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Preference Revelation Games in School Choice When Students Are Naïve or Strategic

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Abstract Abdulkadiroğlu, Che, and Yasuda (2011) consider the preference revelation games induced by the immediate acceptance and deferred acceptance mechanisms (henceforth, the IA and DA games, respectively) when students are naïve or strategic. They study properties of a class of equilibria of the IA game in which some strategic student misrepresents his preferences with positive probability but they do not establish the existence of such equilibria. We show that in fact, depending on the parameters of the model, the IA game may have no such equilibrium. Then we provide a condition, termed richness, on the preference type space that ensures existence. Another result of Abdulkadiroğlu, Che, and Yasuda (2011) is that in a symmetric equilibrium of the IA game, if a strategic student of some preference type reports his preferences truthfully, then a naïve student of the same preference type is at least as well off in that equilibrium of the IA game as in the dominant strategy, truth-telling equilibrium of the DA game. This comparison, however, is silent on the welfare of the other naïve students. We show that some naïve students are indeed worse off under the immediate acceptance mechanism than under the deferred acceptance mechanism.

Keywords immediate acceptance mechanism; deferred acceptance mechanism; preference revelation game

JEL Classification C78, D70

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1. INTRODUCTION

In the design of school choice programs, providing students with the incentive to be truthful about their preferences has emerged as an important issue. Depending on the assignment mechanism in place, a student may find it profitable to misrepresent his preferences. From the designer's viewpoint, such manipulation is problematic because assignments based on the elicited information may not be desirable with respect to the true preferences. In the context of school choice, however, manipulation raises another concern related to fairness. With students exhibiting different levels of strategic sophistication, the sophisticated may benefit at the expense of the naïve.[1] This possibility undermines the designer's ability to achieve one of the main objectives in school choice: to be fair in allocating school seats to students.

Given a mechanism, one can formalize this problem by considering the preference revelation game it induces. The immediate acceptance (henceforth, IA) mechanism, also known as the "Boston mechanism", is not strategy-proof (Abdulkadiroğlu and Sönmez, 2003).[2,3] Thus, we can analyze the game induced by the IA mechanism—let us call it the IA game—to compare the welfare of naïve and strategic students. The set of pure strategy Nash equilibrium outcomes of the IA game coincides with the set of stable matches for the modified economy where strategic students are given higher priority than naïve students at all schools except at those that naïve students rank first (Pathak and Sönmez, 2008). Therefore, by strategizing well, students can effectively attain higher priority than those who do not, thus obtaining a seat at more preferred schools.

Abdulkadiroğlu, Che, and Yasuda (2011) (henceforth, ACY) note that two assumptions play a key role in deriving this result: (i) school priorities are strict; and (ii) students have complete information. Since these assumptions are unlikely to be met in real life, it is difficult to draw practical implications from Pathak and Sönmez (2008). Moreover, under different assumptions on school priorities or the nature of information, naïve students may benefit from other students' strategic behavior. To advance this point, ACY consider a model where no student has priority at any school and students have incomplete information as to whether other students are naïve or strategic. They also assume that students

[1] Abdulkadiroğlu et al. (2006) provide empirical evidence based on the data collected in Boston Public Schools.

[2] The immediate acceptance mechanism is no longer used in Boston Public Schools; it was replaced by the deferred acceptance mechanism in 2005. For this reason, we follow Thomson (2011) and use the term "immediate acceptance mechanism".

[3] A mechanism is strategy-proof if no student ever benefits by lying about his preferences.

have the same ordinal preferences over schools while their relative intensities, or von Neumann-Morgenstern (vNM) preferences (cardinal utilities), may differ. These vNM preferences are privately drawn according to a common distribution that is publicly known.

Under these assumptions, ACY compare symmetric equilibria of the IA game with the dominant strategy, truth-telling equilibrium of the game induced by the deferred acceptance (henceforth, DA) mechanism, or the DA game. They show that if some strategic student manipulates with positive probability in a former equilibrium, then naïve students have a higher probability of obtaining a seat at top schools in the former equilibrium than in the latter equilibrium. Moreover, if a strategic student of some preference type reports his preferences truthfully in a former equilibrium, then a naïve student of the same preference type is at least as well off in the former equilibrium as in the latter equilibrium.

