrium in the asymmetric 2-group best-shot contest has the following properties:*

- *(a) When $\alpha < 2$ and $k < \frac{1}{(\alpha-1)^{1/\alpha}}$, there exists an equilibrium in which the highest-valuation player in each group exerts effort and the other in that group free rides. There may also exist an equilibrium in which the highest-valuation player in a group free rides.*
- *(b) When $\alpha < 2$ and $k \geq \frac{1}{(\alpha-1)^{1/\alpha}}$ or $\alpha \geq 2$, there does not exist the equilibrium in which the highest-valuation player in each group exerts effort and the other in that group free rides. However, there may exist the other equilibrium in which the highest-valuation player in a group free rides.*

Lemma 7 shows that the equilibrium in which the highest-valuation player free rides in his group can still exist, even though there does not exist the equilibrium in which only the highest-valuation player exerts effort and the others in the group do nothing. This is in contrast to the equilibrium characteristics in the CRS and DRS cases which say that there always exists the equilibrium in which only the highest-valuation player exerts efforts in the group and there may exist the other equilibrium in which the highest-valuation player free rides.

Lemma 6 and 7 show the increasing returns to scale of the production function in the best-shot contest may result in different characteristics on the pure-strategy Nash equilibrium of the game, comparing with the CRS and DRS cases. The following proposition presents this.

Proposition 8. *The pure-strategy Nash equilibrium in the best-shot group contest with the IRS production function has the following properties:*

- *(a) Depending on the value of α and the valuation structure of the players, there may or may not exist the equilibrium in which only the highest-valuation player exerts effort and the others in the group free ride.*
- *(b) There may not exist any pure-strategy Nash equilibrium.*

(c) *There may exist the equilibrium in which the highest-valuation player free rides in the group, even though there does not exist the equilibrium in which the highest-valuation player in the group exerts effort and the others free ride.*

4. CONCLUSION

According to the literature on perfect-substitute contests between groups, the shape of the production function of individual players for the contest matters. Specifically, the CRS of the production function brings a Nash equilibrium in which only the highest-valuation players in each group makes efforts and the rest of the players have a free ride, while the DRS of the production function removes the free ride of the players in equilibrium. Each player exerts effort more in proportion to his valuation on the prize. The IRS of the production function brings an equilibrium in which each player expends effort less in proportion to his valuation.

In this paper, we have studied the matter of the production function in the weakest-link and best-shot group contests. We have obtained following main results. First, different from the perfect-substitute group contests, the DRS of the production function does not have any effect on changing the characteristics of the equilibrium of the weakest-link and best-shot group contests with the CRS. Namely, the CRS and DRS of the production function give us the same nature of equilibrium. Second, the IRS of the production function causes the nonconcavity of the players' payoff functions, and it complexifies the payoff-maximizing problems for the players and the way of deriving the best responses of the players. Hence, the close attention is needed to figure out the equilibrium of the weakest-link and best-shot group contests with the IRS. Third, the IRS makes changes in equilibrium properties. Specifically, in the weakest-link contest with the CRS and DRS, there always exist multiple pure-strategy Nash equilibria. However, in case of the IRS, a unique equilibrium may exist. In the best-shot contest with the CRS and DRS, there always exists the equilibrium in which only highest-valuation player in a group expends effort and the others in the group free rides on him, and then there may exist the other equilibrium in which the highest-valuation player free rides on the others in the group. On the contrary, in case of the IRS, the latter equilibrium may exist even in the case where the former does not exist at all. Besides, there may not exist any pure-strategy Nash equilibrium.

In this paper we have examined several numerical examples for the contests with the IRS and based on our heuristic findings in those examples, we found the equilibrium while assuming the symmetry among the lowest or highest valuation players and the restricted number of players. Is there any way of finding the equilibrium, if any, more formally/systematically rather than analyzing each case with specific parameter values or assumptions within the model? This question must be an interesting/meaningful one in the literature on group contests, although now it seems very challenging. We leave it for the future work.

