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Incomplete Markets with Cone Constraints

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Abstract We show the existence of equilibrium in a two-period incomplete financial market with non-ordered preferences where agents are subject to cone portfolio constraints. The main theorem of the paper is built on the consequences of portfolio decomposition provided in Hahn and Won (2007).

Keywords Incomplete markets, cone portfolio constraints, portfolio decomposition, constrained arbitrage, competitive equilibrium.

JEL Classification C62, D51

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1. INTRODUCTION

The purpose of this paper is to show that there exists an equilibrium in a two-period incomplete financial markets where agents are subject to cone portfolio constraints and their preferences may not be complete or transitive. In contrast to unconstrained asset markets, constrained ones may allow redundant assets to play a role in risk sharing.[1] To model the behavior of boundedly rational consumers, the axioms of completeness and transitivity of preferences need to be relaxed.

It is conventional in the literature to assume that individual preferences on the choice set is complete and transitive. In reality, people may not show definite preference over all possible alternatives. In this case, the completeness axiom is a quite strong requirement for individual preferences, as noted by Aumann (1962). Individual preferences may not satisfy the transitivity axiom, which is essential for the numerical representation of preferences over a continuum of alternatives. It is widely recognized that transitive preferences fail to explain important anomalous phenomena such as preference reversal.[2] Moreover, when agents such as households, firms, and financial institutions behave as group decision makers, preferences are frequently nontransitive due to the aggregation problem of individuals' preferences. But such a phenomenon cannot be dismissed as a consequence of irrational choices.

When financial asset markets are free from portfolio constraints, redundant assets do not contribute to risk sharing and therefore useless. Moreover, their prices can be determined by linear pricing rule and thus, there is no fiduciary incentive to produce redundant assets.[3] In reality, however, redundant assets such as futures and options are produced through costly financial innovation and frequently violate the law of one price. This is because financial asset markets are subject to market frictions such as portfolio constraints, which are binding at individuals' optimal portfolio choices involving redundant assets. As demonstrated in Balasko et al. (1990), redundant assets may also cause the unbounded multiplicity of optimal portfolio choices in general. Thus, it is quite important to fully understand the risk-sharing role of redundant assets in the constrained incomplete markets.

[1] Assets are called redundant if their returns are linearly dependent on the returns of other assets.
[2] See Loomes et al. (1991) among others.
[3] See Werner (1985) and Geanakoplos and Polemarchakis (1987) among others.

Siconolfi (1986) investigates the existence problem in incomplete markets under general convex constraints, motivating the research on the indeterminacy problem such as Cass et al. (2001). Won and Hahn (2000) also examine the existence issue in a similar context. But both Siconolfi (1986) and Won and Hahn (2000) impose strong restrictions on the risk-sharing role of redundant assets to circumvent the large multiplicity problem with portfolio choices. Due to these restrictions, Siconolfi (1986), Won and Hahn (2000), and Cass et al. (2001) may not cover the case with nonnegative wealth constraints. On the other hand, Balasko et al. (1990), Benveniste and Ketterer (1992), and Polemarchakis and Siconolfi (1997) deal with a special case where portfolio constraints are represented by linear homogeneous equations. They, however, cannot cover either short-selling nor nonnegative wealth constraints. Recently, Hahn and Won (2007) address the large multiplicity problem of optimal portfolio choices by examining the role of redundant assets in incomplete markets with rational preferences and cone portfolio constraints from a little different perspective. They adopt a distinct approach to treating redundant assets such that portfolios are projected onto the row span of the asset structure (asset return matrix) without losing income spanning capability in equilibrium.

While most of the literature on incomplete markets deal with rational preferences, Werner (1989) examines the equilibrium existence problem in unconstrained incomplete markets without complete or transitive preferences. Angeloni and Cornet (2006) extend the results of Werner (1989) to multiperiod incomplete financial markets with portfolio constraints under the assumption that the attainable set of portfolio choices is bounded. As discussed above, the condition of Angeloni and Cornet (2006) on redundant assets is so restrictive that it may not cover many interesting cases. We extends the results of Hahn and Won (2007) to economies without ordered preferences. It is worth noting that Hahn and Won (2007) extend the classical results of unconstrained incomplete markets to the case of cone constraint portfolios while the current paper attempts the extension of Werner (1989) to the case of cone constraint portfolios. Besides non-ordered preferences, however, this paper differs from Hahn and Won (2007) in that it allows consumption in the first period which, in turn, makes it possible to do without a survival condition with asset markets.

The paper is organized as follows. The economy is described, and constrained arbitrage and portfolio decomposition are discussed in Section 2. The main theorem of the paper is presented in Section 3, followed by concluding remarks.

2. MODEL

2.1. Constrained Incomplete Markets without Ordered Preferences

We consider a two-period incomplete market without production. Consumptions are allowed in both periods, and financial assets are traded in the first period ($t = 0$) and pay monetary returns to the holder in the second period ($t = 1$). Asset payoffs are contingent upon the event $s \in S = \{1, \ldots, \mathbf{s}\}$ which is revealed in the second period. In each state $s \in S_0 = S \cup \{0\}$ where state 0 denotes the first period, there is a market for $\mathbf{l}$ commodities. Thus, there are $\ell := \mathbf{l}(\mathbf{s} + 1)$ commodities traded over the two periods and thus $\mathbb{R}^\ell$ is the commodity space.

