JETEM

Journal of Economic Theory and Econometrics

The Journal of the Korean Econometric Society

(계 량 경 제 학 보)

December 2008

Volume 19. Issue 4. pp. 61-74

Strategy-Proof Risk Sharing for Uncertainty Averse Preferences

Biung-Ghi Ju*

Abstract We consider a model of state-contingent goods allocation. Each agent has a set of beliefs, or priors, and evaluates an allocation according to the minimum expected utility over the set of priors. We study allocation rules that satisfy e¢ ciency, individual rationality, and strategy-proofness (no one can benefit by misrepresenting his preferences). In the aggregate certainty case, the three axioms are compatible and we characterize the non-empty family of rules which contains both Walrasian and non-Walrasian rules. In the aggregate uncertainty case, we show in the 2-agent and 2-state case that the three axioms may or may not be compatible, depending on existence of a degenerate prior.

Keywords Risk sharing, Strategy-proofness, Uncertainty aversion, Maximin expected utility, Efficiency, Individual rationality

JEL Classification C70, D70

---

* Department of Economics, Korea University, Anam-dong 5-1, Seongbuk-gu, Seoul 136-701, Korea.

e-mail: bgju@korea.ac.kr I thank a referee for helpful comments. All remaining errors are mine. This work was supported by Korea Research Foundation under the fund KRF-2008-332-B00040.

Received December 1, 2008, Revised December 15, 2008, Accepted December 30, 2008

1. Introduction

We consider a model of state-contingent goods allocation. Each agent has a set of beliefs, or priors, and evaluates an allocation according to the minimum expected utility over the set of priors. Preferences of this type are known as maximin expected utility preferences and their axiomatic foundation is offered by Gilboa and Schmeidler (1989). Our main objective is to study (allocation) rules satisfying the three well-known axioms, *efficiency*, *individual rationality*, and *strategy-proofness* (Gibbard 1973; Satterthwaite 1975). Numerous authors have shown that the three axioms are incompatible in exchange economies: see Hurwicz (1972), Dasgupta et al. (1979), Zhou (1991), Schummer (1997), and Serizawa (2002), Serizawa and Weymark (2003), Ju (2003, 2005), etc.

Of particular relevance to our work is Ju (2005). This paper considers the model of state-contingent goods allocation where agents have the common prior and expected utility preferences. The main results in Ju (2005) show that when *aggregate certainty* (constant aggregate endowments across states) prevails, the three axioms are compatible; otherwise, they are incompatible. However, the family of rules satisfying the three axioms in the aggregate certainty case is extremely restricted; they are fixed price selections from Walrasian equilibrium allocations. We show that in the case of maximin expected utility preferences, the family is larger due to the well-known indeterminacy of equilibria. We characterize this family imposing the three axioms. It consists of both Walrasian and non-Walrasian rules. When aggregate uncertainty holds, we find results that give mixed messages, yet, closer to the negative side. When there are two states and two agents with the same set of non-degenerate priors, only dictatorial rules are *efficient* and *strategy-proof*. But when there is a degenerate prior, we provide a characterization result similar to the aggregate certainty case.

The rest of this paper is organized as follows. Section 2. explains the model and basic concepts. Section 3. offers the main results.

2. The Model and Basic Concepts

Consider a society consisting of $n \geq 2$ agents, $N \equiv \{1, \cdots, n\}$. There is a finite number S of uncertain *states* with $S \geq 2$. Denote the set of states by $S \equiv \{1, \ldots, S\}$. At each state $s \in S$, a fixed amount Ω_s of a single good, or money, is available in the society, which is the sum of individual amounts owned by each agent $i \in N$. Let $\Omega \equiv (\Omega_s)_{s \in S}$ be the *aggregate endowment*. *Aggregate certainty* holds if the aggregate endowment is composed of a constant amount across states, that is, $\Omega_1 = \cdots = \Omega_S$. Otherwise, *aggregate uncertainty* holds. Let $\mathcal{W} \equiv \{(\omega_i)_N \in \mathbb{R}_+^{S \times N} : \sum_N \omega_i = \Omega\}$ be the set of (individual) *endowments profiles*. For each profile $\omega \equiv (\omega_i)_N \in \mathcal{W}$, the i^{th} component $\omega_i \equiv (\omega_{is})_{s \in S}$ indicates agent i's *endowment*. Note that although we admit variability of individual endowments, we assume the aggregate endowment to be fixed.