Appendix

Proof of Lemma 2. Let us consider the condition under which there are λ active groups in equilibrium, where group i is called active if $X_i > 0$. That is, we consider the following strategy profile:

$$\left(e_{11},\ldots,e^K_{1m_1},\cdots,e_{\lambda 1},\ldots,e^K_{\lambda m_\lambda},0,\ldots,0,\cdots,0,\ldots,0\right),$$

where $e_{i1} = e_{i2} = \cdots = e^K_{im_i} > 0$ for all $i = 1,2,\ldots,\lambda$. Since the key players in the active groups are the lowest-valuation players in each group and they are assumed to be symmetric, i.e., $v_{1m_1} = \cdots = v_{\lambda m_\lambda} = v$, we have the symmetric solutions, $e^K_{1m_1} = e^K_{2m_2} = \cdots = e^K_{\lambda m_\lambda} = \frac{(\lambda-1)\alpha v}{\lambda^2}$, that satisfy the first-order and second-order conditions for maximizing the payoffs of all the key players in the active groups. The above strategy profile is then

$$\left(\frac{(\lambda-1)\alpha v}{\lambda^2},\ldots,\frac{(\lambda-1)\alpha v}{\lambda^2},\cdots,\frac{(\lambda-1)\alpha v}{\lambda^2},\ldots,\frac{(\lambda-1)\alpha v}{\lambda^2},0,\ldots,0,\cdots,0,\ldots,0\right).$$

For this strategy profile to be a Nash equilibrium, every player (including the key players in the active groups) shouldn't have any incentive to change his effort level. First, we consider the non-deviation condition for the key players in the active groups. At the strategy profile above, the key player in each active group has his payoff:

$$\pi^*_{im_i} = v\frac{1}{\lambda} - \frac{(\lambda-1)\alpha v}{\lambda^2} = \frac{(\lambda-\alpha(\lambda-1))v}{\lambda^2} \quad \text{for } i = 1,2,\ldots,\lambda.$$

Denoting $\pi^d_{im_i}(e_{im_i})$ by the payoff the key player obtains when he changes his effort level to the boundary $e_{im_i} = 0$ and $e_{im_i} = \infty$, we have:

$$\pi^d_{im_i}(e_{im_i} = 0) = 0 \ \text{ and } \ \pi^d_{im_i}(e_{im_i} = \infty) = -\infty.$$

So, we obtain the following non-deviation condition for the key players in the active groups:

$$\pi^*_{im_i} > \pi^d_{im_i}(e_{im_i} = 0) \Leftrightarrow \alpha < \frac{\lambda}{\lambda-1}.$$

Second, we consider the deviation incentives for the other players, except for the key player, in each active group. Since the key player has the lowest valuation in the group, the valuations of the other players are greater than or at least equal to the valuation of the key player, and it means that their willingness to exert effort, given the effort levels of the players in the other groups, is greater than or at least equal to the key player's one. Thus, they do not have any incentive to change (decrease) their effort levels in the profile above. Last, we consider the deviation incentives for the players in the inactive groups. In each inactive group, all the players choose zero effort level. Then any player doesn't have an incentive to change (increase) his effort level, due to the perfect complementarity among the players within the group. In short, if $\alpha < \frac{\lambda}{\lambda-1}$, no player in the contest has an unilateral incentive to deviate from the above strategy profile, and the strategy profile constitutes the Nash equilibrium, in which λ groups actively participate in the contest and the rest $(n-\lambda)$ groups are inactive.