Although ACY's results reveal a countervailing strength of the IA mechanism, they do not address two important issues. The first result is concerned with symmetric equilibria of the IA game satisfying a particular property. The existence of *a* symmetric equilibrium is well established. But one cannot expect that the IA game always has a symmetric equilibrium in which *some strategic student misrepresents his preferences with positive probability*. Unless the existence of equilibria with the latter property is proved, AYC's result is vacuous.

Even with the existence issue resolved, however, a proper welfare comparison of mechanisms should focus on the payoffs to students, rather than their probability of being assigned to top schools. The second result of ACY pertains to payoff comparison, but it is limited to the naïve students of those preference types who would report truthfully were they strategic. Unless the validity of a similar statement for the other naïve students is checked, ACY's welfare comparison is incomplete and can be potentially misleading.

Our objective is to complement ACY by addressing these issues. First, we show that in fact, depending on the parameters of the model, the IA game may not admit an equilibrium in which some strategic student manipulates with positive probability. In Example 1, the preference type space, the space from which vNM preferences are drawn, is very coarse, and this forces each strategic student to report his preferences truthfully. Truth-telling is the unique equilibrium (symmetric or not) of the IA game, and therefore, the first result of ACY does not apply. Further, the observations from this example can be generalized.

We identify a condition under which the IA game has a symmetric equilibrium with the property we are interested in. Our condition requires the preference type space to be sufficiently "rich". Specifically, the preference type space is rich if some preference type attaches to two schools with adjacent rankings vNM indices with "small" differences. How close these indices should be depends on the relative capacities of the schools. Theorem 1 shows that if the preference type space is rich, then *each* symmetric equilibrium of the IA game has a strategic student manipulating with positive probability; moreover, the converse is true.

Finally, we provide an example where our concern about the potential welfare loss of naïve students materializes. Example 2 shows that the naïve students who report differently than the strategic students of the same preference type, are indeed worse off under the IA mechanism than under the DA mechanism. Thus, when the IA mechanism is in place, the presence of strategic students is harmful to some naïve students.

2. THE MODEL

We consider the setup of ACY. Let $A \equiv \{1, \cdots, m\}$ ($m \geq 2$) be the set of schools. Schools are denoted by a, b, and so on. For each $a \in A$, school a has capacity $q_a \in \mathbb{N}$. Schools do not have priorities. Let $\Delta(A)$ be the set of all lotteries over A.

Let $N \equiv \{1, \cdots, n\}$ be the set of students. Assume that $n = \sum_{a \in A} q_a$. Students are denoted by i, j, and so on. Let $i \in N$. Student i's type consists of two parts. The first part concerns preferences, so we call it his **preference type**. Student i has vNM preferences over lotteries defined on schools. The preferences are drawn from a finite set $V \subseteq [0,1]^A$ according to a publicly known, common distribution f, and for each $v \in V$, $v_1 > v_2 > \cdots > v_m$. We call V the **preference type space**. Since V is finite, we may assume that for each $v \in V$, $f(v) > 0$. For each $v \in V$ and each $a \in A$, v_a is the vNM index attached to school a. Note that any preference type induces the same ordinal preferences over schools: school 1 is most preferred, school 2 is second most preferred, and so on.

Below we consider assignment mechanisms that allow each student to submit his preferences over schools. The second part of the student's type pertains to how he behaves in expressing his preferences; thus, we call it his **behavior type**. Each student can be either (i) naïve, always reporting his true preferences; or (ii) strategic, possibly lying about his preferences. The probability of being naïve is $\alpha \in (0,1)$. We denote the naïve type by 0 and the strategic type by 1. In sum, $V \times \{0,1\}$ is the common type space. Types are drawn independently across students.