Agents can share their individual risks by allocating the aggregate endowment prior to the realization of a state in S. An *allocation* is a list of state-contingent bundles indexed by agents, $(z_i)_N \in \mathbb{R}_+^{N \times S}$, where for each $i \in N$, the i^{th} bundle z_i indicates what agent i receives in various states. It is *feasible* if the sum of individual bundles is less than or equal to the aggregate endowment, $\sum_N z_i \leq \Omega$.[1] The set of all feasible allocations is denoted by Z, of which the generic element is $z \equiv (z_i)_{i \in N}$.

Each agent has multiple beliefs about uncertain states, or priors, and the set of priors is revealed. For each $i \in N$, denote agent i's set of priors by Π^i. Agent $i \in N$ has the so-called maximin expected utility preferences (Gilboa and Schmeidler 1989), briefly *MEU preferences* R_i that are represented by the set of i's priors Π^i and an index function $u_i \colon \mathbb{R}_+ \to \mathbb{R}$ as follow: for each pair $x, y \in \mathbb{R}_+^S$,

$$x \ R_i \ y \text{ if and only if } \min_{\pi \in \Pi^i} \sum_{s \in S} \pi_s u_i(x_s) \geq \min_{\pi \in \Pi^i} \sum_{s \in S} \pi_s u_i(x_s).$$

We assume that preferences are continuous, monotonic, and convex.[2] Note that by

[1] Notation for vector inequalities are as follows. For each $x, x' \in \mathbb{R}^T$, we write $x \geq x'$ if for all $s \in T$, $x_s \geq x'_s$ and for some $r \in T$, $x_r > x'_r$. We write $x > x'$ if for all $s \in T$, $x_s > x'_s$.

[2] A preference is *continuous* if for each bundle x, both the set of all bundles weakly preferred to x and the set of all bundles to which x is weakly preferred are closed. A preference is *monotonic* if for

continuity, monotonicity and convexity, $u_i(\cdot)$ is continuous, strictly increasing and concave. We further assume that for each $i \in N$, the set of i's priors Π^i is convex and compact and that the set of *common priors* $\Pi^* \equiv \cap_{i\in N}\Pi^i$ is full dimensional, that is, the interior of Π^* in Δ^{S-1} is non-empty. Let $\mathcal{R}_i$ be the family of such preferences of agent $i \in N$. Note that by full dimensionality of Π^*, if x is not a full insurance bundle, then there is $\pi \in \Pi^*$ such that $\sum_{s\in S}\pi_s u_i(x_s) > min_{\pi'^i}\sum_{s\in S}\pi'_s u_i(x_s)$.[3]

An *economy* is characterized by a preference profile $R \equiv (R_i)_{i\in N} \in \times_{i\in N}\mathcal{R}_i$ and a profile of individual endowments $\omega \equiv (\omega_i)_N \in \mathcal{W}$. Let $\mathcal{E} \equiv [\times_{i\in N}\mathcal{R}_i] \times \mathcal{W}$ be the family of economies. A risk sharing rule, or simply a *rule* on $\mathcal{E}$ is a function $\varphi\colon \mathcal{E} \to Z$ associating with each economy a single feasible allocation.