In addition to this equilibrium, if $\alpha < \frac{\lambda}{\lambda-1}$, there also exist the Nash equilibrium in which less than λ number of active groups exist. Namely, λ represents the maximum number of active groups that can exist in equilibrium. Specifically, we have the following:

- If $1 < \alpha < \frac{n}{n-1}$, there exist the Nash equilibria in which there are maximally n active groups.
- If $\frac{n}{n-1} \leq \alpha < \frac{n-1}{(n-1)-1}$, there exist the Nash equilibria in which there are maximally $(n-1)$ active groups.
- If $\frac{n-1}{(n-1)-1} \leq \alpha < \frac{n-2}{(n-2)-1}$, there exist the Nash equilibria in which there are maximally $(n-2)$ active groups.

$$\vdots$$

- If $\frac{3}{2} \leq \alpha < 2$, there exists the Nash equilibrium in which there are 2 active groups.
- If $2 \leq \alpha$, there does not exist the Nash equilibrium in which there are at least 2 active groups. In this case, there exists a unique pure-strategy Nash equilibrium in which all the groups are inactive, i.e., $\left(e^*_{11} = 0, ..., e^*_{1m_1} = 0, \cdots, e^*_{n1} = 0, ..., e^*_{nm_n} = 0\right)$.

■

Proof of Lemma 3. Given our restrictions, we have the following strategy profile:

$$(e_{11}, e^K_{12}, e_{21}, e^K_{22}) = \left(\frac{\alpha k^{\alpha+1}}{(k^\alpha+1)^2}, \frac{\alpha k^{\alpha+1}}{(k^\alpha+1)^2}, \frac{\alpha k^{\alpha}}{(k^\alpha+1)^2}, \frac{\alpha k^{\alpha}}{(k^\alpha+1)^2}\right),$$

where e^K_{12} and e^K_{22} are the solutions that satisfy the first-order and the second-order conditions for maximizing the payoffs for the key players in each group. We check if the above strategy profile constitutes the Nash equilibrium of the game. First, we consider the deviation incentives for the key players. At the above strategy profile, the key players in each group have their payoffs:

$$\pi^*_{12} = k\frac{k^\alpha}{k^\alpha+1} - \frac{\alpha k^{\alpha+1}}{(k^\alpha+1)^2} = \frac{k^{\alpha+1}(k^\alpha+1-\alpha)}{(k^\alpha+1)^2}$$

and

$$\pi^*_{22} = 1\frac{1}{k^\alpha+1} - \frac{\alpha k^\alpha}{(k^\alpha+1)^2} = \frac{1+k^\alpha-\alpha k^\alpha}{(k^\alpha+1)^2}.$$

Denoting $\pi^d_{i2}(e_{i2})$ by the payoff the key player of group i obtains when he changes his effort level to the boundary $e_{i2}=0$ and $e_{i2}=\infty$, we have:

$$\pi^d_{i2}(e_{i2}=0)=0 \;\text{ and }\; \pi^d_{i2}(e_{i2}=\infty)=-\infty \;\text{ for }\; i=1,2$$

So, we obtain the following non-deviation condition for the key players in each group:

$$\pi^*_{i2} > \pi^d_{i2}(e_{i2}=0) \;\text{ for all }\; i=1,2 \Leftrightarrow \alpha<2 \;\text{ and }\; k<\frac{1}{(\alpha-1)^{1/\alpha}}.$$

Next, we consider the deviation incentive for the other player, except for the key player, in each group. Since the key player has the lowest valuation in each group, the valuation of the other player in the group is greater than or at least equal to the valuation of the key player. It means that his willingness to exert effort, given the effort levels of the players in the other group, is greater than or at least equal to the key player's one. Thus, he does not have any incentive to change (decrease) his effort level in the profile above. Hence, if $\alpha<2$ and $k<\frac{1}{(\alpha-1)^{1/\alpha}}$, then no player in the contest has an unilateral incentive to deviate from the above strategy profile, and the strategy profile constitutes the Nash equilibrium. Otherwise, i.e., $\alpha<2$ and $k\geq\frac{1}{(\alpha-1)^{1/\alpha}}$ or $\alpha\geq 2$, this equilibrium disappears. In these cases, there exists a unique pure-strategy Nash equilibrium, $(0,0,0,0)$. ■

