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The Fundamental Theorem of Asset Pricing with Convex Transaction Costs*

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The paper establishes the fundamental theorem of asset pricing (FTAP) with convex transaction costs. The arbitrage pricing rules are independent of the marginal effect of transaction costs and determined by average costs for large transactions. This implies that the pricing rules are characterized by minimal information on the nature of transaction costs. Remarkably, no matter how complex the convex transaction cost functions are, the pricing rules are as simple and concrete as in the case with proportional transaction costs. This result is in sharp contrast to equilibrium pricing theory based upon the knowledge of the marginal effect of transaction costs and therefore, of transaction cost functions. Moreover, the no arbitrage condition is equivalent to viability of asset prices.

Keywords: Arbitrage Pricing, Convex Transaction Costs, Viability

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I . Introduction

Borrowing and lending rates differ in the real world. The spread between them is ascribed to financial intermediation, which is costly due to market frictions. This is an easy example where the law of one price is violated in the face of market frictions.[1] The arbitrage pricing theory which does not take market frictions into account may be unable to characterize asset prices in economies which are far from being ideal. The effect of market frictions on asset pricing must be properly understood to make an asset pricing theory come closer to reality.

The paper establishes the fundamental theorem of asset pricing (FTAP) in markets where transaction costs need not be proportional. The arbitrage pricing rules are independent of the marginal effect of transaction costs and determined by average costs for large transactions. Remarkably, no matter how complex the convex transaction cost functions are, the pricing rules are as simple and concrete as in the case with proportional transaction costs. Thus, asset valuation can be free from the intractability of convex transaction costs. The consequences of the paper are in sharp contrast to equilibrium pricing theory based upon the knowledge of the marginal effect of transaction costs and therefore, of transaction cost functions. The pricing rules are characterized by minimal information on the nature of transaction costs. It is also worth noting that marginal transaction costs are irrelevant to the determination of the pricing rules as far as they differ from average costs for large transactions.

There exists a large body of the literature on arbitrage pricing theory under proportional transaction costs. Garman and Ohlson (1981), Boyle and Vorst (1992), Jouini and Kallal (1995), Kabanov (1999), Kabanov and Stricker (2001), Delbaen, Kabanov, and Valkeila (2002), Zhang, Xu, and Deng (2002), Schachermayer (2004) among others examine the effect of proportional transaction costs on asset pricing. Leland (1985) and Boyle and Vorst (1992)

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1) Evidences of mispricing are abundant in the literature; stock index futures (Canina and Figlewski, 1995), primes and scores (Jarrow and O'Hara, 1989), closed-end funds (Pontiff, 1996), stock options (Conrad, 1989) among others.

investigate option pricing in discrete time with proportional transaction costs. The consequences of the paper will be useful in extending those results to the case with convex transaction costs.

The notion of arbitrage used here is appropriate to examining equilibrium asset prices. Viability of asset prices is an important criterion for asset pricing theory because it is fulfilled in equilibrium.[2] The no arbitrage condition is shown to be equivalent to viability of asset prices. The consequence vindicates the coherence of arbitrage as a conceptual framework for equilibrium analysis. 'Arbitrage' pricing theory would be almost vacuous if it undergoes serious failure in the viability test.[3]

As mentioned earlier, arbitrage pricing theory has informational advantage over equilibrium pricing theory. What is required to capture the pricing rules is information on the average cost for large transactions which is independent of the local behavior of transaction cost functions. This point deserves remarks. The amount of information on the nature of market frictions which is necessary to describe the arbitrage pricing rules depends on the definition of arbitrage. Broadly speaking, the notions of arbitrage for frictional markets can be classified into two categories, 'local arbitrage' and 'global arbitrage'.[4] Local arbitrage is introduced to characterize the properties of optimal portfolios and equilibrium prices. Since they satisfy the first order conditions for utility maximization or cost minimization, the no local arbitrage conditions eventually depend on the current optimal positions. In particular, it requires the knowledge of the marginal effects of market frictions on asset pricing. Local arbitrage is used in most literature with convex transaction costs and taxation schedules.[5]

In contrast, global arbitrage can be used to provide necessary conditions for equilibrium prices in a parsimonious way. The no global arbitrage conditions are supposed to require minimal information on unobservable market

2) Informally speaking, asset prices are viable if they allow agents to make an optimal choice in asset markets.
3) As shown later, the well-known notions of arbitrage do not pass the viability test.
4) 'Global' is used to contrast the latter to the former.
5) For deails, see Ross (1987), and Dybvig and Ross (1986), Prisman (1986), Dermody and Prisman (1993) among others.

data such as risk preferences, the initial wealth, and the exact form of market frictions. In particular, they are free from the knowledge of the marginal effects of market frictions on asset pricing. Notably, Dammon, and Green (1987) characterize equilibrium prices and investigate the existence of the arbitrage pricing rules under progressive taxation from the viewpoint of global arbitrage. This paper takes the same line of research as Dammon and Green (1987).

The two notions of arbitrage usually lead to the same consequence of asset pricing in the case with proportional transaction costs. This is not the case, however, with non-proportional transaction costs or taxes. As illustrated in the main text, asset pricing by local arbitrage tends to extremely underestimate the multiplicity of the pricing rules when transaction costs are convex and extremely overestimate it when they are non-convex.

Moreover, both notions of arbitrage drastically diverge in informational requirement to characterize the pricing rules. The marginal transaction costs or tax rates are placed in the pricing kernel in the presence of transaction costs and taxation. Thus they provide indispensable information to capture the pricing rules. If transaction costs or capital income taxes are proportional to the size of transactions, their marginal effect is constant over all positions. If transaction cost functions are nonlinear, however, the marginal transaction cost depends upon the functional form of market frictions as well as the position to be concerned about. In theory, local arbitrage leads to sharper results than global arbitrage in the case with nonlinear cost functions. In reality, however, it is hard to pick out the pricing rules which meet the status quo of the markets under the unobservable pricing kernel. For example, Ross (1987) and Dybvig and Ross (1986) introduce local tax arbitrage to address arbitrage pricing theory with progressive taxation. In this case, the marginal tax rate is calculable on the basis of the knowledge of both the tax schedule and the current portfolio position of individuals, which do not belong to publicdomain of information in general.

The paper is organized as follows. In Section Ⅱ, the finite-period markets and the structure of transaction costs are discussed. In Section Ⅲ, four types of transaction cost functions are described which display the same average cost for large trades. It is also illustrated that the no arbitrage condition of the literature may not be compatible with viability. Transaction cost functions are characterized in terms of the average costs for large transactions in Section Ⅳ. In Section Ⅴ, the notion of arbitrage with transaction costs is presented and analyzed in terms of the average costs for large transactions. Section Ⅵ is devoted to verifying the equivalence between the no arbitrage condition and the existence of pricing rules. The pricing rules are characterized in a concrete form as in the case with proportional transaction costs. Section Ⅶ shows the triple equivalence between the no arbitrage condition, the existence of pricing rules and viability of asset prices. The paper is concluded with Section Ⅷ.

Ⅱ. The Model

Asset markets are assumed to persist over finite time periods, $t=1, \ldots, T$. Let $\Omega=\{1, 2, \ldots, S\}$ denote a finite set of the states of nature. The revelation of information is described by a collection of partitions of Ω, $F=\{F_0, F_1, \ldots, F_T\}$, where F_T is finer than F_{t-1} (i.e., $\sigma \in F_t$ and $\sigma' \in F_{t-1}$ imply that $\sigma \subset \sigma'$ or $\sigma \cap \sigma' = \varnothing$) for all $t=1, \ldots, T$.[6] We assume that $F_0 = \{\Omega\}$. The information available at time $t=1, \ldots, T$ is described by the set $\sigma \in F_t$ of the states of nature. We set $D=\cup_{t=0}^{T} F_t$ and $D_{-T}=\cup_{t=0}^{T-1} F_t$. An element in D is called a node or an event and D is called an event tree. In particular, σ_t in D denotes an event in F_t. For each $\sigma_t \in F_t$, let σ_t^- denote the event which immediately precedes σ_t, σ_t^+ the set of events which immediately succeed σ_t. For some positive integer n, let $\mathcal{L}(D_{-T}, \mathbb{R}^n)$ denote the collection of all $\mathbb{R}^n$-valued functions on D_{-T}. For brevity, $\mathcal{L}^n$ will be used instead of $\mathcal{L}(D_{-T}, \mathbb{R}^n)$. Let $\#D$ and $\#D_{-T}$ denote the number of elements in

6) For more details on the stochastic economy, see Magill and Shafer (1991) or Magill and Quinzii (1996).