We now define three main axioms for rules. An allocation is *efficient* if it is feasible and no other feasible allocation makes someone better off without making someone else worse off. For each $R \in \times_{i\in N}\mathcal{R}_i$, let $P(R)$ be the set of all *efficient* allocations for R. Note that this set depends not on individual endowments but on the aggregate endowment, which is fixed; thus no extra argument ω. For each $i \in N$, let $P_i(R) \equiv \{z_i : z \in P(R)\}$. An allocation is *individually rational* if each agent receives a bundle that is at least as good as his endowment. A rule φ is *efficient* if it always recommends an efficient allocation. It is *individually rational* if it always recommends an individually rational allocation. It is *strategy-proofness* if no one can ever benefit by misrepresenting his preference, independently of others' representations, that is, for each $(R,\omega) \in \mathcal{E}$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R,\omega)\ R_i\ \varphi_i((R'_i, R_{-i}),\omega)$.

A *Walrasian* (equilibrium) *allocation for* $(R,\omega) \in \mathcal{E}$ is an allocation $z \in Z$ with a price vector $p \in \Delta^{S-1}$ such that $p\cdot z_i \le p\cdot\omega_i$ and for each $i \in N$ and each $z'_i \in \mathbb{R}^S_+$ with $p\cdot z'_i \le p\cdot\omega_i$, $z_i\ R_i\ z'_i$. We call p an *equilibrium price.* For each $(R,\omega) \in \mathcal{E}$, let $W(R,\omega)$

all x, y, x is weakly preferred to y whenever $x \geqq y$ and x is preferred to y whenever $x > y$. It is *strictly monotonic* if for all x, y, x is preferred to y whenever $x \ge y$. Finally, a preference is *convex* if for all x, the set of all bundles preferred to x is convex.

[3]If for each $\pi \in \Pi^*$, $\sum_{s\in S}\pi_s u_i(x_s) = \min_{\pi'\in\Pi^i}\sum_{s\in S}\pi'_s u_i(x_s)$, then Π^* is a subset of $\{\pi \in \Delta^{S-1} : \sum_{s\in S}\pi_s u_i(x_s) = \min_{\pi'\in\Pi^i}\sum_{s\in S}\pi'_s u_i(x_s)\}$ which has no interior point in Δ^{S-1} because of the constraint $\sum_{s\in S}\pi_s u_i(x_s) = \min_{\pi'\in\Pi^i}\sum_{s\in S}\pi'_s u_i(x_s)$. Note that this constraint is independent of $\sum_{s\in S}\pi_s = 1$ because x is not a full insurance bundle.

be the set of Walrasian allocations. We refer to the set valued function $W\colon \mathcal{E} \twoheadrightarrow Z$ as the *Walrasian correspondence*. Since preferences are continuous, monotonic, and convex, it is non-empty valued.[4] A *selection from the Walrasian correspondence* is a rule associating with each economy a single Walrasian allocation. When the selection is made using a fixed equilibrium price, it is called a *fixed price selection*. Clearly, any selection from the Walrasian correspondence is *efficient* and *individually rational*. Some of our results will explain when there exist *strategy-proof* selections.

A rule φ is *dictatorial* if for each $\omega \in \mathcal{W}$, there is an agent $i \in N$, the "dictator", who always receives the entire aggregate endowment Ω independently of preferences, that is, for each $R \in \times_{i\in N}\mathcal{R}_i$, $\varphi_i(R,\omega) = \Omega$. Note that the dictator may vary across endowments profiles. When preferences are strictly monotonic,[5] any dictatorial rule is *efficient*. Since for each $\omega \in \mathcal{W}$, dictatorial rules are constant across economies with endowments profile ω, they are *strategy-proof*. However, since at least one agent receives 0, no dictatorial rule is *individually rational*. Another simple example is the *no-trade rule* that always recommends endowments profiles. Clearly, no-trade rule is *individually rational* and *strategy-proof*. However, it is not *efficient* because no-trade allocations are not always *efficient*.