Proof of Lemma 6. Let us consider the condition under which there are λ active groups in equilibrium and the active player in each active group is the highest-valuation player in that group. That is, we consider the following strategy profile:

$$\left(e^A_{11},0,\ldots,0,e^A_{21},0,\ldots,0,\cdots,e^A_{\lambda 1},0,\ldots,0,\quad 0,\ldots,0,\cdots,0,\ldots,0\right).$$

Since the active players in the active groups are the highest-valuation players in each group and they are assumed to be symmetric, i.e., $v_{11} = \cdots = v_{\lambda 1} = v$, we have the symmetric solutions, $e^A_{11} = e^A_{21} = \cdots = e^A_{\lambda 1} = \frac{(\lambda-1)\alpha v}{\lambda^2}$, that satisfy the first-order and second-order conditions for maximizing the payoffs of all the active players in the active groups.

For the above strategy profile to be a Nash equilibrium, every player (including the active players in the active groups) shouldn't have any incentive to change his effort level. First, we consider the non-deviation condition for the active players in each active groups. At the strategy profile above, the active player in each active group has his payoff:

$$\pi^*_{i1} = \frac{(\lambda - \alpha(\lambda-1))v}{\lambda^2} \ \text{ for } \ i = 1, 2, \ldots, \lambda.$$

Denoting $\pi^d_{i1}(e_{i1})$ by the payoff the active player obtains when he changes his effort level to the boundary $e_{i1} = 0$ and $e_{i1} = \infty$, we have:

$$\pi^d_{i1}(e_{i1} = 0) = 0 \ \text{ and } \ \pi^d_{i1}(e_{i1} = \infty) = -\infty.$$

We then obtain the following non-deviation condition for the active players in the active groups:

$$\pi^*_{i1} > \pi^d_{i1}(e_{i1} = 0) \Leftrightarrow \alpha < \frac{\lambda}{\lambda - 1}.$$

Second, we consider the deviation incentives for the other players, except for the active player, in each active group. Since the active player has the highest valuation in the group, the other players' valuation are less than or at most equal to the valuation of the active player, and it means that their willingness to exert effort, given the effort levels of the players in the other groups, is less than or at most equal to the active player's one. Thus, they do not have any incentive to change (increase) their effort levels in the profile above. Last, we consider the deviation incentives for the players in the inactive groups. In each inactive group, all the players including the highest-valuation player choose zero effort level. So, unless the highest-valuation player has an incentive to increase his effort level, the other players in the group do not have any deviation incentive, either. The non-deviation condition for the highest-valuation player in each inactive group is $\alpha \geq \frac{\lambda+1}{\lambda}$. To sum up, if $\frac{\lambda+1}{\lambda} \leq \alpha < \frac{\lambda}{\lambda-1}$, no player in the contest has an unilateral incentive to deviate from the above strategy profile, and the strategy profile constitutes the Nash equilibrium, in which λ groups actively participate in the contest and the active player in each active group is the highest-valuation player in the group. Here λ represents the number of active groups existing in equilibrium. Specifically, we have the following:

- If $1 < \alpha < \frac{n}{n-1}$, there exist the Nash equilibria in which there are n active groups and the highest-valuation player in each group exerts effort and the others free ride on him. And, depending on parameter values (the structure of valuations for the players) within the model, there may exist different type of equilibrium in which the highest-valuation players in the groups free ride.

- If $\frac{n}{n-1} \leq \alpha < \frac{n-1}{(n-1)-1}$, there exist the Nash equilibria in which there are $(n-1)$ active groups and the highest-valuation player in each active group exerts effort and the others free ride. And the other equilibrium mentioned above may exist.