D and $\#D_{-T}$. Then $\mathcal{L}^n$ is the Euclidean space of dimension $(\#D_{-T})\times n$. Let $\mathcal{L}$ denote the set of all real-valued functions defined on D. We set $\mathcal{L}_+ = \{x \in L : x(\sigma) \geq 0, \sigma \in D\}$ and $\mathcal{L}_{++} = \{x \in L : x(\sigma) > 0, \sigma \in D\}$.

There are J long-lived assets issued at time 0 and traded in each state of time $t = 0, \ldots, T-1$. Allowing for some notational abuse, we also denote the set of assets by J. A price process of asset j is a function $q_j : D_{-T} \to \mathbb{R}$ and a trading strategy is a function $\theta : D_{-T} \to \mathbb{R}^J$. Thus, $q = (q_1, \ldots, q_J)$ and θ are a point in $\mathcal{L}^J$. More specifically, $q^j(\sigma)$ and $\theta^j(\sigma)$ denote a price and a position of asset j, and $q(\sigma) \in \mathbb{R}^J$ and $\theta(\sigma) \in \mathbb{R}^J$ denote prices and positions of J assets at the node $\sigma \in D$. For a price-event pair (q, σ) in $\mathcal{L}^J \times D$, let $R(\cdot, q; \sigma) : \mathcal{L}^J \to \mathbb{R}$ denote the net return schedule which is derived from deducting transaction costs from the gross return. Specifically, if a trading strategy $\theta \in \mathcal{L}^J$ is chosen at the price q, the net return $R(\theta, q; \sigma)$ will be delivered to the investor in the event σ. For a price $q \in \mathcal{L}^J$, let $R(\cdot, q)$ denote the function which assigns each $\sigma \in D$ to $R(\cdot, q; \sigma)$. Thus, for a trading strategy $\theta \in \mathcal{L}^J$, $R(\theta, q)$ is a $\#L$-dimensional net return vector.

Transaction costs are incurred in buying and selling assets. For a net trade and a price (v, q) in $\mathcal{L}^J \times \mathcal{L}^J$, let $C^j_\sigma(v^j(\sigma), q^j(\sigma))$ denote the transaction cost for the positional changes $v^j(\sigma)$ with asset $j \in J$ in the event $\sigma \in D$. We set[7)]

$$C(v(\sigma), q(\sigma); \sigma) = \sum_{j \in J} C^j_\sigma(v^j(\sigma), q^j(\sigma)).$$

The function $C(v(\sigma), q(\sigma); \sigma)$ indicates the variable transaction costs for the portfolio changes $v(\sigma)$ in the event $\sigma \in D$. We exclude the effect of fixed transaction costs on asset pricing.[8)]

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7) Dermody and Prisman(1993) indicate that transaction costs on trading each individual asset is a function of the number of shares traded, and transaction costs on a trade is the sum of the transaction costs on trading each individual stock.

8) This exclusion is not totally unrealistic. If there exist investors or financial institutions which are rich enough to cover fixed transaction costs, then they can always benefit from any portfolios which could provide arbitrage opportunities in

For each $\sigma \in D$, let $R_\sigma \in \mathbb{R}^J$ denote the gross returns of assets which are available before transaction costs are deducted. Then for each $(\theta, q) \in \mathcal{L}^J \times \mathcal{L}^J$, the net return process $R(\theta, q;\sigma)$ is represented by

$$R(\theta, q;\sigma) = \begin{cases} -q(\sigma)\cdot\theta(\sigma) - C(\theta(\sigma), q(\sigma);\sigma), & \sigma = \sigma_0 \\ R_\sigma\cdot\theta(\sigma^-) - q(\sigma)\cdot(\theta(\sigma)-\theta(\sigma^-)) & \\ \qquad - C(\theta(\sigma)-\theta(\sigma^-), q(\sigma);\sigma), & \sigma \in D_{-T} \setminus \{\sigma_0\} \\ R_\sigma \cdot\theta(\sigma^-), & \sigma \in F_T \end{cases}$$

Ⅲ. Examples

Four transaction cost functions are presented in the first example which look quite distinct but give the same average cost for large transactions. As shown later, they give the same consequence in terms of asset pricing. It is illustrated in the second example that the no arbitrage condition of Dermody and Prisman (1993) may be far from being a necessary condition for viability when the transaction cost function is convex, piece-wise linear and differentiable at zero. Specifically, the no arbitrage condition of Dermody and Prisman (1993) explains only a ‘small’ part of viable prices.[9)]

Example 1. Transaction cost functions are given which locally differ but behave asymptotically in the same manner. As shown later, they are indistinguishable in terms of arbitrage pricing. Let q denote the price of an asset. For each $i = 1, \ldots, 4$, we define $C_i(\cdot\,, q) : \mathbb{R} \to \mathbb{R}$ as the transaction cost function for the asset. They are depicted in <Figure 1>.

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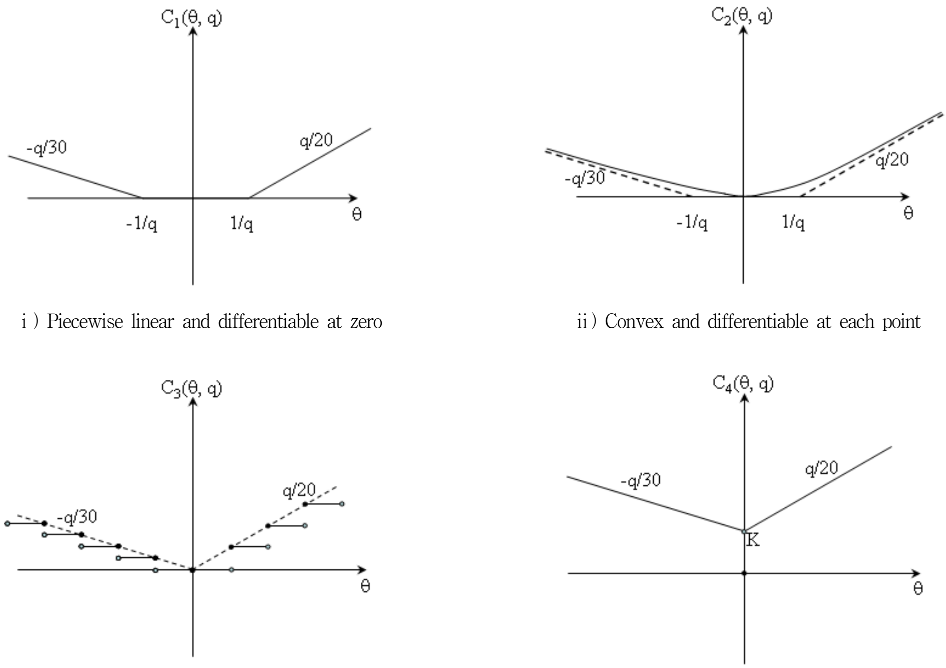
frictionless markets. In this case, the presence of fixed transaction costs does not matter to arbitrage pricing.

9) This result is true in general when transaction cost functions are convex and differentiable at zero.



i) Piecewise linear and differentiable at zero

ii) Convex and differentiable at each point

iii) Indivisible Assets

iv) Non-convex with jump at zero

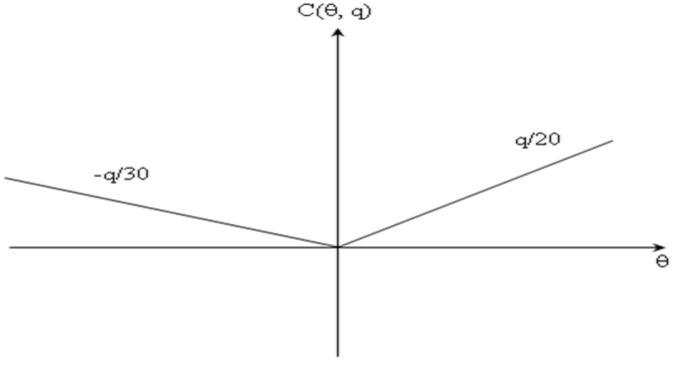
〈Figure 1〉

The function $C_1(\cdot, q)$ is piecewise linear with kinks at $-1/q$ and $1/q$, and flat at zero while $C_2(\cdot, q)$ is strictly convex and differentiable. The function $C_3(\cdot, q)$ jumps regularly and represents transaction costs with indivisible assets. In contrast to the first three functions which are free from fixed cost component, $C_4(\cdot, q)$ has the fixed transaction cost denoted by K. If K is ignored in $C_4(\cdot, q)$, it is the same as $C_3(\cdot, q)$.