When there are only two states, for each agent $i \in N$, the following two priors in Π^i play crucial roles. Let $\pi^{i1} \equiv \arg\max\{\pi_1 : \pi \in \Pi^i\}$ be the prior with the greatest probability of state 1. Let $\pi^{i2} \equiv \arg\max\{\pi_2 : \pi \in \Pi^i\}$ be the prior with the greatest probability of state 2. Then for all index functions $u_i\colon \mathbb{R}_+ \to \mathbb{R}$ and all $x \in \mathbb{R}^2_+$, if $x_1 > x_2$, $\min_{\pi\in\Pi^i} \sum_{s\in S} \pi_s u_i(x_s) = \sum_{s\in S} \pi^{i2}_s u_i(x_s) < \sum_{s\in S} \pi^{i1}_s u_i(x_s)$ and if $x_1 < x_2$, $\min_{\pi\in\Pi^i} \sum_{s\in S} \pi_s u_i(x_s) = \sum_{s\in S} \pi^{i1}_s u_i(x_s) < \sum_{s\in S} \pi^{i2}_s u_i(x_s)$. Thus each MEU preference R_i coincides with the EU (expected utility) preference with prior π^{i2} over $\{x \in \mathbb{R}^2_+ : x_1 \geq x_2\}$ and with the EU preference with prior π^{i1} over $\{x \in \mathbb{R}^2_+ : x_1 \leq x_2\}$.

The following concepts or notation will be used later. A *full insurance bundle* is a state-contingent bundle consisting of a constant amount of money across states. A

[4]See Mas-Colell, Whinston, and Green (1995).
[5]Preferences are strictly monotonic, when each agent's priors are non-degenerate.

full insurance allocation is an allocation consisting of only full insurance bundles. Let B_{FI} be the set of all full insurance bundles. Let A_{FI} be the set of all full insurance allocations. For all $p \in \mathbb{R}^S_+$ and all $\omega \in \mathbb{R}^S_+$, let $B(p,\omega) \equiv \{y \in \mathbb{R}^S_+ : p \cdot y \leq p \cdot \omega\}$. For all $X \subseteq \mathbb{R}^S_+$, all $i \in N$, and all $R_i \in \mathcal{R}_i$, let $Max[R_i, X]$ be the set of all best bundles for R_i in X. Let $p \in \mathbb{R}^S_{++}$, $R \in \times_{i\in N}\mathcal{R}_i$, and $i \in N$. For each $x \in \mathbb{R}^S_+$, let $H(p,x) \equiv \{y \in \mathbb{R}^S_+ : p \cdot y \geq p \cdot x\}$, $UC(R_i,x) \equiv \{y \in \mathbb{R}^S_+ : y\ R_i\ x\}$, $SUC(R_i,x) \equiv \{y \in \mathbb{R}^S_+ : y\ P_i\ x\}$, and $LC(R_i,x) \equiv \{y \in \mathbb{R}^S_+ : x\ R_i\ y\}$.[6] For each $x \in \mathbb{R}^S_+$, p *supports* R_i *at* x if $UC(R_i,x) \subseteq H(p,x)$. For each $z \equiv (z_i)_N \in \mathbb{R}^{N\times S}$, p *supports* R *at* z if for each $i \in N$, p supports R_i at z_i.

3. Results

3.1. Aggregate Certainty

In this section, we consider the aggregate certainty case. Note that for each agent $i \in N$ and each full insurance bundle $z_i \in B_{\mathrm{FI}}$, if there is a prior $\pi \in \Pi^i$ such that $\pi \cdot z_i \geq \pi \cdot \omega_i$, then z_i is *individually rational* for all MEU preferences associated with Π^i and vice versa.[7] Thus, we call such a bundle *always rationalizable.* For each $i \in N$ and each $\omega_i \in \mathbb{R}^l_+$, let $B^i_{\mathrm{arFI}}(\omega_i) \equiv \{z_i \in B_{\mathrm{FI}} : \text{for some } \pi \in \Pi^i,\ \pi \cdot z_i \geq \pi \cdot \omega_i\}$ be the set of always rationalizable full insurance bundles of agent i with endowment ω_i. Let $A_{\mathrm{arFI}}(\omega) \equiv \{z \in Z : \text{for all } i \in N,\ z_i \in B^i_{\mathrm{arFI}}(\omega_i)\}$ be the set of *always rationalizable full insurance allocations under endowments profile* ω. If ω has aggregate certainty and there is a common prior (i.e., $\Pi^* \neq \emptyset$), then $A_{\mathrm{arFI}}(\omega) \neq \emptyset$. This is because by aggregate certainty, any full insurance allocation where each agent $i \in N$ receives the expected income under a common prior $\pi \in \Pi^*$ for sure is feasible.