Let $I = \{1, 2, \ldots, m\}$ denote the set of agents, and $J = \{1, 2, \ldots, \mathbf{j}\}$ the set of financial assets. Each agent $i \in I$ has the consumption set X_i and an initial endowment of goods $e_i \in X_i$. A preferences relation $\succ_i$ of agent $i \in I$ is a binary relation on $X_i \times X_i$ and is irreflexive and may be neither complete nor transitive. For a consumption bundle $x_i \in X_i$, we define the preference correspondence $P_i : X_i \to 2^{X_i}$ by $P_i(x_i) = \{x'_i \in X_i : x'_i \succ_i x_i\}$, which is the set of consumption bundles of consumer i being preferred to x_i. Consumer i is initially endowed with a consumption bundle $e_i \in X_i$. For a collection of points $\{y(0), \ldots, y(S)\}$ in $\mathbb{R}^L$, we set $y = (y(0), \ldots, y(S))$. For each $s \in S_0$, let us define the state-wise preference correspondence $P_i(\cdot, s) : X_i \to 2^{X_i}$ by

$$P_i(x_i, s) = \{x'_i \in X_i : x'_i \in P_i(x_i),\, x'_i(s') = x_i(s'),\, \forall\, s' \in S_0 \setminus \{s\}\}.^4$$

The portfolio constraint facing a consumer in incomplete markets is represented by the opportunity set Θ_i of portfolios in $\mathbb{R}^{\mathbf{j}}$. We set $X := \prod_{i \in I} X_i$ and $\Theta := \prod_{i \in I} \Theta_i$. The set of attainable consumption-portfolio allocations is defined by

$$A = \left\{(x, \theta) \in X \times \Theta : \textstyle\sum_{i \in I} x_i = \sum_{i \in I} e_i,\ \sum_{i \in I} \theta_i = 0\right\}.$$

[4] To explain $P_i(x_i, s)$, let us define the vector $x_i(-s) := (x_i(0), x_i(1), \ldots, x_i(s-1), x_i(s+1), \ldots, x_i(S))$. Then $P_i(x_i, s)$ just denotes the set of consumption bundles y_i's in X_i such that $y_i := (y_i(s), x_i(-s))$ is preferred to $x_i := (x_i(s), x_i(-s))$. In other words, if consumer i changes the consumption bundle $x_i(s)$ to $y_i(s)$ only at the state s and he prefers $y_i = (y_i(s), x_i(-s))$ to x_i, then we say $y_i \in P_i(x_i, s)$. For example, suppose that P_i is state-wise monotonic. Then $P_i(x_i, s)$ is the collection of consumption bundles y_i's in X_i which satisfy $y_i(s) > x_i(s)$ and $y_i(-s) = x_i(-s)$.

and $\hat{X}_i$ denotes the coordinate projection of A on X_i. Let $\hat{X} := \prod_{i\in I} \hat{X}_i$.

Each asset $j \in J$ pays $r_j(s)$ in state s. The vector of asset returns in state s is given by a J-dimensional row vector $r(s) = (r_j(s))_{j\in J}$ and the return of asset j by a S-dimensional column vector $r_j = (r_j(s))_{s\in S}$. The asset payoffs are described by an $S \times J$ matrix $R = [(r(s))_{s\in S}]$. Here either $S \geq J$ or $S < J$ holds. This means that it does not matter whether financial markets are potentially complete or not. For vectors p and $x_i \in \mathbb{R}^\ell$, we set

$$p \,\square\, (x_i - e_i) = \begin{bmatrix} p(0)\cdot(x_i(0) - e_i(0)) \\ p(1)\cdot(x_i(1) - e_i(1)) \\ \vdots \\ p(\mathbf{s})\cdot(x_i(\mathbf{s}) - e_i(\mathbf{s})) \end{bmatrix}, \quad W(q) = \begin{bmatrix} -q \\ R \end{bmatrix},$$

where $p(s)$, $x_i(s)$, and $e_i(s)$ are the price vector, consumer i's consumption, and consumer i's endowment, respectively at state $s \in S_0$. The open budget correspondence $\mathcal{B}_i : \mathbb{R}^\ell \times \mathbb{R}^{\mathbf{J}} \to 2^{X_i\times\Theta_i}$ of agent i is defined by

$$\mathcal{B}_i(p, q) := \{(x_i, \theta_i) \in X_i \times \Theta_i : p \,\square\, (x_i - e_i) \ll W(q)\cdot\theta_i\}$$ [5]

and the budget correspondence $c\ell\mathcal{B}_i : \mathbb{R}^\ell \times \mathbb{R}^{\mathbf{J}} \to 2^{X_i\times\Theta_i}$ of agent i is defined by

$$c\ell\mathcal{B}_i(p, q) := \{(x_i, \theta_i) \in X_i \times \Theta_i : p \,\square\, (x_i - e_i) \leq W(q)\cdot\theta_i\}.$$

For a given pair $(p, q) \in \mathbb{R}^\ell_+ \times \mathbb{R}^{\mathbf{J}}$, each agent $i \in I$ chooses his consumption and portfolio $(x_i, \theta_i) \in c\ell\mathcal{B}_i(p, q)$ satisfying $(P_i(x_i) \times \Theta_i) \cap c\ell\mathcal{B}_i(p, q) = \emptyset$. Let us define a competitive equilibrium of the economy $\mathcal{E}$ as follows.[6]

DEFINITION 2.1. : A *competitive equilibrium* of the economy $\mathcal{E}$ is a profile $(p^*, q^*, x^*, \theta^*) \in \mathbb{R}^\ell \times \mathbb{R}^{\mathbf{J}} \times A$ such that (i) $(x_i^*, \theta_i^*) \in c\ell\mathcal{B}_i(p^*, q^*)$, $\forall\, i \in I$ and (ii) $(P_i(x_i^*) \times \Theta_i) \cap c\ell\mathcal{B}_i(p^*, q^*) = \emptyset$, $\forall\, i \in I$.

Let $V = span\{r(s) : s \in S\}$ and $V^\perp = \{\theta \in \mathbb{R}^{\mathbf{J}} : R\cdot\theta = 0\}$. Redundant assets exist if and only if $V^\perp \neq \{0\}$. In particular, some assets are redundant if the rank of the return

[5]Let x and y be vectors in $\mathbb{R}^k$. We say that $x \geq y$ if $x - y \in \mathbb{R}^k_+$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x - y \in \mathbb{R}^\ell_{++}$.

[6]Let A be a nonempty subset of an Euclidean space. We denote the closure of A by $c\ell(A)$, the interior of A by $int(A)$, the boundary of A by ∂A, and the convex hull of A by coA.

matrix R is less than the minimum of J and S. Portfolios in $V^{\perp}$ are called ***zero-income portfolios***, which generate zero income transfer in each state of the second period. In particular, portfolios in $\Theta_i \cap V^{\perp}$ are called *constrained zero-income portfolios* for agent i. A portfolio θ in $\mathbb{R}^{\mathbf{J}}$ has the direct sum $\hat{\theta} + \tilde{\theta}$ where $\hat{\theta} \in V$ and $\tilde{\theta} \in V^{\perp}$. The portfolio $\tilde{\theta}$ does not affect the size of income transfers but may matter to the feasibility of θ under the portfolio constraints.