Despite their local difference, they have the same asymptotic property. We set $\overline{C}_i(\theta, q) = \lim_{\lambda \to \infty} C_i(\lambda\theta, q)/\lambda$. It is easy to see that for each $i = 1, \ldots, 4$, $\overline{C}_i(\theta, q)$ is equal to the function

$$\overline{C}(\theta, q) = \begin{cases} \dfrac{q}{20}\theta, & \text{if } \theta \geq 0 \\ -\dfrac{q}{30}\theta, & \text{if } \theta < 0 \end{cases} \tag{1}$$

This function represents proportional transaction costs in such a way that $\frac{q}{20}$ is a transaction cost for buying one unit of the asset and $\frac{q}{30}$ is a transaction cost for selling one unit of the asset. The average transaction cost of the four cases for large transactions is all the same as that of the proportional transaction cost function $\overline{C}(\cdot, q)$. <Figure 2>.



〈Figure 2〉

Example 2. It is illustrated that the no arbitrage condition of Dermody and Prisman (1993) explains only a small part of viable prices. This means that a significant portion of viable prices may allow arbitrage in the sense of Dermody and Prisman (1993). They use the following notion of arbitrage in a two-period world.

Definition DP. The price $q \in \mathbb{R}^J$ admits *no arbitrage* if it satisfies

$$max - \theta \cdot q - C(\theta, q) : R \cdot \theta \geq 0 = 0 \text{ and}$$

$$R \cdot \overline{\theta} = 0 \text{ for all optimal solution } \overline{\theta} \in \mathbb{R}^J.$$ [10]

10) Let x and x' be vectors in a Euclidean space. Then $x \geq x'$ implies x is greater than or equal to x' in a component-wise manner; $x > x'$ implies that $x \geq x'$ and $x \neq x'$; $x \gg x'$ implies that each component of x is greater than the counterpart of x'.

To be more specific, we consider a two-asset one-state economy with convex transaction costs. Both assets pay one dollar in the state of the next period. Thus the gross return structure is represented by the 1×2 matrix $R=[1 \quad 1]$. Let q^1 and q^2 denote the price of the two assets. We assume that transaction costs are incurred only for trading the second asset according to the cost schedule

$$C_f(\theta^2, q^2) = \begin{cases} (q^2\theta^2 - 1)/20, & \text{if } \theta^2 \geq 1/q^2 \\ 0, & \text{if } -1/q^2 \leq \theta^2 < 1/q^2 \\ -(q^2\theta^2 + 1)/30, & \text{if } \theta^2 < -1/q^2 \end{cases} \tag{2}$$

This function is depicted in the first diagram of <Figure 1>. Clearly, $C_f(0, q^2) = 0$ and $C_f(\theta^2, q^2)$ is piece-wise linear, continuous and convex.

Let Λ_{DP} denote the set of prices which admit no arbitrage in the sense of Definition DP. Then $(q^1, q^2) \in \Lambda_{DP}$ if and only if it satisfies

$$max\{-\theta^1 q^1 - \theta^2 q^2 - C_f(\theta^2, q^2) : \theta^1 + \theta^2 \geq 0\} = 0 \text{ and}$$
$$\overline{\theta^1} + \overline{\theta^2} = 0 \text{ for all optimal solution } (\overline{\theta^1}, \overline{\theta^2}).$$

We claim that

$$\Lambda_{DP} = \{(q^1, q^2) \in \mathbb{R}^2_{++} : q^1 = q^2\}.$$

Let $(\overline{\theta^1}, \overline{\theta^2})$ denote the solution to $max\{-\theta^1 q^1 - \theta^2 q^2 - C_f(\theta^2, q^2) : \theta^1 + \theta^2 \geq 0\} = 0$. Suppose that $q^1 = q^2$. Then it is easy to see that $\overline{\theta^1} + \overline{\theta^2} = 0$ and $max\{-\theta^1 q^1 - \theta^2 q^2 - C_f(\theta^2, q^2) : \theta^1 + \theta^2 \geq 0\} = 0$. Therefore, (q^1, q^2) satisfies the no arbitrage condition of Definition DP.

We show that any (q^1, q^2) with $q^1 \neq q^2$ admits an arbitrage in the sense of Definition DP. Suppose that $q^1 > q^2$. We set $\eta^1 = -1/2q^2$ and $\eta^2 = -1/2q^2$. Then $-1/q^2 \leq \eta^2 < 1/q^2$ and $\eta^1 + \eta^2 \geq 0$. Moreover,

$$-\eta^1 q^1 - \eta^2 q^2 - C_f(\eta^2, q^2) = -\eta^1 q^1 - \eta^2 q^2$$
$$= -\frac{1}{2}(1 - \frac{q^1}{q^2}) > 0 \tag{3}$$

Suppose that $q^1 < q^2$. We set $\eta^1 = 1/2q^2$ and $\eta^2 = -1/2q^2$. Then $-1/q^2 \leq \eta^2 < 1/q^2$ and $\eta^1 + \eta^2 \geq 0$. Moreover,

$$-\eta^1 q^1 - \eta^2 q^2 - C_f(\eta^2, q^2) = -\eta^1 q^1 - \eta^2 q^2$$
$$= \frac{1}{2}(1 - \frac{q^1}{q^2}) > 0 \tag{4}$$

It follows that for any (q^1, q^2) with $q^1 \neq q^2$,

$$max\{-\theta^1 q^1 - \theta^2 q^2 - C_f(\theta^2, q^2) : \theta^1 + \theta^2 \geq 0\} > 0.$$

Thus, whenever $q^1 \neq q^2$, (q^1, q^2) does not satisfies the no arbitrage condition of Definition DP.

To do a viability test with prices in Λ_{DP}, we introduce the set Λ_V of viable prices. Any prices in Λ_V would allow agents with monotonic preferences to have an optimal choice in asset markets. We claim that

$$\Lambda_V = \{(q^1, q^2) \in \mathbb{R}^2_{++} : (29/30)q^2 \leq q^1 \leq (21/20)q^2\}.$$

(This claim is verified in the Appendix.) Prices in Λ_V are viable but any with $q^1 \neq q^2$ does not satisfy the no arbitrage condition of Dermody and Prisman (1993). Thus, the no arbitrage condition of Dermody and Prisman (1993) explains a very small part of viable prices.

Ⅳ. Transaction Costs in the Large

Transaction cost functions are characterized in terms of average costs for large transactions. We demonstrate that there exists the proportional transaction cost which shares arbitrage pricing rules with the original nonlinear transaction cost functions. The proportional function is determined independently of the local curvature of the original functions. We assume the convexity of transaction cost functions.

Assumption 1 : For all $q \in \mathcal{L}^J$, $j \in J$ and $\sigma \in D$, $C_\sigma^j(\cdot, q^j(\sigma))$ is a convex function and $C_\sigma^j(0, q^j(\sigma)) = 0$.

This condition states that transaction cost functions are convex and no transactions cost nothing. For each $q \in \mathcal{L}^J$, $j \in J$ and $\sigma \in D$, we set

$$b^j(q^j(\sigma);\sigma) = \lim_{z \to \infty} \frac{C_\sigma^j(z, q^j(\sigma))}{z} \text{ and } s^j(q^j(\sigma);\sigma) = \lim_{z \to -\infty} \frac{C_\sigma^j(z, q^j(\sigma))}{z}.$$

Then $b^j(q^j(\sigma);\sigma)$ and $s^j(q^j(\sigma);\sigma)$ are the average transaction cost for large long positions and short positions in asset j, respectively. If average transaction costs are unbounded at infinity, then marginal costs increase progressively and eventually, go to infinity. This means that nobody wants to possess long or short positions beyond a certain large size of trade volumes even when asset prices are negative. In this case, any asset prices, whether positive or negative, do not admit arbitrage opportunity. Thus, the case with unbounded average costs at infinity is trivial in terms of arbitrage pricing theory because any prices are arbitrage-free.