An **assignment** is a profile $x \equiv (x_i)_{i \in N}$ such that (i) for each $i \in N$, $x_i \in \Delta(A)$; and (ii) for each $a \in A$, $\sum_{i \in N} x_{ia} \leq q_a$. Let X be the set of all assignments. Let

$\mathscr{R}$ be the set of all ordinal preferences over A. An (assignment) **mechanism** is a mapping from $\mathscr{R}^N$ to X, associating with each preference profile an assignment. Denote a generic mechanism by φ. We are interested in two well-known mechanisms: the immediate acceptance and deferred acceptance mechanisms, denoted by *IA* and *DA*, respectively. Refer to Abdulkadiroğlu and Sönmez (2003) for the definition of these mechanisms. Since schools do not have priorities, all ties are broken randomly when these mechanisms are applied.

Each mechanism φ induces a Bayesian preference revelation game $\Gamma(\varphi)$, defined as follows. Observing his type, each student reports his ordinal preferences over schools. The naïve type reports truthfully while the strategic type reports any preferences. The mechanism φ is applied to the elicited preference profile, to obtain an assignment. Then each student receives a lottery over schools and his payoff is determined by his preference type. We are interested in symmetric (Bayesian Nash) equilibria of $\Gamma(\varphi)$. Since the game is finite, a symmetric equilibrium exists. Also, since the DA mechanism is strategy-proof (Abdulkadiroğlu and Sönmez, 2003), truth-telling is a dominant strategy equilibrium of $\Gamma(DA)$.

3. COMPARISON OF THE IA AND DA MECHANISMS

Now we compare symmetric equilibria of $\Gamma(IA)$ and $\Gamma(DA)$ in terms of (i) the probability that a naïve student is assigned to top schools; and (ii) the welfare of naïve students. The following proposition is due to ACY (Theorem 2).

Proposition 1. *Let σ^{IA} be a symmetric equilibrium of $\Gamma(IA)$ and σ^{DA} the dominant strategy, truth-telling equilibrium of $\Gamma(DA)$.*

(1) Suppose that some strategic student misrepresents his preferences with positive probability in σ^{IA}. Then for some $k \in A$, each naïve student is assigned to each of the top k schools, $\{1, \cdots, k\}$, with at least as high probability and to some school in that set with higher probability when σ^{IA} is played in $\Gamma(IA)$ than when σ^{DA} is played in $\Gamma(DA)$.

(2) Let $v \in V$ and suppose that a strategic student of preference type v reports his preferences truthfully in σ^{IA}. Then a naïve student of preference type v is at least as well off when σ^{IA} is played in $\Gamma(IA)$ as when σ^{DA} is played in $\Gamma(DA)$.

Part (1) of Proposition 1 examines the property of particular symmetric equilibria of $\Gamma(IA)$, without establishing their existence (the existence of *a* symmetric equilibrium follows from a standard argument). However, depending on the parameters of the model, $\Gamma(IA)$ may not have such an equilibrium, in which case part (1) of Proposition 1 is vacuous (if truth-telling is the unique equilibrium of

$\Gamma(IA)$, then each type of student receives the same payoff in $\Gamma(IA)$ and $\Gamma(DA)$, and therefore, part (2) of Proposition 1 is not so meaningful, either). We demonstrate this point with the following example.

Example 1. There are three schools, each with unit capacity, and three students. Let $\varepsilon \in (0, \frac{1}{4})$. Let $v^1 \equiv (1, \varepsilon, 0)$ and $v^2 \equiv (1, 2\varepsilon, 0)$. Let $V = \{v^1, v^2\}$ be the preference type space. Each student can be type v^1 or v^2 with equal probability; i.e., $f(v^1) = f(v^2) = \frac{1}{2}$. Also, he can be either naïve or strategic with the same probability; i.e., $\alpha = \frac{1}{2}$. As shown in Section 4, truth-telling is the unique equilibrium (symmetric or not) of $\Gamma(IA)$, and therefore, part (1) of Proposition 1 does not apply. △

The main reason why no strategic type manipulates in Example 1 is that the preference type space V is too coarse, so that for any preference type, the difference between the vNM indices attached to schools 1 and 2 is large enough. Thus, a strategic student never finds it profitable to rank school 2 above school 1. Generalizing this observation, we now define a condition that guarantees the existence of an equilibrium of $\Gamma(IA)$ such that some strategic student manipulates with positive probability.