Let A be a nonempty convex subset of an Euclidean space. The recession cone of A is the set $\Gamma(A) = \{v \in E : A + v \subset A\}$. An element $v \in \Gamma(A)$ is a direction of recession of A. It is easy to see that $v \in \Gamma(A)$ if and only if $x + \lambda v \in A$ for some $x \in A$ and all $\lambda \geq 0$. Note that $\Gamma(A) \subset A$ whenever $0 \in A$ and that $\Gamma(A)$ is closed if A is closed. The lineality space $\mathcal{L}(A)$ of A is the maximal subspace contained in the recession cone of A.[7]

We set $C_i = \Gamma(\Theta_i)$ and $L_i = \mathcal{L}(\Theta_i)$ for each $i \in I$. A portfolio in $C_i \cap V^{\perp}$ is a zero-income portfolio that is a direction of recession of Θ_i. Let N denote the lineality space of the cone $\sum_{i\in I}(C_i \cap V^{\perp})$. If $N \neq \{0\}$, there exists a set of nonzero portfolios in $C_i \cap V^{\perp}$ that jointly span N. As shown later, if $N \neq \{0\}$, then there exists a large multiplicity of optimal portfolios in equilibrium. Let M denote the orthogonal complement of N in $V^{\perp}$, where $N = \mathcal{L}(\sum_{i\in I}(C_i \cap V^{\perp}))$. For each $i \in I$, let $\hat{\Theta}_i$ denote the projections of Θ_i onto $V + M$ and $\Gamma(\hat{\Theta}_i)$ the recession cone of $\hat{\Theta}_i$.

Characteristics of each agent are assumed to satisfy the following properties.

ASSUMPTIONS : For every $i \in I$,

A1. X_i is closed, convex, bounded from below in $\mathbb{R}^{\ell}$.

A2. $x_i \notin coP_i(x_i)$ for every $x_i \in X_i$.[8]

A3. P_i is lower hemicontinuous on X_i.

A4. P_i is open-valued on X_i.

A5. $\forall\, x_i \in \hat{X}_i, \quad x_i \in \partial P_i(x_i, s), \forall\, s \in S_0$.

A6. $e_i \in intX_i$.

[7] For more details on the analysis of convex sets, see Rockafellar (1970).

[8] The correspondence $coP_i : X_i \to 2^{X_i}$ is defined by $coP_i(x_i) = co(P_i(x_i)), \forall\, x_i \in X_i$.

A7. Θ_i **is a closed convex cone with vertex in** $\mathbb{R}^{\mathbf{J}}$ **with** $0 \in \Theta_i$**, and** $\hat{\Theta}_i$ **is closed.**[9]

Assumption A1 is a standard assumption in incomplete markets. Assumption A2 is a weak convexity assumption and implies that P_i is irreflexive, i.e., $x_i \notin P_i(x_i)$. Gale and Mas-Colell (1979) suggest Assumption A3, which implies that coP_i is lower hemicontinuous. Assumption A4 means that, if z_i is sufficiently close to y_i which is preferred to x_i, then $z_i \in P_i(x_i)$. Local nonsatiation at every state $s \in S_0$ is represented by Assumption A5, as in Werner (1989). However, Assumption A5 can be replaced by a nonsatiation assumption that, for every $x_i \in X_i$ and for every $s \in S_0$, $P_i(x_i, s) \neq \emptyset$. In this case, on behalf of P_i, we can employ an augmented preference correspondence $\hat{P}_i$ defined by $\hat{P}_i(x_i) = co\,(\{x_i\} \cup coP_i(x_i)) \setminus \{x_i\}$. Note that the preference correspondence $\hat{P}_i$ inherits the properties of P_i and is convex-valued. Assumption A6 is a survival condition in goods markets. Finally, Assumption A7 means that no trading of securities is possible. Note that market frictions such as short-selling constraints, bid-ask spreads, and proportional transaction costs can be represented as a convex cone with vertex (See Luttmer (1996)). The linear constraints of Balasko et al. (1990), Benveniste and Ketterer (1992), and Polemarchakis and Siconolfi (1997) are a special case of Assumption A7.

In addition to the convexity of the portfolio constraints, Siconolfi (1986) and Won and Hahn (2000) impose the extra requirement that $C_i \cap V^{\perp} = \{0\}$ for each $i \in I$. Notice that if $C_i \cap V^{\perp} = \{0\}$ for each $i \in I$, then $N = \{0\}$. Thus, they cannot cover the case where portfolio constraints lead to the large portfolio multiplicity problem, i.e., $N \neq \{0\}$.

2.2. Portfolio Decomposition

Zero-income portfolios contain information on how one asset is replicated by other assets. In frictionless markets, they have no value in equilibrium and therefore, are a realization of linear pricing. In the case where asset markets are subject to portfolio constraints, zero-income portfolios may not follow the linear pricing rule but still have important information on asset pricing in equilibrium. However, since redundant

[9] A set A in an Euclidean space E is a cone if $\lambda v \in A$ for all $v \in A$ and all $\lambda \geq 0$. It is a cone with vertex if $A - v$ is a cone for some $v \in E$.

assets are involved in risk sharing and create the unbounded set of zero-income portfolios, it is impossible in general to remove them without perturbing equilibrium. A systematic way of decomposing portfolios is developed to treat the large multiplicity of alternative portfolios with the same value. The consequence of portfolio decomposition enables us to construct the artificial economy which is free from the large multiplicity of zero-value portfolios in equilibrium and has the same set of equilibrium allocations of goods as the original economy. These results will play a crucial role in investigating the existence of equilibrium with the original economy.

If $N \neq \{0\}$, a large multiplicity of equilibrium prices arise. Specifically, N consists of zero-income portfolios which turn out to have no value in equilibrium and therefore, cause the large multiplicity problem. In general, we can consider two ways of handling the problem with zero-income portfolios: the projection method and the exclusion method. The former involves projecting away the zero-income portfolios which cause the large multiplicity problem. The latter is to simply remove redundant assets from the asset structure and then price them by the law of one price. This is the way of handling redundant assets in the classical literature of incomplete markets such as Werner (1985) and Balasko and Cass (1989). In unconstrained asset markets, both ways of handling zero-income portfolios are the same in that they do not affect the opportunity set of income transfers. As shown in Example 3.1. of Won and Hahn (2003), however, the exclusion method is not appropriate in the presence of portfolio constraints because it changes drastically the opportunity set of income transfers.