To be more specific, suppose that $b^j(q^j(\sigma);\sigma) = \infty$ for some $j \in J$ and $\sigma \in D_{-T}$. Let θ be a nonzero portfolio with $\theta(\sigma) - \theta(\sigma^-) > 0$, i.e., non-trivial long position with asset j in the event σ. For any $\lambda > 0$, the net return of asset j in σ generated by $\lambda\theta$ is

$$\lambda\left[R_\sigma^j \theta^j(\sigma^{-1}) - q^j(\sigma)[\theta(\sigma) - \theta(\sigma^{-1})] - \frac{C_\sigma^j[\lambda(\theta^j(\sigma^{-1}) - \theta(\sigma)), q^j(\sigma)]}{\lambda}\right]. \quad (5)$$

Since $C_\sigma^j[\lambda(\theta^j(\sigma^{-1}) - \theta(\sigma)), q^j(\sigma)]/\lambda = \infty$, for sufficiently large λ the term inside the bracket of (5) is negative and therefore, the net return in (5) goes to minus infinity as $\lambda \to \infty$. Such a consequence holds for any θ and q in $\mathcal{L}^J$.

In particular, as the size of long positions with asset j increases, the net return goes to minus infinity even at prices $q^j(\sigma) < 0$ [11]. This is quite unrealistic. The same is true for the case with $s^j(q^j(\sigma);\sigma) = \infty$. To exclude these unrealistic cases, we can assume that for each $q \in \mathcal{L}^J$, $j \in J$ and $\sigma \in D$,

$$-\infty < s^j(q^j(\sigma);\sigma) \leq b^j(q^j(\sigma);\sigma) < \infty. \tag{6}$$

We define the function

$$\overline{C}^j_\sigma(z, q^j(\sigma)) = \begin{cases} b^j(q^j(\sigma);\sigma)z, & z \geq 0 \\ s^j(q^j(\sigma);\sigma)z, & z < 0 \end{cases} \tag{7}$$

This function can be considered a proportional transaction cost for the change of the position z for asset j in markets where $b^j(q^j(\sigma);\sigma)$ and $s^j(q^j(\sigma);\sigma)$ are charged as the unit transaction cost for long and short positions, respectively. Remarkably, the function in (7) displays the asymptotic behavior of C^j_σ for a given position z because for a nonzero z,

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{C^j_\sigma(\lambda z, q^j(\sigma))}{\lambda} &= \lim_{\lambda\to\infty} \frac{C^j_\sigma(\lambda z, q^j(\sigma))}{\lambda z} z \\ &= \begin{cases} b^j(q^j(\sigma);\sigma)z, & z > 0 \\ s^j(q^j(\sigma);\sigma)z, & z < 0 \end{cases} \\ &= \overline{C}^j_\sigma(z, q^j(\sigma)) \end{aligned} \tag{8}$$

For each q, v in $\in \mathcal{L}^J$ and $\sigma \in D$, we set

$$\overline{C}(v(\sigma), q(\sigma);\sigma) = \lim_{\lambda\to\infty} \frac{C(\lambda v(\sigma), q(\sigma);\sigma)}{\lambda}. \tag{9}$$

It follows that

11) Any price $q^j(\sigma)$ whether it is positive or negative would not admit an arbitrage opportunity because sufficiently large long position with asset j necessarily generates loss.

$$\overline{C}(v(\sigma), q(\sigma); \sigma) = \sum_{j \in J} \overline{C}^j_\sigma(v^j(\sigma), q^j(\sigma)). \tag{10}$$

We define the following notion of transaction cost function.

Definition 4.1. For each $q \in \mathcal{L}^J$ and $\sigma \in D$, $\overline{C}^j_\sigma(\cdot, q^j(\sigma)) : \mathbb{R} \rightarrow \mathbb{R}$ and $\overline{C}(\cdot, q(\sigma); \sigma) : \mathbb{R}^J \rightarrow \mathbb{R}$ are the *transaction cost function in the large (LTC function)* for asset $j \in J$ and for the asset structure J, respectively.

The LTC function represents average transaction costs for large net trades. Clearly, it is a convex and positively homogeneous function.

Lemma 4.1. *For each* $q \in \mathcal{L}^J$, $j \in J$ *and* $\sigma \in D$, $\overline{C}^j_\sigma(\cdot, q^j(\sigma))$ *is convex and* $\overline{C}^j_\sigma(\lambda z, q^j(\sigma)) = \lambda \overline{C}^j_\sigma(z, q^j(\sigma))$ *for all* $z \in \mathbb{R}$ *and* $\lambda \geq 0$.

If the transaction cost function is proportional, it coincides with the LTC function. Let $\bar{b}^j(\sigma) \in [0, 1)$ and $\bar{s}^j(\sigma) \in [0, 1)$ denote the transaction cost rate for buying and selling j at σ. Clearly, $\bar{b}^j(\sigma) q^j(\sigma) = b^j(q^j(\sigma); \sigma)$ and $\bar{s}^j(\sigma) q^j(\sigma) = - s^j(q^j(\sigma); \sigma)$ for all $q \in \mathcal{L}^J$, $j \in J$ and $\sigma \in D$. In this case, the LTC function has the form

$$\overline{C}^j_\sigma(z, q^j(\sigma)) = \begin{cases} \bar{b}^j(\sigma) q^j(\sigma) z, & z \geq 0 \\ \\ -\bar{s}^j(\sigma) q^j(\sigma) z, & z < 0 \end{cases} \tag{11}$$

For each q, θ in $\mathcal{L}^J$ and $\sigma \in D$, we define the set

$$G(\theta, q; \sigma) = \{v \in \mathcal{L}^J : R(\theta + \lambda v, q; \sigma) \geq R(\theta, q; \sigma) \quad \forall \lambda \geq 0\}.$$

The set $G(\theta, q; \sigma)$ contains portfolios represented by a direction in which the net return is improved from the current position in the event σ. By convexity of transaction cost functions, $R(\cdot, q; \sigma)$ is a concave function for all q and

$\sigma \in D$. Thus, it coincides with the recession cone of the level set $\{\theta \in \mathcal{L}^J : R(\theta, q\ ;\ \sigma) \geq c\}$.[12] For notational ease, we set $G(q\,;\sigma) = G(0, q\ ;\ \sigma)$ for each $q \in \mathcal{L}^J$. By Theorem 8.7 of Rochafellar (1970), $G(\theta, q\,;\sigma) = G(q\ ;\ \sigma)$ for each $\theta \in \mathcal{L}^J$. We set $G(q) = \cap_{\sigma \in D} G(q\,;\sigma)$. A nonzero $v \in G(q)$ represents a change of the position which adds nonnegative income to any positions in each state if the positional change is sufficiently large in the direction v. Clearly, $G(q)$ is a cone. We make the following assumption.

Assumption 2 : For each $q \in \mathcal{L}^J$ and $\sigma \in D\ \backslash_{-T}$, let v be a point in $G(q\ ;\ \sigma)$ with $R(v, q\,;\sigma) > 0$. Then the following holds true.

$$\lim_{\lambda \to \infty} \frac{\lambda \overline{C}(v(\sigma), q(\sigma);\sigma) - C(\lambda v(\sigma), q(\sigma);\sigma)}{R(\lambda v, q\,;\sigma)} < 1.$$

This condition requires that as the size of v increases indefinitely, the asymptotic proportion of the difference between the proportional and original transaction costs to the net returns is less than one. Assumption 2 trivially holds when the transaction cost function C is proportional to the positional changes. The following provides a class of transaction cost functions which satisfy Assumption 2.

Lemma 4.2. For each $q \in \mathcal{L}^J$ and $\sigma \in D\ \backslash_{-T}$, let v be a point in $G(q\ ;\ \sigma)$ with $R(v, q\,;\sigma) > 0$. Suppose that $\lim_{\lambda \to \infty} R(\lambda v, q\,;\sigma) \to \infty$ and there exists $\delta(\,\cdot\,, q(\sigma);\sigma) : \mathbb{R}^j \to \mathbb{R}$ such that $sup_{z \in \mathbb{R}^J} |\delta(z, q(\sigma);\sigma)| < \infty$ and for all $x \in \mathbb{R}^J$,

$$\overline{C}(z, q(\sigma);\sigma) = C(z, q(\sigma);\sigma) + \delta(z, q(\sigma);\sigma).$$

Then Assumption 2 holds true.

12) For a convex set C in a Euclidean space E, $v \in E$ is a direction of recession of C if $v + x \in C$ for all $x \in C$. The recession cone of C is a set of directions of recession of C. For details, see Rockafellar (1970).