Definition. The preference type space V is **rich** if there are $v \in V$ and $a \in A$ such that

$$v_{a+1} > \frac{q_a}{q_a + q_{a+1} + \cdots + q_m} v_a + \frac{q_{a+1}}{q_a + q_{a+1} + \cdots + q_m} v_{a+1} + \cdots + \frac{q_m}{q_a + q_{a+1} + \cdots + q_m} v_m. \tag{1}$$

To illustrate the richness condition, suppose that there are four schools; i.e., $A = \{1, 2, 3, 4\}$. Then V is rich if there is $v \in V$ such that

$$\begin{aligned} v_2 &> \frac{q_1}{n} v_1 + \frac{q_2}{n} v_2 + \frac{q_3}{n} v_3 + \frac{q_4}{n} v_4; \text{ or} \\ v_3 &> \frac{q_2}{q_2 + q_3 + q_4} v_2 + \frac{q_3}{q_2 + q_3 + q_4} v_3 + \frac{q_4}{q_2 + q_3 + q_4} v_4. \end{aligned}$$

These inequalities can be interpreted as follows. Compute a weighted average of vNM indices for the schools, where the weights are relative capacities (the righthand side of the first inequality). If the index attached to school 2, the second most preferred school, is larger than the weighted average, then V is rich. Otherwise, consider schools 2, 3, and 4. Compute a similar weighted average of vNM indices for these schools, where the weights are relative capacities with respect to the three schools only (the righthand side of the second inequality). If the index attached to school 3, the second most preferred school among the three schools, is larger than the weighted average, then V is rich. Loosely speaking, if V consists of diverse preferences, so that some preference type values two schools with adjacent rankings quite closely, then V is rich.

Richness, we believe, is a mild requirement on the preference type space, especially in light of the realistic aspect of school choice problems: while students may agree on the rankings of schools, the relative intensities of their vNM preferences tend to vary greatly. Yet, as the following theorem shows, richness is sufficient and necessary for *each* symmetric equilibrium of $\Gamma(IA)$ to have some strategic student manipulating with positive probability.

Theorem 1. *If the preference type space V is rich, then in each symmetric equilibrium of $\Gamma(IA)$, some strategic student misrepresents his preferences with positive probability. The converse also holds.*

Proof. Assume that V is rich. Suppose, by contradiction, that there is a symmetric equilibrium σ^{IA} of $\Gamma(IA)$ such that each strategic student reports his preferences truthfully with probability 1. When σ^{IA} is played, each type of student submits the same preferences. Let $v \in V$ and $a \in A$ satisfy Inequality (1). Since all ties are broken randomly, the payoff to a strategic student of preference type v is $\sum_{b\in A} \frac{q_b}{n} v_b$. Now suppose that the student reports the preferences that are the same as his true preferences, except that the rankings of schools a and $a+1$ are switched. Then his payoff is

$$\frac{q_1}{n}v_1 + \cdots + \frac{q_{a-1}}{n}v_{a-1} + \frac{n-q_1-\cdots-q_{a-1}}{n}v_{a+1},$$

which exceeds the equilibrium payoff $\sum_{b\in A} \frac{q_b}{n} v_b$, a contradiction.

Next, to prove the converse, we prove its contrapositive. Suppose that V is not rich. Let σ be the truth-telling strategy profile; i.e., for each $i \in N$ and each $v \in V$, strategic student i of preference type v reports his preferences truthfully. The proof is complete if we show that σ is an equilibrium. Let $i \in N$ and $v \in V$. When σ is played, the payoff to strategic student i of preference type v is $\sum_{a\in A} \frac{q_a}{n} v_a$. As shown below, this is an upper bound to all the payoffs student i of preference type v can get by reporting any other preferences.