A rule $\varphi\colon \mathcal{E} \to Z$ is a *selection from always rationalizable full insurance allocations*

[6] We use H for "Half space", UC for "Upper Contour set", SUC for "Strict Upper Contour set", and LC for "Lower Contour set".

[7] Since $\pi{\cdot}z_i \geq \pi{\cdot}\omega_i$ for some $\pi \in \Pi^i$ and $u_i(\cdot)$ is strictly increasing and concave, $u_i(\pi{\cdot}z_i) \geq u_i(\pi{\cdot}\omega_i) \geq \sum_{s\in S} \pi_s u_i(\omega_{is}) \geq \min_{\pi'\in\Pi^i} \sum_{s\in S} \pi'_s u_i(\omega_{is})$. Because z_i is a symmetric bundle, then for each $\pi' \in \Pi^i$, $\sum_{s\in S} \pi'_s u_i(z_{is}) = u_i(\pi' \cdot z_i)$. Therefore, $\min_{\pi'\in\Pi^i} \sum_{s\in S} \pi'_s u_i(z_{is}) \geq \min_{\pi'\in\Pi^i} \sum_{s\in S} \pi'_s u_i(\omega_{is})$.

if for each $(R,\omega) \in \mathcal{E}$, $\varphi(R,\omega) \in A_{\text{arFI}}(\omega)$. It is *own preference invariant* if for each $i \in N$, each $(R,\omega) \in \mathcal{E}$, and each $R'_i \in \mathcal{R}_i$, $\varphi_i((R'_i, R_{-i}), \omega) = \varphi_i(R,\omega)$.

The following simple characterization of *efficiency* is useful.

Lemma 1. *Under aggregate certainty, an allocation is efficient if and only if it is a full insurance allocation.*

Proof. Assume aggregate certainty. Let $R \in \times_{i\in N}\mathcal{R}_i$. For each $i \in N$, let $u_i \colon \mathbb{R}_+ \to \mathbb{R}$ be an index function of R_i. Since each common prior $\pi \in \Pi^*$ supports R at any full insurance allocation. then full insurance allocations are *efficient*. Suppose that $z \in Z$ is not a full insurance allocation. Let $i \in N$ be such that $z_i \notin B_{\text{FI}}$. Then by the full dimensionality of Π^*, there is $\bar{\pi} \in \Pi^*$ such that $\sum_{s\in S} \bar{\pi}_s u_i(z_{is}) > \min_{\pi\in\Pi^i} \sum_{s\in S} \pi_s u_i(z_{is})$ (see Footnote 3). Let $\bar{\mu}_i \equiv \bar{\pi} \cdot z_i$. Then by concavity of u_i, $u_i(\bar{\mu}_i) \geq \sum_{s\in S} \bar{\pi}_s u_i(z_{is})$ and so $(\bar{\mu}_i, \cdots, \bar{\mu}_i)\ P_i\ z_i$. For each $j \neq i$, let $\bar{\mu}_j \equiv \bar{\pi} \cdot z_j$. Then by concavity of u_j, $u_j(\bar{\mu}_j) \geq \sum_{s\in S} \bar{\pi}_s u_j(z_{js})$. Since $\bar{\pi} \in \Pi^j$, $\sum_{s\in S} \bar{\pi}_s u_j(z_{js}) \geq \min_{\pi\in\Pi} \sum_{s\in S} \pi_s u_j(z_{js})$. Hence $\left(\bar{\mu}_j, \cdots, \bar{\mu}_j\right)\ R_j\ z_j$. Let $\bar{z}$ be the allocation where each agent $j \in N$ receives full insurance bundle $(\bar{\mu}_j, \ldots, \bar{\mu}_j)$. By aggregate certainty, $\bar{z}$ is feasible. Therefore, z is not *efficient*, contradicting the initial assumption. □

Under "strict concavity" and "differentiability" assumptions on utility indices, the same characterization of *efficient* allocation follows from the main result by Billot et al. (1998) (Theorem 1). When preferences are represented by concave, yet non-strictly concave utility indices, without any restriction on the common prior set, there might be some *efficient* non-full insurance allocations. Thus, the full dimensionality of the common prior set is crucial in Lemma 1.