PROOF : Since $\lim_{\lambda\to\infty} R(\lambda v, q; \sigma) \to \infty$ and $sup_{z\in\mathbb{R}^J}\delta(z;\sigma) < \infty$, it is easy to see that

$$\lim_{\lambda\to\infty} \frac{\lambda\overline{C}(v(\sigma), q(\sigma);\sigma) - C(\lambda v(\sigma), q(\sigma);\sigma)}{R(\lambda v, q; \sigma)} = \lim_{\lambda\to\infty} \frac{\delta(\lambda v(\sigma), q(\sigma);\sigma)}{R(\lambda v, q ; \sigma)} = 0 \qquad \square$$

For each $q \in \mathcal{L}^J$ and $\sigma \in D$, we consider the function $V(\cdot, q; \sigma) : \mathcal{L}^J \to \mathbb{R}$ defined by

$$V(\theta, q; \sigma) = \begin{cases} -q(\sigma)\cdot\theta(\sigma) - \overline{C}(\theta(\sigma), q(\sigma);\sigma), & \sigma = \sigma_0 \\ R(\sigma)\cdot\theta(\sigma^-) - q(\sigma)\cdot(\theta(\sigma) - \theta(\sigma^-)) & \\ \qquad - \overline{C}(\theta(\sigma) - \theta(\sigma^-), q(\sigma); \sigma), & \sigma \in D_{-T} \setminus \sigma_0 \\ R_\sigma \cdot \theta(\sigma^-), & \sigma \in F_T \end{cases} \tag{12}$$

Recalling that $\overline{C}(v(\sigma), q(\sigma); \sigma) = \lim_{\lambda\to\infty} C(\lambda v(\sigma), q(\sigma)/\lambda$ for each $v \in \mathcal{L}^J$ and $\sigma \in D$, we see that for all $\theta \in \mathcal{L}^J$,

$$V(\theta, q ; \sigma) = \lim_{\lambda\to\infty} \frac{R(\lambda\theta, q; \sigma)}{\lambda}. \tag{13}$$

The function $V(\cdot, q; \sigma)$ represents the net return schedule at the price q in the event σ if transaction costs were to be charged according to the LTC function $\overline{C}(\cdot, q(\sigma); \sigma)$.

Definition 4.2. For each $q \in \mathcal{L}^J$ and $\sigma \in D$, $V(\cdot, q; \sigma)$ is the *return function in the large (L-return function)* for the asset structure J.

The following properties of the L-return function $V(\cdot, q; \sigma)$ are immediate from Lemma 4.1.

Lemma 4.3. *For each* $q \in \mathcal{L}^J$ *and* $\sigma \in D$, $V(\cdot, q; \sigma)$ *is concave and* $V(\lambda\theta, q; \sigma) = \lambda V(\theta, q ; \sigma)$ *for all* $\theta \in \mathcal{L}^J$ *and* $\lambda \geq 0$.

V. Arbitrage with Transaction Costs

The following provides an extension of the notion of arbitrage with frictionless markets to the markets with transaction costs.

Definition 5.1. An asset price $q \in \mathcal{L}^J$ admits *no arbitrage opportunities* if there is no $\theta \in G(q)$ which satisfies $R(\theta, q) > 0$.

The notion of arbitrage in Definition 5.1 has several desired properties. First, it will allow us to characterize as easily the pricing rules in markets with non-proportional transaction costs as in markets with proportional transaction costs. Second, it turns out to exactly match viability of asset prices. This is one of the virtues that no arbitrage conditions must satisfy as a conceptual framework for equilibrium analysis. Third, the no arbitrage condition does not depend on the initial position. This property is particularly useful in characterizing the pricing rules when marginal transaction costs are an information not to be observed. If transaction costs is nonproportional, information on the cost function and the initial position is usually required to calculate marginal transaction costs. But such information is specific to individuals and does not belong to the public domain of information. Finally, it subsumes as a special case the existing notions of arbitrage with proportional transaction costs used in Garman and Ohlson (1981) and Zhang, Xu, and Deng (2002) among others.

The set of no arbitrage prices is given by

$$\Lambda = \{q \in \mathcal{L}^J : R(v, q) \not> \text{ for all } v \in G(q)\},$$

where $\not>$ denotes the negation of the vector inequality $>$.

We show that the no arbitrage condition can be characterized by the L-return function.

Proposition 5.1. *Under Assumptions 1-2,* $q \in \Lambda$ *if and only if there exists no nonzero* $v \in \mathcal{L}^J$ *which satisfies* $V(v, q) > 0$.

PROOF : Suppose that there exists a nonzero $v \in \mathcal{L}^J$ such that $V(v, q) > 0$. By the convexity of C, we have $C(\lambda v(\sigma), q(\sigma); \sigma) \geq \lambda C(v(\sigma), q(\sigma); \sigma)$ for all $\lambda > 1$ and therefore,

$$\lim_{\lambda \to \infty} C(\lambda v(\sigma), q(\sigma); \sigma)/\lambda = \overline{C}(v(\sigma), q(\sigma); \sigma) \geq C(v(\sigma), q(\sigma); \sigma). \tag{14}$$

Thus, it follows that for $\sigma = \sigma_0$,

$$\begin{aligned} V(v, q; \sigma) &= -q(\sigma) \cdot v(\sigma) - \overline{C}(v(\sigma), q(\sigma); \sigma) \\ &\leq -q(\sigma) \cdot v(\sigma) - C(v(\sigma), q(\sigma); \sigma) \end{aligned} \tag{15}$$

This implies that for $\sigma = \sigma_0$, $R(v, q; \sigma) \geq V(v, q; \sigma)$.

By applying the same arguments to the case with $\sigma \in D \setminus \{\sigma_0\}$, we can show that $R(v, q; \sigma) \geq V(v, q; \sigma)$. Recalling that $V(v, q) > 0$, we have $R(v, q) > 0$. By Lemma 4.3, $V(\lambda v, q) = \lambda V(v, q) > 0$ and therefore, $R(\lambda v, q) > 0$ for all $\lambda > 0$. This implies that $v \in G(q; \sigma)$. Thus, v provides an arbitrage opportunity, which contradicts the fact that $q \in \Lambda$.

Suppose that there exists a nonzero $v \in G(q)$ such that $R(v, q) > 0$. Since $v \in G(q)$ and $R(0, q; \sigma) = 0$ for each $\sigma \in D$, $R(\lambda v, q; \sigma) > 0$ for all $\lambda > 0$. This implies that for all $\sigma \in D$,

$$V(v, q; \sigma) = \lim_{\lambda \to \infty} R(\lambda v, q; \sigma)/\lambda \geq 0. \tag{16}$$

Since $V(v, q) \geq 0$, we have only to show that there exists $\sigma \in D$ such that $V(v, q; \sigma) > 0$. Recalling that $R(v, q) > 0$, we can pick $\bar{\sigma} \in D$ such that $R(v, q; \bar{\sigma}) > 0$. If $\bar{\sigma} \in F_T$, we are done because

$$V(v, q; \bar{\sigma}) = R(v, q; \bar{\sigma}) = R_{\bar{\sigma}} \cdot v(\bar{\sigma}^-). \tag{17}$$

Thus without loss of generality, we can assume that $\bar{\sigma} \in D \setminus F_T$.

We consider the case with $\bar{\sigma} = \sigma_0$.

$$
\begin{aligned}
0 <\ & R(\lambda v, q; \sigma_0) \\
&= -\lambda q(\sigma_0)\cdot v(\sigma_0) - C(\lambda v(\sigma_0), q(\sigma_0); \sigma_0) \\
&= -\lambda q(\sigma_0)\cdot v(\sigma_0) - \lambda \overline{C}(v(\sigma_0), q(\sigma_0); \sigma_0) \\
&\qquad\qquad + [\lambda \overline{C}(v(\sigma_0), q(\sigma_0); \sigma_0) - C(\lambda v(\sigma_0), q(\sigma_0); \sigma_0)] \\
&= \lambda[-q(\sigma_0)\cdot v(\sigma_0) - \overline{C}(v(\sigma_0), q(\sigma_0); \sigma_0)] \\
&\qquad\qquad + [\lambda \overline{C}(v(\sigma_0), q(\sigma_0); \sigma_0) - C(\lambda v(\sigma_0), q(\sigma_0); \sigma_0)]
\end{aligned}
$$

Then it follows from Assumption 2 that for sufficiently large $\lambda > 0$,

$$
\frac{\lambda}{R(\lambda v, q; \sigma_0)}[-q(\sigma_0)\cdot v(\sigma_0) - \overline{C}(v(\sigma_0), q(\sigma_0); \sigma_0)] =
$$

$$
1 - \frac{\lambda \overline{C}(v(\sigma_0), q(\sigma_0); \sigma_0) - C(\lambda v(\sigma_0), q(\sigma_0); \sigma_0)}{R(\lambda v, q; \sigma)} > 0
$$

This result implies that $V(v, q; \sigma_0) = -q(\sigma_0)\cdot v(\sigma_0) - \overline{C}(v(\sigma_0), q(\sigma_0); \sigma_0) > 0$. By applying the same arguments to the case with $\bar{\sigma} \neq \sigma_0$ we can show that $V(v, q; \bar{\sigma}) > 0$. Consequently, we have $V(v, q) > 0$ □

By Proposition 5.1, the notion of arbitrage in Definition 5.1 is equivalent to the following.