Suppose that student i of preference type v ranks school $b \in A\backslash\{1\}$ first. Because all the other students report truthfully, he is assigned to school b with probability 1, getting payoff v_b, which cannot exceed v_2. But because Inequality (1) is violated for school b, $v_2 \leq \sum_{a\in A} \frac{q_a}{n} v_a$. Thus, student i of preference type v cannot profitably deviate by top-ranking any school other than school 1.

Suppose that student i of preference type v ranks school 1 first and school $b \in A \backslash \{1,2\}$ second. Again, because all the other students report truthfully, he is assigned to school 1 with probability $\frac{q_1}{n}$ and to school b with probability $1-\frac{q_1}{n}$, getting payoff $\frac{q_1}{n} v_1 + \left(1-\frac{q_1}{n}\right) v_b$, which cannot exceed $\frac{q_1}{n} v_1 + \left(1-\frac{q_1}{n}\right) v_3$ (since $v_b \leq v_3$). But because Inequality (1) is violated for school b, $\frac{q_1}{n} v_1 + \left(1-\frac{q_1}{n}\right) v_3 \leq \sum_{a \in A} \frac{q_a}{n} v_a$. Thus, student i of preference type v cannot profitably deviate by ranking school 1 first and any school other than school 2 second.

Clearly, the above argument can be repeated to show: ranking school 1 first, school 2 second, and school $b \in A \backslash \{1,2,3\}$ third is not profitable; ranking school 1 first, school 2 second, school 3 third, and school $b \in A \backslash \{1,2,3,4\}$ fourth is not profitable; and so on. Thus, σ is an equilibrium. □

Part (2) of Proposition 1 compares the welfare of naïve students under the IA and DA mechanisms, only for those preference types that report truthfully even when they are strategic. However, it may very well be the case that as a consequence of manipulation by strategic students, other naïve students are, in fact, worse off under the IA mechanism than under the DA mechanism.[4] We illustrate this possibility with the following example.

Example 2. Consider Example 1, with the modification that $\varepsilon \in \left(\frac{5}{14}, \frac{1}{2}\right)$. Let σ^{IA} be the strategy profile such that (i) each strategic student of preference type v^1 reports his preferences truthfully; and (ii) each strategic student of preference type v^2 reports the preferences that rank school 2 first, school 1 second, and school 3 third. It is easy to check that σ^{IA} is an equilibrium of $\Gamma(IA)$. Now, for each $(v,k) \in V \times \{0,1\}$, let $U^{IA}_{(v,k)}$ be the interim payoff to a student of type (v,k) when σ^{IA} is played in $\Gamma(IA)$. Then $U^{IA}_{(v^1,0)} = U^{IA}_{(v^1,1)} = \frac{7+3\varepsilon}{16}$, $U^{IA}_{(v^2,0)} = \frac{7+6\varepsilon}{16}$, and $U^{IA}_{(v^2,1)} = \frac{1+74\varepsilon}{48}$.

Now let σ^{DA} be the dominant strategy, truth-telling equilibrium of $\Gamma(DA)$. For each $(v,k) \in V \times \{0,1\}$, let $U^{DA}_{(v,k)}$ be the interim payoff to a student of type (v,k) when σ^{DA} is played in $\Gamma(DA)$. Then $U^{DA}_{(v^1,0)} = U^{DA}_{(v^1,1)} = \frac{1+\varepsilon}{3}$ and $U^{DA}_{(v^2,0)} = U^{DA}_{(v^2,1)} = \frac{1+2\varepsilon}{3}$. Note that $\frac{7+3\varepsilon}{16} > \frac{1+\varepsilon}{3}$ and $\frac{1+74\varepsilon}{48} > \frac{1+2\varepsilon}{3} > \frac{7+6\varepsilon}{16}$. Therefore, naïve students of preference type v^2 are worse off when σ^{IA} is played in $\Gamma(IA)$ than when σ^{DA} is played in $\Gamma(DA)$; and the opposite is true for students of all other types. △

[4]One cannot use the example in ACY (p. 407) to make this point because it does not fit with the model. Students are not symmetric in the example, with some students always naïve and others always strategic. In the model, students are fully symmetric, which makes it compelling to focus on symmetric equilibria.