Hahn and Won (2007) reports the following proposition presenting a consequence of portfolio decomposition.

PROPOSITION 2.1. : The following results hold true under Assumption A7.

(i) $\Theta = N + \sum_{i \in I} \hat{\Theta}_i$,

(ii) $\left(\sum_{i \in I}(\Gamma(\hat{\Theta}_i) \cap M)\right) \cap \left(-\sum_{i \in I}(\Gamma(\hat{\Theta}_i) \bigcap M)\right) = \{0\}$,

(iii) $N = \mathcal{L}(\sum_{i \in I}(\Theta_i \cap V^{\perp}))$.

The results of Proposition 2.1. will play a crucial role in characterizing equilibrium prices and proving the existence of equilibrium. The following proposition states that

equilibrium prices are in $V + M$.

PROPOSITION 2.2. : Let Q^* be the set of equilibrium asset prices of the economy $\mathcal{E}$. Then it holds true that $Q^* \subset V + M$.[10]

[10]For its proof, see Hahn and Won (2007).

2.3. Constrained Arbitrage

No arbitrage conditions in the literature provide a useful framework for asset pricing and equilibrium analysis. First, we consider a typical form of arbitrage for unconstrained asset markets.

DEFINITION 2.2. : Suppose that $\Theta_i = \mathbb{R}^{\mathbf{j}}$ for all $i \in I$. Then an asset price $q \in \mathbb{R}^{\mathbf{j}}$ admits *no arbitrage* if there is no $v \in \mathbb{R}^{\mathbf{j}}$ which satisfies $W(q) \cdot v > 0$.

The no arbitrage condition of Definition 2.2. provides a suitable conceptual framework for studying asset pricing, portfolio choice problem, and equilibrium in frictionless markets. Specifically, Werner (1985) among others shows the existence of equilibrium with incomplete markets by taking advantage of Definition 2.2.

When asset markets face portfolio constraints, however, the no arbitrage conditions for frictionless markets are no longer useful because the extended law of one price fails. We introduce a notion of arbitrage which is appropriate in characterizing equilibrium prices in constrained asset markets.

DEFINITION 2.3. : A price vector $q \in \mathbb{R}^{\mathbf{j}}$ admits *no constrained arbitrage* for agent i in the economy $\mathcal{E}$ if there is no $v_i \in C_i$ such that $W(q) \cdot v_i > 0$. A price vector $q \in \mathbb{R}^{\mathbf{j}}$ admits *no constrained arbitrage* for the economy $\mathcal{E}$ if it admits no constrained arbitrage for every agent $i \in I$.

The above definition is used in Luttmer (1996) among others. Let Q_i denote the set of prices which admit no constrained arbitrage prices for agent i. We set $Q = \bigcap_{i \in I} Q_i$. The set Q denotes the set of prices which admits no constrained arbitrage for the economy $\mathcal{E}$. We define the sets of normalized prices.

$$
\begin{aligned}
\Delta_0 &= \left\{ (p(0), q) \in \mathbb{R}^{\mathbf{l}} \times c\ell(Q) : \|(p_0, q)\| \leq 1 \right\}, \\
\Delta_s &= \left\{ p(s) \in \mathbb{R}^{\mathbf{l}} : \|p(s)\| \leq 1 \right\}, \\
\Delta_1 &= \textstyle\prod_{s \in S} \Delta_s, \\
\Delta &= \Delta_0 \times \Delta_1.
\end{aligned}
$$

where $\| \cdot \|$ denotes the Euclidean norm.

3. EXISTENCE OF EQUILIBRIUM

When proving the existence of a competitive equilibrium, we will apply the following fixed point theorem which is given by Gale and Mas-Colell (1975, 1979).

LEMMA 3.1. (Gale and Mas-Colell, 1975, 1979) : Let T_k be a nonempty compact convex subset of the finite dimensional Euclidean space and $T = \prod_{k \in K} T_k$. Let $\varphi_k : T \to 2^{T_k}$ be lower hemicontinuous with convex values. Then there is $t^* \in T$ such that $t_k^* \in \varphi_k(t^*)$ or $\varphi_k(t^*) = \emptyset$ for every k.

Let $\mathcal{E}$ denote the original economy and $\hat{\mathcal{E}}$ its projected economy which is the same as $\mathcal{E}$ except that each Θ_i is replaced by $\hat{\Theta}_i$. That is to say, the projected economy is given by $\hat{\mathcal{E}} = \{(X_i, \hat{\Theta}_i, \succ_i, e_i)\}$. Let us define the open budget correspondence $\hat{\mathcal{B}}_i : \mathbb{R}^\ell \times \mathbb{R}^{\mathbf{j}} \to 2^{X_i \times \hat{\Theta}_i}$ of agent i in $\hat{\mathcal{E}}$ by

$$\hat{\mathcal{B}}_i(p,q) := \left\{(x_i, \hat{\theta}_i) \in X_i \times \hat{\Theta}_i : p \,\square\, (x_i - e_i) \ll W(q) \cdot \hat{\theta}_i\right\}$$

and the budget correspondence $c\ell\hat{\mathcal{B}}_i : \mathbb{R}^\ell \times \mathbb{R}^{\mathbf{j}} \to 2^{X_i \times \hat{\Theta}_i}$ of agent i in $\hat{\mathcal{E}}$ by

$$c\ell\hat{\mathcal{B}}_i(p,q) := \left\{(x_i, \hat{\theta}_i) \in X_i \times \hat{\Theta}_i : p \,\square\, (x_i - e_i) \leq W(q) \cdot \hat{\theta}_i\right\}.$$

The set of attainable consumption-portfolio allocations is defined by

$$\hat{A} = \left\{(x, \hat{\theta}) \in X \times \hat{\Theta} : \textstyle\sum_{i \in I}(x_i - e_i) = 0,\ \sum_{i \in I}\hat{\theta}_i = 0\right\}.$$

One can notice that $\hat{A}$ has the same coordinate projection $\tilde{X}_i$ on X_i as A. Recall that $\tilde{X}_i$ is compact and therefore $\tilde{X}$ is compact. Let us define a competitive equilibrium for the projected economy $\hat{\mathcal{E}}$ as follows.