- If $\frac{n-1}{(n-1)-1} \leq \alpha < \frac{n-2}{(n-2)-1}$, there exist the Nash equilibria in which there are $(n-2)$ active groups and the highest-valuation player in each active group exerts effort and the others free ride. And the other equilibrium may exist.

$$\vdots$$

- If $\frac{3}{2} \leq \alpha < 2$, there exists the Nash equilibrium in which there are 2 active groups and the highest-valuation player in each active group exerts effort and the others free ride. And the other equilibrium may exist.

- If $2 \leq \alpha$, there does not exist the Nash equilibrium in which there are at least 2 active groups and the highest-valuation players are the active ones. And the other equilibrium, in which the highest-valuation player free rides, does not exist, either. That is, in this case, there does not exist any pure-strategy Nash equilibrium of the game.

■

Proof of Lemma 7. Given our restrictions, we have the following strategy profile:

$$(e_{11}^A, e_{12}, e_{21}^A, e_{22}) = \left(\frac{\alpha k^{\alpha+1}}{(k^\alpha+1)^2}, \frac{\alpha k^{\alpha+1}}{(k^\alpha+1)^2}, \frac{\alpha k^\alpha}{(k^\alpha+1)^2}, \frac{\alpha k^\alpha}{(k^\alpha+1)^2} \right),$$

where e_{11}^A and e_{21}^A are the solutions that satisfy the first-order and the second-order conditions for maximizing the payoffs for the active players in each group.

We check if the above strategy profile constitutes the Nash equilibrium of the game. First, we consider the deviation incentives for the active players. At the above strategy profile, the active players in each group have their payoffs:

$$\pi_{11}^{*} = \frac{k^{\alpha+1}(k^{\alpha}+1-\alpha)}{(k^{\alpha}+1)^{2}}$$

and

$$\pi_{21}^{*} = \frac{1+k^{\alpha}-\alpha k^{\alpha}}{(k^{\alpha}+1)^{2}}.$$

Denoting $\pi_{i1}^{d}(e_{i1})$ by the payoff the active player of group i obtains when he changes his effort level to the boundary $e_{i1}=0$ and $e_{i1}=\infty$, we have:

$$\pi_{i1}^{d}(e_{i1}=0)=0 \text{ and } \pi_{i1}^{d}(e_{i1}=\infty)=-\infty \text{ for } i=1,2$$

So, we obtain the following non-deviation condition for the active players in each group:

$$\pi_{i1}^{*} > \pi_{i1}^{d}(e_{i1}=0) \text{ for all } i=1,2 \Leftrightarrow \alpha<2 \text{ and } k<\frac{1}{(\alpha-1)^{1/\alpha}}.$$

Next, we consider the deviation incentive for the other player, except for the active player, in each group. Since the active player has the highest valuation in each group, the other player's valuation is less than or at most equal to the valuation of the active player. It means that his willingness to exert effort, given the effort levels of the players in the other group, is less than or at most equal to the active player's one. Thus, he does not have any incentive to change (increase) his effort level in the profile above. Hence, if $\alpha<2$ and $k<\frac{1}{(\alpha-1)^{1/\alpha}}$, then no player in the contest has an unilateral incentive to deviate from the above strategy profile, and the strategy profile constitutes the Nash equilibrium. Otherwise, i.e., $\alpha<2$ and $k\geq\frac{1}{(\alpha-1)^{1/\alpha}}$ or $\alpha\geq 2$, this equilibrium disappears. However, there may still exist the other equilibrium in which the highest-valuation player in a group free rides on the other in his group. See the following example.

Suppose that $\alpha=1.5$, $k=1.6$, $v_{12}=v_{22}=0.9$. Given these parameters, we have the strategy profiles:

$$(e_{11}^{A},e_{12},e_{21}^{A},e_{22})=(0.531212781,0,0.332007988,0)$$

and

$$(e_{11},e_{12}^{A},e_{21},e_{22}^{A})=(0,0.3375,0,0.3375).$$

Checking if each of these strategy profiles constitutes the Nash equilibrium or not, we find that the first one is not the Nash equilibrium but the second one is the equilibrium. ■

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