Definition 5.1′. An asset price $q \in \mathcal{L}^J$ admits *no arbitrage opportunities* if there is no $\theta \in \mathcal{L}^J$ which satisfies $V(v, q) > 0$.

The set of no arbitrage prices is expressed as

$$
\Lambda = \{q \in \mathcal{L}^J : V(v, q) \not> 0 \text{ for } all\ v \in \mathcal{L}^J\}.
$$

It is worth noting that the effect of transaction costs on $V(v, q)$ is determined by the LTC function which describes the behavior of average costs for large transactions. Thus, transaction costs affect asset pricing only via the LTC function. Any other parts of transaction costs beyond proportionality are irrelevant to the determination of pricing rules.

Ⅵ. Arbitrage and the Existence of Pricing Rules

It is demonstrated here that the no arbitrage condition of Definition 5.1 allows us to extend the classical fundamental theorem of asset pricing to the case with non-proportional transaction costs.[13] For each $q \in \Lambda$, we define the set

$$Z(q) = \{y \in L : y \leq V(v, q) \text{ for } some\ v \in L^J\}.$$

We introduce the following technical lemma.

Lemma 6.1. *For all $q \in \mathcal{L}^J$, the set $Z(q)$ is closed.*

PROOF : See the appendix.

The following result shows the equivalence between the no arbitrage condition and the existence of pricing rules.

Theorem 6.1. *Under Assumptions 1–2, $q \in \Lambda$ if and only if there exists $\pi \in L_{++}$ such that $\pi \cdot V(v, q) \leq 0$ for all $v \in \mathcal{L}^J$.*

PROOF : Suppose that $q \notin \Lambda$. Then by Proposition 5.1 there exists $v \in \mathcal{L}^J$ such that $V(v, q) > 0$. Since $\pi \in L_{++}$, this implies that $\pi \cdot V(v, q) > 0$, which leads to a contradiction.

Suppose that $q \in \Lambda$. By Lemmas 4.3 and 6.1, $Z(q)$ is a closed, convex cone. Let Δ denote the set $\{y \in L_+ : \sum_{\sigma \in D} y(\sigma) = 1\}$. Clearly, Δ is compact and convex. Then $q \in \Lambda$ is equivalent to the condition that $Z(q) \cap (L_+ \setminus \{0\}) = \varnothing$ or $Z(q) \cap \Delta = \varnothing$. Since $Z(q)$ is a closed, convex cone, by the separating hyperplane theorem there exists a nonzero $\pi \in L$ such that

13) Magill and Quinzii (1996), and Dybvig and Ross (1989) are a great reference to the fundamental theorem of asset pricing in a frictionless market.

$$sup_{v \in \mathcal{L}^J} \pi \cdot V(v,q) < inf_{y \in \Delta} \pi \cdot y. \tag{18}$$

In particular, we see that

$$0 = \pi \cdot V(0,q) \leq sup_{v \in \mathcal{L}^J} \pi \cdot V(v,q) < inf_{y \in \Delta} \pi \cdot y \tag{19}$$

Thus we have $inf_{y \in \Delta} \pi \cdot y > 0$, which implies that $\pi \in L_{++}$. Let $v \in \mathcal{L}^J$. Then for each $\lambda > 0$, we have $\pi \cdot V(\lambda v, q) = \lambda \pi \cdot V(v,q) < inf_{y \in \Delta} \pi \cdot y$, or

$$\pi \cdot V(v,q) < inf_{y \in \Delta} (\pi \cdot y)/\lambda \tag{20}$$

By letting $\lambda \to \infty$, we have $\pi \cdot V(v,q) \leq 0$. □

The pricing rules which satisfy the no arbitrage condition are characterized in a concrete form as follows.

Theorem 6.2. *Under Assumptions 1-2,* $q \in \Lambda$ *if and only if there exists* $\pi \in L_{++}$ *such that for each* $\sigma \in D$ *and* $j \in J$,

$$\pi(\sigma)[q^j(\sigma) + s^j(q^j(\sigma);\sigma)] \leq \sum_{\hat{\sigma} \in D_\sigma \setminus \{\sigma\}} \pi(\hat{\sigma}) R^j_{\hat{\sigma}} \leq \pi(\sigma)[q^j(\sigma) + b^j(q^j(\sigma);\sigma)] \tag{21}$$

PROOF : For each $q \in \mathcal{L}^J$, $j \in J$ and $\sigma \in D$, the LTC function $\overline{C}^j_\sigma(\cdot, q^j(\sigma))$ in $V(\cdot, q;\sigma)$ is proportional to the parameter which is equal to $b^j(q^j(\sigma);\sigma)$ if the changes of position are nonnegative and to $s^j(q^j(\sigma);\sigma)$ if the changes of position are negative. Thus the proof of the theorem can be done by applying the same argument made in the proof of Theorem 3.1 and 3.2 of Zhang, Xu and Deng (2002) which assume that the original transaction cost function $C^j_\sigma(\cdot, q^j(\sigma))$ is proportional to the unit cost of transactions. □

Theorems 6.1 and 6.2 shows that the pricing rules with convex transaction costs are as simple as in the case with proportional transaction costs.

The following examples show that compared to Definition 5.1, the notion of arbitrage used in Dermody and Prisman (1993) underestimates the multiplicity of the pricing rules when transaction cost functions are convex and overestimates it when they are non-convex.

Example 3. We consider a two-asset one-state economy which is the same as in Example 2 except for the transaction cost functions. Both assets pay one dollar in the state. Then the return function is a 1×2 matrix $R=[1\ 1]$. Let q^2 denote the price of the second asset. We assume that the transaction cost function for trading the second asset is one of the four one of the four $C_i(\cdot, q^j(\sigma))'$s depicted in <Figure 1>. For each $i=1,\ldots,4$, let Λ_i denote the set of no arbitrage prices with C_i. Then it follows by Theorem 6.2 that for each $i=1,\ldots,4$.

$$\Lambda_i=\{(q^1, q^2)\in\mathbb{R}^2_{++}:(29/30)q^2\le q^1\le(21/20)q^2\}.$$

It is worth noting that in the case with C_1 and C_2, Λ_{DP} defined in Example 2 is equal to $\{(q^1, q^2)\in\mathbb{R}^2_{++}: q^1=q^2\}$ and therefore, is much smaller than Λ_1. Thus the no arbitrage condition of Dermody and Prisman (1993) extremely underestimates the multiplicity of the pricing rules which are shown to be viable in the next section.[14)]

Example 4. We consider a two-asset one-state economy which is the same as in Example 2 except that the transaction cost function is replaced by the following function

$$C_{nc}(\theta^2, q^2)=\begin{cases}(q^2/2)|\theta^2|, & \text{if } |\theta^2|<1\\ (q^3/3)|\theta^2|, & \text{if } 1\le|\theta^2|<2\\ (q^2/2)|\theta^2|, & \text{if } |\theta^2|<1\end{cases}\tag{22}$$

This function is continuous and locally non-convex. We set

14) Such underestimation is a general phenomenon when transaction cost functions are convex and smooth at the origin.

$$\overline{C}^{j}_{nc}(\theta^2, q^2) = \lim_{\lambda\to\infty} C_{nc}(\lambda\theta^2, q^2)/\lambda \tag{23}$$

Clearly, $\overline{C}_{nc}(\theta^2, q^2) = (q^2/4)|\theta^2|$.

By Theorem 6.2, we have

$$\Lambda = \{(q^1, q^2) \in \mathbb{R}^2_{++} : (3/4)q^2 \leq q^1 \leq (5/4)q^2\}.$$

Let Λ_{DP} denote the set of no arbitrage prices in the sense of Dermody and Prisman (1993). It is easy to see that

$$\Lambda_{DP} = \{(q^1, q^2) \in \mathbb{R}^2_{++} : q^2/2 \leq q^1 \leq (3/2)q^2\}.$$

Clearly, $\Lambda \subset \Lambda_{DP}$. Thus, the no arbitrage condition of Dermody and Prisman (1993) overestimates the multiplicity of the pricing rules.[15]

Ⅶ. Arbitrage and Viability

Most literature on asset valuation by arbitrage focuses on verifying the equivalence between the no arbitrage conditions and the existence of pricing functionals. If the notions of arbitrage do not pass viability test, however, they fail to exactly characterize asset pricing in equilibrium. We shows that the no arbitrage condition of Definition 5.1 is equivalent to viability. Thus the notion of arbitrage in Definition 5.1 provides a coherent conceptual framework for studying asset pricing, portfolio choice problem, or equilibrium in markets with general transaction cost structures.