DEFINITION 3.1. : A *competitive equilibrium* of the projected economy $\hat{\mathcal{E}}$ is a profile $(p^*, q^*, x^*, \hat{\theta}^*) \in \Delta \times \hat{A}$ such that (i) $(x_i^*, \hat{\theta}_i^*) \in c\ell\hat{\mathcal{B}}_i(p^*, q^*), \forall\, i \in I$ and (ii) $(P_i(x_i^*) \times \hat{\Theta}_i) \cap c\ell\hat{\mathcal{B}}_i(p^*, q^*) = \emptyset, \forall\, i \in I$.

One can show that both the economies $\mathcal{E}$ and $\hat{\mathcal{E}}$ have the same set of equilibria except for the nominal difference in asset allocations.

THEOREM 3.1. : The pair $(p, q, x, \hat{\theta})$ is an equilibrium of $\hat{\mathcal{E}}$ if and only if there exists $\eta = (\eta_i)$ with $\eta_i \in N$ for each $i \in I$ such that $(p, q, x, \hat{\theta} + \eta)$ is an equilibrium of $\mathcal{E}$.[11]

To work in the sequence of truncated economies, take an increasing sequence $\{(K_n, M_n)\}$ of compact convex cube pairs with center 0 such that $K_n \subset \mathbb{R}^\ell$ with $\hat{X}_i \subset intK_1$ for all $i \in I$ and $M_n \subset \mathbb{R}^{\mathbf{J}}$ satisfying $\bigcup_n K_n = \mathbb{R}^\ell$ and $\bigcup_n M_n = \mathbb{R}^{\mathbf{J}}$. Let us define $X_i^n := X_i \cap K_n$ and $\Theta_i^n := \hat{\Theta}_i \cap M_n$. In addition, we need to define a new preference correspondence $P_i^n : X_i^n \to 2^{X_i^n}$ by $P_i^n(x_i) := coP_i(x_i) \cap X_i^n$. Consider the sequence of truncated projected economies $\{\mathcal{E}^n = (X_i^n, \Theta_i^n, P_i^n, e_i)_{i \in I}\}$. In the economy $\mathcal{E}^n$, each agent i has a nonempty compact convex choice set $X_i^n \times \Theta_i^n$. Define $X^n := \prod_{i \in I} X_i^n$ and $\Theta^n := \prod_{i \in I} \Theta_i^n$. Let us define $\gamma : \Delta \to \mathbb{R}$ by

$$\gamma_s(p, q) = \begin{cases} 1 - \|(p(0), q)\|, & \textbf{if } s = 0, \\ 1 - \|p(s)\|, & \textbf{if } s \in S. \end{cases}$$

For every $i \in I$, define the modified open budget correspondence $\mathcal{B}_i^n : \Delta \to 2^{X_i^n \times \Theta_i^n}$ by

$$\mathcal{B}_i^n(p, q) = \left\{ (x_i, \hat{\theta}_i) \in X_i^n \times \Theta_i^n : p \,\square\, (x_i - e_i) \ll W(q) \cdot \hat{\theta}_i + \gamma(p, q) \right\}$$

and the modified budget correspondence $c\ell\mathcal{B}_i^n : \Delta \to 2^{X_i^n \times \Theta_i^n}$ by

$$c\ell\mathcal{B}_i^n(p, q) = \left\{ (x_i, \hat{\theta}_i) \in X_i^n \times \Theta_i^n : p \,\square\, (x_i - e_i) \leq W(q) \cdot \hat{\theta}_i + \gamma(p, q) \right\}.$$

In addition, we define a correspondence $\mathfrak{B}_i^n : \Delta \to 2^{X_i^n \times \Theta_i^n}$ by

$$\mathfrak{B}_i^n(p, q) = \begin{cases} \{(e_i, 0)\}, & \textbf{if } \mathcal{B}_i^n(p, q) = \emptyset, \\ c\ell\mathcal{B}_i^n(p, q), & \textbf{if } \mathcal{B}_i^n(p, q) \neq \emptyset. \end{cases}$$

LEMMA 3.2. : Under Assumptions A1, A6, and A7, for every $i \in I$ and for every $n \in \mathbb{N}$, $\mathcal{B}_i^n$ is lower hemicontinuous with convex values on Δ.

Proof : It is clear that $\mathcal{B}_i^n$ is convex-valued. Moreover, since $\mathcal{B}_i^n$ obviously has open graph, it is lower hemicontinuous. ■

LEMMA 3.3. : Under Assumptions A1, A6, and A7, for every $i \in I$ and for every $n \in \mathbb{N}$, $c\ell\mathcal{B}_i^n$ is lower hemicontinuous with convex values on $\Delta_i^n := \{(p, q) \in \Delta : \mathcal{B}_i^n(p, q) \neq \emptyset\}$.

[11] For its proof, see Hahn and Won (2007).

Proof : Since $\mathcal{B}_i^n$ is nonempty-valued on Δ_i^n, it is the case that $c\ell\mathcal{B}_i^n(p,q) = c\ell[\mathcal{B}_i^n(p,q)]$. Therefore Lemma 3.2. implies that $c\ell\mathcal{B}_i^n$ is lower hemicontinuous with convex values on Δ_i^n. ■

LEMMA 3.4. : Under Assumptions A1, A6, and A7, for every $i \in I$ and for every $n \in \mathbb{N}$, $\mathfrak{B}_i^n$ is lower hemicontinuous with nonempty convex values on Δ.