Remark 1. The observation in Example 2 can be generalized.[5] In a setup with three students and three schools (each with unit capacity), normalize, without loss of generality, the preference type space V so that for each $v \in V$, $v_1 = 1$ and $v_3 = 0$. Let $V = \{v^1, v^2\}$, where $v^1 \equiv (1, t_1, 0)$ and $v^2 \equiv (1, t_2, 0)$, with $t_1 < t_2$ (thus, students of preference type v^2 has stronger preferences for school 2 than students of preference type v^1). Let σ^{IA} be the strategy profile in which only strategic students of preference type v^2 manipulates, ranking school 2 first, school 1 second, and school 3 third. It can be shown that if $t_1 < \frac{5}{7} < t_2$, (i) σ^{IA} is an equilibrium of $\Gamma(IA)$; and (ii) naïve students of preference type v^2 are worse off when σ^{IA} is played in $\Gamma(IA)$ than when σ^{DA} (truth-telling equilibrium) is played in $\Gamma(DA)$. Therefore, some naïve students suffer due to the presence of strategic students.[6]

This three-student, three-school argument can be embedded in an environment with arbitrary numbers of students, schools, and capacities. Consider the preference type space V consisting of v's such that $v = (\delta^{m-3}, \delta^{m-2}, \cdots, \delta, 1, t, 0)$, where $\delta > 1$. For each $v \in V$, the ratio of vNM indices for two schools with adjacent rankings is δ. Thus, as $\delta \to \infty$, all students, naïve and strategic, truthfully report their preferences over the top $m-3$ schools in any equilibrium of $\Gamma(IA)$. Thus, welfare consequences of equilibria in the IA and DA games rely only on how students rank the bottom 3 schools in the IA game. To analyze the latter behavior, we can use the three-student, three-school argument above. △

4. UNIQUENESS OF AN EQUILIBRIUM IN EXAMPLE 1

In this section, we prove that truth-telling is the unique equilibrium of the game $\Gamma(IA)$ specified in Example 1. First, denote by "*abc*" the preferences that rank school a first, school b second, and school c third. Let $\sigma \equiv (\sigma_i)_{i \in N}$ be an equilibrium of $\Gamma(IA)$ (σ may not be symmetric). Without loss of generality, we show that when students 2 and 3 play (σ_2, σ_3), it is optimal for student 1 to report his preferences truthfully whatever his type is.

Step 1: *For each $i \in N$ and each $(v, 1) \in V \times \{0, 1\}$, σ_i assigns zero probability to student i of type $(v, 1)$ reporting any of the preferences* 132, 231, 312, *and* 321.

Suppose, by contradiction, that there are $i \in N$ and $(v, 1) \in V \times \{0, 1\}$ such

[5] I thank an anonymous referee for encouraging me to explore this direction.

[6] Nevertheless, in this example, students' ex ante payoffs (i.e., social welfare) is higher in the IA game than in the DA game. When σ^{IA} is played in $\Gamma(IA)$, each student receives an ex ante payoff of $\frac{1}{96}(9t_1 + 23t_2 + 32)$; when truth-telling is played in $\Gamma(DA)$, each student receives an ex ante payoff of $\frac{1}{6}(t_1 + t_2 + 2)$. Since $t_1 < t_2$, $\frac{1}{96}(9t_1 + 23t_2 + 32) > \frac{1}{6}(t_1 + t_2 + 2)$.

that student i of type $(v,1)$ reports 132 with positive probability. It is clear that reporting 123 weakly dominates reporting 132 for student i of type $(v,1)$. Suppose that he plays an action that is the same as $\sigma_i(v,1)$, except that it prescribes reporting 123 whenever $\sigma_i(v,1)$ prescribes reporting 132. Since the students in $N\backslash\{i\}$ report $(123,123)$ with at least probability $\frac{1}{4}$, this latter action increases the interim payoff to type $(v,1)$, a contradiction.