To examine viability of arbitrage-free prices, we introduce an agent who has the endowment of consumptions $e \in L_+$ and preferences represented by a utility function $u: L_+ \to \mathbb{R}$.[16] For a price $q \in \mathcal{L}^J$, the agent chooses

15) Such overestimation is a general phenomenon when transaction cost functions are not convex and smooth at the origin.

16) It is implicitly assumed that a single consumption good is available in each state of the economy.

$(x^*, \theta^*) \in L_+ \times \mathcal{L}^J$ which solves the optimization problem :

$$max_{(x,\theta)} u(x)$$

subject to the budget set

$$B(q) \in \{(x, \theta) \in L_+ \times \mathcal{L}^J : x - e \leq R(\theta, q)\}.$$

The demand correspondence $\xi(q)$ is the set of optimal choices in $L_+ \times \mathcal{L}^J$ which solve the above optimization problem.

Definition 7.1. An asset price $q \in \mathcal{L}^J$ is *viable* if $\xi(q) \neq \varnothing$.

To investigate the relationship between the no arbitrage condition and viability, we make the following assumptions.

Assumption 3 : For each $q \in \mathcal{L}^J$ and $\sigma \in D$, let v be a point in $G(q; \sigma)$ with $R(\nu, q; \sigma) > 0$. Then for each $\theta \in L^J$, there exists $\lambda > 0$ such that

$$R(\theta + \lambda v, q; \sigma) > R(\theta, q; \sigma).$$

Assumption 4 : u is continuous, strictly increasing and quasiconcave.

Assumption 5 : For a price $q \in \Lambda$, the following set is closed.

$$X(q) = \{x \in L_+; x - e \leq R(\theta, q) \text{ for } some\, \theta \in \mathcal{L}^J\}.$$

Let θ be a point in L^J. If $v \in G(q; \sigma)$, by definition $R(\theta + \lambda v, q; \sigma) \geq R(\theta, q; \sigma)$ for all $\lambda > 0$. Assumption 3 requires that the inequality be strict for some $\lambda > 0$. Assumption 4 is standard. Assumption 5 requires that the set of income transfers be closed. If $V(\cdot, q) = R(\cdot, q)$ for all $q \in \mathcal{L}^J$, then Lemma 6.1 shows that Assumption 5 holds true. By the same argument used in proving Lemma 6.1, we can show that Assumption 5 is satisfied in the case where is $R(\cdot, q)$ is piece-wise linear with finitely many kinks. It is also satisfied with transaction cost functions with fixed cost component as in the fourth diagram of <Figure 4>.[17)]

The following result shows that the no arbitrage condition is fully compatible with viability of the pricing rules.

Theorem 7.1. *Under Assumptions 1–5,* $q \in \mathcal{L}^J$ *if and only if* $\xi(q) \neq \varnothing$.

PROOF : ($\leftarrow$) For a price $q \in \mathcal{L}^J$, suppose that $\xi(q) \neq \varnothing$. Then there exists a point $(x, \theta) \in \xi(q)$. Suppose that there exists a nonzero $v \in G(q)$ such that $R(v, q) > 0$. Then $R(\theta + \lambda v, q) \geq R(\theta, q)$ for all $\lambda > 0$. Let σ be the event in D with $R(v, q;\sigma) > 0$. By Assumption 3, there exists $\lambda > 0$ such that $R(\theta + \lambda v, q;\sigma) \geq R(\theta, q;\sigma)$. By the strict monotonicity of u, there exists $x' \in L_+$ such that $u(x') > u(x)$ and $(x, \theta + \lambda v) \in B(q)$, which contradicts the optimality of (x, θ) in $B(q)$.

($\rightarrow$) Let q be a price in Λ. We set

$$X(q) = \{x \in L_+; x - e \leq R(\theta, q) \text{ for } some\ \theta \in \mathcal{L}^J\}.$$

First, we show that $X(q)$ is compact. By Assumption 5, $X(q)$ is closed. Thus, we have to show that $X(q)$ is bounded. Suppose that it is unbounded. Then $B(q)$ is unbounded. Since $B(q)$ is closed and convex, by Theorem 8.4 of Rockafellar (1970), there must exist a direction of recession $(y, \eta) \in L_+ \times L^J$ of $B(q)$ such that $y > 0$, $\eta \neq 0$, and for all $(x, \eta) \in B(q)$ and $\lambda > 0, (y, \theta) + \lambda(y, \eta) \in B(q)$. In particular, $(e, 0) + \lambda(y, \eta) \in B(q)$ for all $\lambda > 0$. Then for all $\lambda > 0$, $\lambda y \leq R(\lambda\eta, q)$. Thus, we have $y \leq lim_{\lambda\to\infty} R(\lambda\eta, q)/\lambda$

17) Assumption 5 holds true with transaction cost functions which is lower-semi-continuous and piece-wise linear with finitely many jumps. This is the case with transaction cost functions with fixed cost component. Assumption 5, however, may be violated with indivisible assets. Transaction costs with indivisible assets are usually upper-semicontinuous as shown in the third diagram of <Figure 1>, implying that the net return functions are lower-semicontinuous. In this case, $X(q)$ is not closed in general for a price $q \in \mathcal{L}^J$.

$= V(\eta, q)$. This implies that $V(\eta, q) > 0$, which contradicts $q \in \Lambda$. Therefore, $X(q)$ is bounded.

Since $X(q)$ is compact, there exists $x \in X(q)$ which maximizes u over $X(q)$, and therefore, $\theta \in \mathcal{L}^J$ such that $(x, \theta) \in \xi(q)$. Thus, q is viable. □

Theorem 7.1 shows the equivalence between the no arbitrage condition and viability of asset prices. Thus, Theorems 6.2 and 7.1 lead to an extension of the fundamental theorem of asset pricing stated in Harrison and Kreps (1979) and Dybvig and Ross (1989) to the case with transaction costs.

Theorem 7.2. *Under Assumptions 1-5, the following statements are equivalent.*
(*i*) $q \in \Lambda$.
(*ii*) *There exists* $\pi \in L_{++}$ *such that for each* $\sigma \in D$ *and* $j \in J$,

$$\pi(\sigma)[q^j(\sigma) + s^j(q^j(\sigma);\sigma)] \le \sum_{\hat{\sigma} \in D_\sigma \setminus \{ \backslash \sigma \}} \pi(\hat{\sigma}) R^j_{\hat{\sigma}} \le \pi(\sigma)[q^j(\sigma) + b^j(q^j(\sigma);\sigma)] \tag{24}$$

(*iii*) $\xi(q) \neq \phi$

Ⅷ. Conclusion

The fundamental theorem of asset pricing is proved in the presence of transaction costs. It displays the triple equivalence among the no arbitrage condition, the existence of pricing rules, and the viability of asset prices. Transaction cost functions are convex and need not be proportional.

It is shown that no matter how complex the transaction cost structure looks like, the pricing rules which do not admit arbitrage opportunities are as simple as in the case with proportional transaction costs. In particular, the pricing rules do not depend on transaction choices and the local behavior of the transaction cost functions. The fundamental theorem of asset pricing makes sure that informational requirement to capture the form of pricing rules is minimal. Thus, the paper is differentiated from the literature which uses local arbitrage as a conceptual framework for asset pricing in the presence of convex transaction costs. Another advantage of the paper comes from the equivalence between the no arbitrage condition and the viability of asset prices, which implies the exactness of the no arbitrage condition in explaining equilibrium prices.

The consequence of the paper has important applications for asset valuation. For example, option pricing theories in discrete time with proportional transaction costs which are addressed in Leland (1985) and Boyle and Vorst (1992) among others can be straightforwardly extended to the case with convex transaction costs.