Proof : It is obvious that $\mathfrak{B}_i^n$ is convex-valued. To show that it is lower hemicontinuous, take any open set V in $X_i^n \times \Theta_i^n$. Let $G_i^n = \{(p,q) \in \Delta : \mathfrak{B}_i^n(p,q) \cap V \neq 0\}$ and $\Delta_i^n = \{(p,q) \in \Delta : \mathcal{B}_i^n(p,q) \neq \emptyset\}$. Observe that Δ_i^n is open in Δ since $\mathcal{B}_i^n$ lower hemicontinuous on Δ. Now define

$$\begin{aligned} G_i^n(1) &= \{(p,q) \in \Delta_i^n : c\ell\mathcal{B}_i^n(p,q) \cap V \neq 0\}, \\ G_i^n(2) &= \{(p,q) \in \Delta : \{e_i, 0\} \cap V \neq 0\}. \end{aligned}$$

Taking into account the fact that $\{e_i, 0\} \subset c\ell\mathcal{B}_i^n(p,q)$ for all $(p,q) \in \Delta$, we see that $G_i^n = G_i^n(1) \cup G_i^n(2)$. Since $c\ell\mathcal{B}_i^n$ is lower hemicontinuous on Δ_i^n by Lemma 3.3, the set $G_i^n(1)$ is open in Δ_i^n and therefore is open in Δ. On the other hand, obviously $\{e_i, 0\}$ is lower hemicontinuous, $G_i^n(2)$ is also open in Δ. Consequently, it follows that $G_i^n = G_i^n(1) \cup G_i^n(2)$ is open in Δ, which means that $\mathfrak{B}_i^n$ is lower hemicontinuous. ■

Let $z(s) = \sum_{i \in I}[x_i(s) - e_i(s)]$. Now construct the following correspondences $\varphi_0^n : \Delta \times X^n \times \Theta^n \to 2^{\Delta}$ and $\varphi_i : \Delta \times X^n \times \Theta^n \to 2^{X_i^n \times \Theta_i^n}$ for every $i \in I$:

$$\begin{aligned} \varphi_0^n(p,q,x,\hat{\theta}) &= \{(\hat{p},\hat{q}) \in \Delta : \textstyle\sum_{s \in S_0}[\hat{p}(s) - p(s)] \cdot z(s) + (\hat{q} - q) \cdot \sum_{i \in I}\hat{\theta}_i > 0\}, \\ \varphi_i^n(p,q,x,\hat{\theta}) &= \begin{cases} \mathfrak{B}_i^n(p,q), & \textbf{if } (x_i, \hat{\theta}_i) \notin c\ell\mathcal{B}_i^n(p,q), \\ (P_i^n(x_i) \times \Theta_i^n) \cap \mathcal{B}_i^n(p,q), & \textbf{if } (x_i, \hat{\theta}_i) \in c\ell\mathcal{B}_i^n(p,q). \end{cases} \end{aligned}$$

LEMMA 3.5. : Under Assumptions A1–A7, φ_i^n is lower hemicontinuous with convex values for every $i \in I_0 := I \cup \{0\}$ and for every n.

Proof : It is obvious that φ_i^n is convex-valued for every $i \in I_0$ and that φ_0^n is lower hemicontinuous. To show φ_i^n is lower hemicontinuous for every $i \in I$, take any open set V in $X_i^n \times \Theta_i^n$. Let $D_i^n = \{(p,q,x,\hat{\theta}) \in \Delta \times X^n \times \Theta^n : \varphi_i^n(p,q,x,\hat{\theta}) \cap V \neq 0\}$ and $C_i^n = \{(p,q,x,\hat{\theta}) \in \Delta \times X^n \times \Theta^n : x_i \notin c\ell\mathcal{B}_i^n(p,q)\}$. Observe that C_i^n is open in $\Delta \times X^n \times \Theta^n$

because $c\ell\mathcal{B}_i^n$ clearly has closed graph on Δ. Now define

$$\begin{aligned} D_i^n(1) &= \{(p,q,x,\hat{\theta}) \in C_i^n : \mathfrak{B}_i^n(p,q) \cap V \neq 0\}, \\ D_i^n(2) &= \{(p,q,x,\hat{\theta}) \in \Delta \times X^n \times \Theta^n : [P_i^n(x_i) \cap \mathcal{B}_i^n(p,q)] \cap V \neq 0\}. \end{aligned}$$

Taking into account the fact that $[P_i^n(x_i) \cap \mathcal{B}_i^n(p,q)] \subset \mathfrak{B}_i^n(p,q)$ for all $(p,q,x,\hat{\theta}) \in \Delta \times X^n \times \Theta^n$, we see that $D_i^n = D_i^n(1) \cup D_i^n(2)$. Since $\mathfrak{B}_i^n$ is lower hemicontinuous on Δ by Lemma 3.4, the set $D_i^n(1)$ is open in C_i^n and therefore is open in $\Delta \times X^n \times \Theta^n$. Observe that the correspondence $(P_i^n \cap \mathcal{B}_i^n) : \Delta \times X^n \times \Theta^n \to 2^{X_i^n}$ defined by $(P_i^n \cap \mathcal{B}_i^n)(p,q,x,\hat{\theta}) = P_i^n(x_i) \cap \mathcal{B}_i^n(p,q)$ is lower hemicontinuous since P_i^n is lower hemicontinuous on X_i^n and $\mathcal{B}_i^n$ has open graph in Δ. Thus it follows that $D_i^n(2)$ is open in $\Delta \times X^n \times \Theta^n$. As a consequence, it is concluded that $D_i^n = D_i^n(1) \cup D_i^n(2)$ is open in $\Delta \times X^n \times \Theta^n$, which implies that φ_i^n is lower hemicontinuous. ■

LEMMA 3.6. : If Assumptions A1–A7 are satisfied, for each n, there is a profile $(p^n, q^n, x^n, \theta^n) \in \Delta \times X^n \times \Theta^n$ such that, for every $i \in I$,

(1) $(x_i^n, \theta_i^n) \in c\ell\mathcal{B}_i^n(p^n, q^n)$**,**

(2) $(P_i^n(x_i^n) \times \Theta_i^n) \cap \mathcal{B}_i^n(p^n, q^n) = \emptyset$**,**

(3) $\forall\, (p,q) \in \Delta$**, it holds true that**

$$\textstyle\sum_{s\in S_0} p^n(s) \cdot z^n(s) + q^n \cdot \sum_{i\in I} \theta_i^n \geq \sum_{s \in S_0} p(s) \cdot z^n(s) + q \cdot \sum_{i \in I} \theta_i^n,$$

where $z^n(s) := \sum_{i\in I}(x_i^n(s) - e_i(s))$ **for every** $s \in S_0$**,**

(4) $z^n = 0$ **and** $\sum_{i\in I} \theta_i^n = 0$**.**

Proof : Applying Lemma 3.1. to φ_i^n, we obtain $(p^n, q^n, x^n, \theta^n) \in \Delta \times X^n \times \Theta^n$ satisfying (1), (2), and (3) for every $i \in I$.