Similarly, using the fact that (i) reporting 132 weakly dominates reporting 312; and (ii) reporting 231 weakly dominates reporting 321, we can show that 312 and 321 are reported with zero probability in σ.

Suppose, by contradiction, that there are $i \in N$ and $(v,1) \in V \times \{0,1\}$ such that student i of type $(v,1)$ reports 231 with positive probability. It is clear that reporting 213 weakly dominates reporting 231 for student i of type $(v,1)$. If any of the students in $N\backslash\{i\}$ reports 213 or 231 with positive probability, then the action that replaces 231 by 213 in $\sigma_i(v,1)$ gives a higher interim payoff than $\sigma_i(v,1)$ does, a contradiction. Thus, each student in $N\backslash\{i\}$ reports each of the preferences 213 and 231 with zero probability. Moreover, by the arguments in the previous paragraphs, each student in $N\backslash\{i\}$ reports each of the preferences 132, 312, and 321 with zero probability. This means that the students in $N\backslash\{i\}$ report $(123,123)$ with probability 1, in which case, by our choice of ε, student i of type $(v,1)$ is better off reporting 123 with probability 1 than playing $\sigma_i(v,1)$, a contradiction.

Step 2: *For each* $(v,1) \in V \times \{0,1\}$*, student 1 of type* $(v,1)$ *reports* 123 *with probability* 1.

To simplify notation, for each $i \in N$ and each $k \in \{1,2\}$, let p_{ik} be the probability that student i of type $(v^k,1)$ reports 123 when σ_i is played. Then $1-p_{ik}$ is the probability that student i of type $(v^k,1)$ reports 213 when σ_i is played. To compute the interim payoff to student 1 of type $(v^2,1)$ when σ is played, we distinguish nine cases, depending on the types of students 2 and 3 he faces.

Case 1: *The type profile for students 2 and 3 is one of* $\big((v^1,0),(v^1,0)\big)$*,* $\big((v^1,0),(v^2,0)\big)$*,* $\big((v^2,0),(v^1,0)\big)$*, and* $\big((v^2,0),(v^2,0)\big)$.

This case occurs with probability $\frac{1}{4}$ and the payoff to student 1 of type $(v^2,1)$ is

$$U_{11} \equiv \frac{1+2\varepsilon}{3}p_{12} + 2\varepsilon(1-p_{12}).$$

Case 2: *The type profile for students 2 and 3 is one of* $\big((v^1,0),(v^1,1)\big)$ *and* $\big((v^2,0),(v^1,1)\big)$.

This case occurs with probability $\frac{1}{8}$ and the payoff to student 1 of type $(v^2, 1)$ is

$$U_{12} \equiv p_{12}\left[\frac{1+2\varepsilon}{3}p_{31} + \frac{1}{2}(1-p_{31})\right] + (1-p_{12})\left[2\varepsilon p_{31} + \varepsilon(1-p_{31})\right].$$

Case 3: *The type profile for students 2 and 3 is one of* $\big((v^1,0),(v^2,1)\big)$ *and* $\big((v^2,0),(v^2,1)\big)$.

This case occurs with probability $\frac{1}{8}$ and is similar to Case 2. Replacing p_{31} by p_{32} in the definition of U_{12}, we obtain the payoff to student 1 of type $(v^2, 1)$, U_{13}.

Case 4: *The type profile for students 2 and 3 is one of* $\big((v^1,1),(v^1,0)\big)$ *and* $\big((v^1,1),(v^2,0)\big)$.

This case occurs with probability $\frac{1}{8}$ and is similar to Case 2. Replacing p_{31} by p_{21} in the definition of U_{12}, we obtain the payoff to student 1 of type $(v^2, 1)$, U_{14}.