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Appendix

Proof of the Claim in Example 2 : We show that

$$\Lambda_V = \{(q^1, q^2) \in \mathbb{R}^2_{++} : (29/30)q^2 \le q^1 \le (21/20)q^2\}.$$

We consider an agent who has preferences represented by a utility function $u(x^0, x^1) = \sqrt{x^0} + \sqrt{x^1}$ and the initial endowment of consumption goods $(1, 1)$.[18] Then he faces the following optimization problem

$$\max\left\{\sqrt{x^0} + \sqrt{x^1} \mid \begin{array}{c} x^0 - 1 \le -\theta^1 q^1 - \theta^2 q^2 - C(\theta^2, q^2) \\ x^1 - 1 \le \theta^1 + \theta^2 \end{array}\right\}.$$

The above maximization problem is reduced to the following

$$\max \sqrt{1 - \theta^1 q^1 - \theta^2 q^2 - C(\theta^2, q^2)} + \sqrt{1 + \theta^1 + \theta^2}.$$

Let $(\hat{\theta}^1, \hat{\theta}^2)$ denote the solution to the maximization problem.

i) $21 q^1 = 20 q^2$.

Clearly, we have

$$(\hat{\theta}^1, \hat{\theta}^2) \in \left\{(\theta^1, \theta^2) : \theta^1 + \theta^2 = \frac{21/20 - (q^1)^2}{(q^1)^2 + q^1}, \theta^2 \ge \frac{1}{q^2}\right\}.$$

ii) $30 q^1 = 29 q^2$

Similarly, we have

$$(\hat{\theta}^1, \hat{\theta}^2) \in \left\{(\theta^1, \theta^2) : \theta^1 + \theta^2 = \frac{31/30 - (q^1)^2}{(q^1)^2 + q^1}, \theta^2 \ge \frac{-1}{q^2}\right\}.$$

iii) $(29/30)q^2 < q^1 < (21/20)q^2$.

The maximization problem has a solution because the following set is compact.

$$\{(\theta^1, \theta^2) \in \mathbb{R}^2 : \theta^1 q^1 + \theta^2 q^2 + C(\theta^2, q^2) \le 1, \theta^1 + \theta^2 \ge -1\}. \qquad \square$$

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18) The argument below do not depend on the form of utility functions and the size of the endowments as far as the utility functions are monotonic.

Proof of Lemma 6.1 : Let q be a price in. First we show that the following set is closed.

$$Y(q)=\{y\in L: y=V(v,q), v\in \mathcal{L}^J\}.$$

Let v be a point in $\mathcal{L}^J$. For some $j\in J$ and $\sigma\in D$, it follows that if $v^j(\sigma)-v^j(\sigma^-)\geq 0$, then

$$\overline{C}^j_\sigma\,(v^j(\sigma)-v^j(\sigma^-),\,q^j(\sigma))=b^j(q^j(\sigma);\sigma)(v^j(\sigma)-v^j(\sigma^-))$$
$$\overline{C}^j_\sigma\,(v^j(\sigma^-)-v^j(\sigma),\,q^j(\sigma))=s^j(q^j(\sigma);\sigma)(v^j(\sigma)-v^j(\sigma^-)),$$

and if $v^j(\sigma)-v^j(\sigma^-)<0$, then

$$\overline{C}^j_\sigma\,(v^j(\sigma)-v^j(\sigma^-),\,q^j(\sigma))=s^j(q^j(\sigma);\sigma)(v^j(\sigma)-v^j(\sigma^-))$$
$$\overline{C}^j_\sigma\,(v^j(\sigma^-)-v^j(\sigma),\,q^j(\sigma))=b^j(q^j(\sigma);\sigma)(v^j(\sigma)-v^j(\sigma^-)).$$

(In the above, we follow the notational convention that $v^j(\sigma_0^-)=0$ for all $j\in J$.) Let $\{y^n\}$ be a sequence in $Y(q)$ which converges to a point y. Since $0\in Y(q)$, without loss of generality we may assume that $y\neq 0$. Then $y^n\neq 0$ for sufficiently large n. For each n we choose v^n in $\mathcal{L}^J$ such that $y^n=V(v^n,q)$. Since D is finite, there exists a subsequence $\{v^m\}$ such that $\{v^{jm}(\sigma)-v^{jm}(\sigma^-)\}$ has the same sign for a given pair $(j,\sigma)\in J\times D$. Thus there exists a $(\#D)\times[J\times(\#D_{-T})]$ matrix Ψ such that $V(v^m,q)=\Psi\cdot v^m$.
We define the sets

$$\Theta^+(q)=\{v\in \mathcal{L}^J|\ v^j(\sigma)-v^j(\sigma^-)\geq 0 \text{ for } \textit{each } (j,\sigma) \textit{ with } v^{jm}(\sigma)-v^{jm}(\sigma^-)\geq 0 \text{ for } \textit{all } m\}$$
$$\Theta^-(q)=\{v\in \mathcal{L}^J|\ v^j(\sigma)-v^j(\sigma^-)\leq 0 \text{ for } \textit{each } (j,\sigma) \textit{ with } v^{jm}(\sigma)-v^{jm}(\sigma^-)< 0 \text{ for } \textit{all } m\}.$$

Since $J\times D$ consists of finitely many elements, $\Theta^+(q)$ and $\Theta^-(q)$ are the intersection of finitely many closed half spaces which contain the origin on the boundary, and therefore, they are a polyhedral cone. By construction, $\{v^m\}$

is in $\Theta^+(q)\cap\Theta^-(q)$. We set

$$Y^{\pm}(q)=\{y\in L: y=V(v,q) \text{ for } some \ v\in\Theta^+(q)\cap\Theta^-(q)\}.$$

Since $\Psi\cdot v=V(v,q)$ for any $v\in\Theta^+(q)\cap\Theta^-(q)$, by Theorem 19.3 of Rockafellar (1970) the set $Y^{\pm}(q)$ is a polyhedral cone. In particular, it is closed. Since $\{y^m\}$ is in $Y^{\pm}(q)$, y is in $Y^{\pm}(q)$. Noting that $Y^{\pm}(q)\subset Y(q)$, y is in $Y(q)$. Thus, $Y(q)$ is closed.

Now show that the set $Z(q)=\{y\in L: y\leq V(v,q), v\in \mathcal{L}^J\}$. Let $\{y^n\}$ be a sequence in $Z(q)$ which converges to a point y. For each n we choose v^n in $\mathcal{L}^J$ such that $y^n\leq V(v^n,q)$.
For each n, we set $z^n=V(v^n,q)$. We claim that $\{z^n\}$ is bounded. Suppose that $\|z^n\|\to\infty$. By positive homogeneity of $V(\cdot,q)$, we have

$$z^n/\|z^n\| = V(v^n,q)/\|z^n\| = V(v^n/\|z^n\|,q).$$

This implies that $z^n/\|z^n\|\in Y(q)$ for each n. Clearly, $\{z^n/\|z^n\|\}$ is bounded. Thus, it has a subsequence convergent to a point $\dot{z}$. Since $Y(q)$ is closed, $\dot{z}$ is in $Y(q)$. Thus there exists $v\in\mathcal{L}^J$ such that $V(v,q)=\dot{z}$. On the other hand, we have $y^n/\|z^n\|\leq z^n/\|z^n\|$. Since $y^n\to y$ and $\|z^n\|\to\infty$, $y^n/\|z^n\|\to 0$. By passing to the limit we have $\dot{z}\geq 0$. Recalling that $\dot{z}\neq 0$, we must have $V(v,q)>0$, which contradicts the fact that $q\in\Lambda$. Since $\{z^n\}$ is bounded, it has a subsequence convergent to a point z in $Y(q)$. Recalling that $y^n\leq z^n$ for each n, we have $y\leq z$. Thus $y\in Z(q)$. □

〔Abstract〕

볼록거래비용하에서 자산가격결정의 기본정리

원 동 철

본 논문의 목적은 볼록거래비용하에서 자산가격결정의 기본정리를 정립하는 것이다. 아비트라지 가격결정함수는 거래비용의 한계적 영향을 받지 않으며 대형 거래의 평균비용에 의해서 결정된다. 이러한 사실은 가격결정원리가 거래비용의 성질에 관한 최소한의 정보에 의해 파악될 수 있음을 의미한다. 또한 볼록거래비용함수가 아무리 복잡할지라도 가격결정원리는 비례적 거래비용의 경우처럼 단순하고 구체적 형태로 도출된다. 이러한 결과는 거래비용의 한계적 영향, 즉 거래비용함수의 국지적 형태에 좌우되는 균형가격결정이론과 큰 대조를 이룬다. 더구나 아비트라지 부재조건은 자산가격의 존속성과 일치한다.

핵심용어 : 아비트라지 가격결정, 볼록거래비용, 존속성