To prove (4), suppose that $z^n(0) \neq 0$ or $\sum_{i\in I} \theta_i^n \neq 0$. Then (3) implies $\|(p^n(0), q^n)\| = 1$ and $p^n(0) \cdot z^n(0) + q^n \cdot \sum_{i\in I} \theta_i^n > 0$. However, (1) implies that $p^n(0) \cdot z^n(0) + q^n \cdot \sum_{i\in I} \theta_i^n \leq m(1 - \|(p^*(0), q^n)\|) = 0$, which is a contradiction. If $z^n(s) \neq 0$ for some $s \in S$, then (3) implies $\|p^n(s)\| = 1$ and $p^n(s) \cdot z^n(s) > 0$. However, (1) implies that $p^n(s) \cdot z^n(s) \leq W_s(q^n) \cdot \sum_{i\in I} \theta_i^n = 0$, which is a contradiction. ■

Before we state the main theorem, it is necessary to provide the following preliminary existence theorem for the projected economy $\hat{\mathcal{E}}$.

THEOREM 3.2. : If Assumptions A1–A7 are satisfied, the economy $\hat{\mathcal{E}}$ has a competitive equilibrium.

Proof : Take a sequence $\{(p^n, q^n, x^n, \theta^n)\}$, each of which is obtained in Lemma 3.6.

Claim 1 : The sequence $\{(p^n, q^n, x^n, \theta^n)\}$ is bounded, by which, without loss of generality, we can assume that $(p^n, q^n, x^n, \theta^n)$ converges a point $(p^*, q^*, x^*, \hat{\theta}^*) \in \Delta \times X \times \hat{\Theta}$.

Proof : First observe that the sequence $\{(p^n, q^n, x^n)\}$ is bounded, since $z^n = 0$ and therefore $x^n \in \hat{X}$ for every n. To show that the sequence $\{\theta^n\}$ is bounded, suppose to the contrary that it is unbounded, implying $a_n^{-1} := \sum_{i \in I} \|\theta_i^n\| \to \infty$. Since the sequence $\{a_n \theta^n\}$ is bounded and the sequence $\{(p^n, q^n, x^n)\}$ is bounded, without loss of generality, we can assume that the sequence $(p^n, q^n, x^n, a_n\theta^n)$ converges to (p^*, q^*, x^*, v) with $v_i \in V + M$. Since $a_n[p^n \,\square\, (x_i^n - e_i)] \leq W(q^n) \cdot (a_n \theta_i^n)$, in the limit, we obtain $W(q^*) \cdot v_i \geq 0$ for each $i \in I$. Observe that $\sum_{i \in I} \|v_i\| = 1$ and that (4) of Proposition 3 implies that $\sum_{i \in I} v_i = 0$. Accordingly, it follows that $W(q^*) \cdot v_i = 0$ and therefore $v_i \in V^{\perp}$ for every $i \in I$. Since $v_i \in V + M, \forall\, i \in I$, we have $v_i \in M$ for every $i \in I$. Moreover, for every $i \in I$, since $v_i \in \Gamma(\hat{\Theta}_i)$ it holds that $v_i \in (\Gamma(\hat{\Theta}_i) \cap M)$ and therefore $v_i \in \sum_{j \in I}(\Gamma(\hat{\Theta}_j) \cap M)$. On the other hand, $v_i = -\sum_{j \neq i} v_j \in -\sum_{j \neq i}(\Gamma(\hat{\Theta}_j) \cap M)$. By (ii) of Proposition 2.1, we see that $v_i = 0$ for every $i \in I$, which is a contradiction. It is clear that $(p^*, q^*, x^*) \in \Delta \times X$. Assumption A7 ensures that $\hat{\theta}^* \in \hat{\Theta}$. □

We will show that $(p^*, q^*, x^*, \hat{\theta}^*)$ is an equilibrium for $\hat{\mathcal{E}}$. Let us define

$$\hat{\mathcal{B}}_i^{\circ}(p^*, q^*) = \left\{(x_i, \hat{\theta}_i) \in X_i \times \hat{\Theta}_i : p^* \,\square\, (x_i - e_i) \ll W(q^*) \cdot \hat{\theta}_i + \gamma(p^*, q^*)\right\},$$

$$c\ell\hat{\mathcal{B}}_i^{\circ}(p^*, q^*) = \left\{(x_i, \hat{\theta}_i) \in X_i \times \hat{\Theta}_i : p^* \,\square\, (x_i - e_i) \leq W(q^*) \cdot \hat{\theta}_i + \gamma(p^*, q^*)\right\}.$$

Claim 2 : $(p^*, q^*, x^*, \hat{\theta}^*)$ satisfies the following: for every $i \in I$,

(*a*) $(x_i^*, \hat{\theta}_i^*) \in c\ell\hat{\mathcal{B}}_i^{\circ}(p^*, q^*)$,

(*b*) $(P_i(x_i^*) \times \hat{\Theta}_i) \cap \hat{\mathcal{B}}_i^{\circ}(p^*, q^*) = \emptyset$,

(c) $\forall\,(p,q) \in \Delta$, **it holds true that**

$$\textstyle\sum_{s\in S_0} p^*(s)\cdot z^*(s) + q^*\cdot \sum_{i\in I}\hat{\theta}_i^* \geq \sum_{s\in S_0} p(s)\cdot z^*(s) + q\cdot \sum_{i\in I}\hat{\theta}_i^*,$$

where $z^*(s) := \sum_{i\in I}(x_i^*(s) - e_i(s))$ **and for every** $s \in S_0$**.**