Case 5: *The type profile for students 2 and 3 is* $\big((v^1,1),(v^1,1)\big)$.

This case occurs with probability $\frac{1}{16}$ and the payoff to student 1 of type $(v^2, 1)$ is

$$\begin{aligned} U_{15} \equiv\ & p_{12}\left[\frac{1+2\varepsilon}{3}p_{21}p_{31} + \frac{1}{2}p_{21}(1-p_{31}) + \frac{1}{2}(1-p_{21})p_{31} + (1-p_{21})(1-p_{31})\right] \\ & +(1-p_{12})\Big[2\varepsilon p_{21}p_{31} + \varepsilon p_{21}(1-p_{31}) + \varepsilon(1-p_{21})p_{31} \\ & + \frac{1+2\varepsilon}{3}(1-p_{21})(1-p_{31})\Big]. \end{aligned}$$

Case 6: *The type profile for students 2 and 3 is* $\big((v^1,1),(v^2,1)\big)$.

This case occurs with probability $\frac{1}{16}$ and is similar to Case 5. Replacing p_{31} by p_{32} in the definition of U_{15}, we obtain the payoff to student 1 of type $(v^2, 1)$, U_{16}.

Case 7: *The type profile for students 2 and 3 is one of* $\big((v^2,1),(v^1,0)\big)$ *and* $\big((v^2,1),(v^2,0)\big)$.

This case occurs with probability $\frac{1}{8}$ and is similar to Case 2. Replacing p_{31} by p_{22} in the definition of U_{12}, we obtain the payoff to student 1 of type $(v^2, 1)$, U_{17}.

Case 8: *The type profile for students 2 and 3 is* $\big((v^2,1),(v^1,1)\big)$.

This case occurs with probability $\frac{1}{16}$ and is similar to Case 5. Replacing p_{21} by p_{22} in the definition of U_{15}, we obtain we obtain the payoff to student 1 of type $(v^2, 1)$, U_{18}.

Case 9: *The type profile for students 2 and 3 is* $\big((v^2,1),(v^2,1)\big)$.

This case occurs with probability $\frac{1}{16}$ and is similar to Case 5. Replacing p_{21} by p_{22} and p_{31} by p_{32} in the definition of U_{15}, we obtain we obtain the payoff to student 1 of type $(v^2, 1)$, U_{19}.

Therefore, when σ is played, the interim payoff to student 1 of type $(v^2, 1)$ is

$$U_1 \equiv \frac{1}{4}U_{11} + \frac{1}{8}U_{12} + \frac{1}{8}U_{13} + \frac{1}{8}U_{14} + \frac{1}{16}U_{15} + \frac{1}{16}U_{16} + \frac{1}{8}U_{17} + \frac{1}{16}U_{18} + \frac{1}{16}U_{16}.$$

Simple calculation shows that the coefficient on p_{12} in U_1 is

$$\frac{1}{24}\left[12 - 24\varepsilon - (p_{21} + p_{22} + p_{31} + p_{32})(1 + 2\varepsilon)\right].$$

Since $\varepsilon \in (0, \frac{1}{4})$, the coefficient is positive. Since σ_1 is a best response to (σ_2, σ_3), it follows that $p_{12} = 1$; i.e., student 1 of type $(v^2, 1)$ reports his preferences truthfully with probability 1.

We can repeat the above computation for student 1 of type $(v^1, 1)$. Denote the resulting payoff by $\hat{U}_1$.[7] Then the coefficient on p_{11} in $\hat{U}_1$ is

$$\frac{1}{24}\left[12 - 12\varepsilon - (p_{21} + p_{22} + p_{31} + p_{32})(1 + \varepsilon)\right],$$

which is positive. Thus, $p_{11} = 1$; i.e., student 1 of type $(v^1, 1)$ reports his preferences truthfully with probability 1.

[7] To obtain $\hat{U}_1$, we can replace p_{12} by p_{11} and ε by $\frac{1}{2}\varepsilon$ in the preceding argument.

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