Proof : It is straightforward to verify (a) **and** (c)**. For** (b)**, we will first show that** $(coP_i(x_i^*) \times \hat{\Theta}_i) \cap \hat{\mathcal{B}}_i^\circ(p^*, q^*) = \emptyset$**. Suppose otherwise. Then we can take a choice** $(\bar{x}_i, \bar{\theta}_i) \in (coP_i(x_i^*) \times \hat{\Theta}_i) \cap \hat{\mathcal{B}}_i^\circ(p^*, q^*)$**. Since the correspondence** coP_i **is lower hemicontinuous and** $\hat{\mathcal{B}}_i^\circ$ **has open graph, the correspondence** $(p, q, x_i) \mapsto (coP_i(x_i) \times \hat{\Theta}_i) \cap \hat{\mathcal{B}}_i^\circ(p, q)$ **is lower hemicontinuous. Thus there is a sequence** $\{(\bar{x}_i^n, \bar{\theta}_i^n)\}$ **converging to** $(\bar{x}_i, \bar{\theta}_i)$ **such that** $(\bar{x}_i^n, \bar{\theta}_i^n) \in (coP_i(x_i^n) \times \hat{\Theta}_i) \cap \hat{\mathcal{B}}_i^\circ(p^n, q^n)$**. For each** $\alpha \in (0, 1)$ **and each** $n \in \mathbb{N}$**, we set** $v_i^n(\alpha) := (\alpha\bar{x}_i^n + (1-\alpha)x_i^n, \alpha\bar{\theta}_i^n + (1-\alpha)\theta_i^n)$**. Notice that for sufficiently large** n **and all** $\alpha \in (0, 1)$**, we have** $v_i^n(\alpha) \in X_i^n \times \Theta_i^n$**. Therefore, for** α **sufficiently close to** 1 **and sufficiently large** n**, we have** $v_i^n(\alpha) \in (P_i^n(x_i^n) \times \Theta_i^n) \cap \mathcal{B}_i^n(p^n, q^n)$**, which contradicts (2) of Lemma 3.6. Therefore it follows that** $(coP_i(x_i^*) \times \hat{\Theta}_i) \cap \hat{\mathcal{B}}_i^\circ(p^*, q^*) = \emptyset$**. Since** $P_i(x_i^*) \subset coP_i(x_i^*)$**, we conclude that** $(P_i(x_i^*) \times \hat{\Theta}_i) \cap \hat{\mathcal{B}}_i^\circ(p^*, q^*) = \emptyset$**.** □

Claim 3 : $z^* = 0$ **and** $\sum_{i\in I}\hat{\theta}_i^* = 0$**.**

Proof : This follows from (4) of Lemma 3.6. and Claim 1. □

Claim 4 : $\gamma(p^*, q^*) = 0$**.**

Proof : Assumption A5 with (a) **of Claim 2 implies that**

$$p^* \,\square\, (x_i^* - e_i) = W(q^*)\cdot\hat{\theta}_i^* + \gamma(p^*, q^*),\ \forall\, i \in I,$$

Summing this up over $i \in I$**, we have** $m\gamma(p^*, q^*) = 0$**, giving** $\gamma(p^*, q^*) = 0$**.** □

Claim 5 : $(x_i^*, \theta_i^*) \in c\ell\hat{\mathcal{B}}_i(p^*, q^*)$**.**

Proof : This is immediate from (a) **of Claim 2 and Claim 4.** □

Claim 6 : $(P_i(x_i^*) \times \Theta_i) \cap \hat{\mathcal{B}}_i(p^*, q^*) = \emptyset$**.**

Proof : This is immediate from (b) **of Claim 2 and Claim 4.** □

Up to now, we have shown that $(p^*, q^*, x^*, \hat{\theta}^*)$ **is a quasi-equilibrium for** $\hat{\mathcal{E}}$**. To prove that it is an equilibrium for** $\hat{\mathcal{E}}$**, we need the following claim.**

Claim 7 : $p^*(0) \neq 0$ **and** $q^* \in Q$**.**

Proof : If $p^*(0) = 0$ **holds, then Assumption A5 implies that** $(P_i(x_i^*) \times \hat{\Theta}_i) \cap \hat{\mathcal{B}}_i(p^*, q^*) \neq \emptyset$**, a contradiction. To see** $q^* \in Q$**, suppose to the contrary that there is a** $\hat{\theta}_i \in \hat{\Theta}_i$ **satisfying** $W(q^*) \cdot \hat{\theta}_i > 0$ **for some** $i \in I$**. Again, Assumption A5 implies that** $(P_i(x_i^*) \times \hat{\Theta}_i) \cap \hat{\mathcal{B}}_i(p^*, q^*) \neq \emptyset$**, a contradiction.** □

Claim 8 : For every $i \in I$**,** $(P_i(x_i^*) \times \Theta_i) \cap c\ell\hat{\mathcal{B}}_i(p^*, q^*) = \emptyset$**.**

Proof : First, observe that $p^*(s) \neq 0, \forall s \in S_0$ **by Claim 4 and Claim 7. Therefore Assumption A6 implies** $\hat{\mathcal{B}}_i(p^*, q^*) \neq \emptyset$**. Now suppose to the contrary that, for some** $i \in I$**, there exists** $(x_i, \theta_i) \in (P_i(x_i^*) \times \Theta_i) \cap c\ell\hat{\mathcal{B}}_i(p^*, q^*)$**. Now one can take** $(x_i', \theta_i') \in \hat{\mathcal{B}}_i(p^*, q^*)$**. Then Assumption A4 implies that, for** $\alpha \in (0, 1)$ **sufficiently close to 1,** $\alpha(x_i, \theta_i) + (1-\alpha)(x_i', \theta_i') \in P_i(x_i^*) \cap \hat{\mathcal{B}}_i(p^*, q^*)$**, which is a contradiction to Claim 6.** □

Consequently, $(p^*, q^*, x^*, \hat{\theta}^*)$ **is an equilibrium for** $\hat{\mathcal{E}}$**.** ■

Now we are ready to provide the main existence theorem for the original economy $\mathcal{E}$**. The following main theorem is immediately follows from Theorem 3.1 and Theorem 3.2.**

THEOREM 3.3. : Under Assumptions A1–A7, there exists a competitive equilibrium in the economy $\mathcal{E}$**.**

3. CONCLUDING REMARKS

We have extended the main results of Won and Hahn (2007) to the case where preferences need neither complete nor transitive. It is assumed in this paper that asset markets are subject to portfolio constraints represented by a cone with vertex. Cone constraints cover many interesting cases such as short-selling restrictions, nonnegative wealth constraints, and restricted market participation. In contrast to the literature, no extra restrictions on feasible portfolios are introduced to limit the risk-sharing capability of redundant assets. In particular, redundant assets are allowed to give rise to the large multiplicity of optimal portfolio choices in equilibrium. The approach of the paper can be extended to the multiperiod incomplete markets with convex constraints

where preferences need neither complete nor transitive.

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