Now we are ready to characterize *efficient, individually rational,* and *strategy-proof* rules.

Theorem 2. *Under aggregate certainty, a rule is efficient, individually rational, and strategy-proof if and only if it is an own preference invariant selections from always rationalizable full insurance allocations.*

Proof. Suppose aggregate certainty. By Lemma 1, every selection from $A_{\text{arFI}}(\cdot)$ is *efficient.* As we showed above, for each $\omega \in \mathcal{W}$, all allocations in $A_{\text{arFI}}(\omega)$ are *individually rational* for all $R \in \times_{i \in N} \mathcal{R}_i$. If a selection from $A_{\text{arFI}}(\cdot)$ is own preference invariant in addition, then it is *strategy-proof.*

In order to prove the converse, let φ satisfy the three axioms. Let $(R, \omega) \in \mathcal{D}$ and $z \equiv \varphi(R, \omega)$. Since φ is *efficient*, then by Lemma 1, z is a full insurance allocation. Suppose by contradiction $z \notin A_{\text{arFI}}(\omega)$. Then there exists $i \in N$ such that for all $\pi \in \Pi^i$, $\pi \cdot z_i < \pi \cdot \omega_i$. Let $u_i^{neut}\colon \mathbb{R}_+ \to \mathbb{R}$ be a linear index function and R_i^{neut} be the MEU preference represented by u_i^{neut}. Then $\omega_i \; P_i^{neut} \; z_i$. By Lemma 1, $\varphi_i((R_i^{neut}, R_{-i}), \omega)$ is also a full insurance allocation. Thus by *strategy-proofness*, $\varphi_i((R_i^{neut}, R_{-i}), \omega) = z_i$, contradicting *individual rationality.* Therefore, φ is a selection from $A_{\text{arFI}}(\omega)$. Own preference invariance is a direct consequence of Lemma 1 and *strategy-proofness.* □

Remark 3. *When aggregate certainty holds, all Walrasian equilibrium allocations are full insurance allocations. Therefore, any own preference invariant selection from the Walrasian correspondence is efficient, individually rational, and strategy-proof. When there is no agent whose endowment is a full insurance bundle, every own preference invariant selection from the Walrasian correspondence is a fixed price selection. Moreover, such selections are constant functions. When there is an agent with a full insurance endowment, we can define other non-fixed price selections from the Walrasian correspondence (we skip the definition of such selections; it is similar to the following non-Walrasian rule). There are also other non-constant and non-Walrasian rules that are efficient, individually rational, and strategy-proof. We give an example of such rules following the same construction as in Satterthwaite and Sonnenschein (1981) and Zhou (1991). Let* φ *be a rule defined as follows. For simplicity, suppose* $N \equiv \{1, 2, 3\}$ *and* $S \equiv \{1, 2\}$. *Let* $\omega \in \mathcal{W}$. *Let* $\bar{\pi} \in \Pi^*$ *and* $\bar{z}_3 \in H(\bar{\pi}, \omega_3) \cap B_{FI}$. *Let* $A_{1,2}(\omega) \equiv \{(z_1, z_2) : (z_1, z_2, \bar{z}_3) \in Z,\ z_1 \in B^1_{arFI}(\omega_1),\ \text{and}\ z_2 \in B^2_{arFI}(\omega_2)\}$. *If we let* $\bar{z}_1 \in H(\bar{\pi}, \omega_1) \cap B_{FI}$ *and* $\bar{z}_2 \in H(\bar{\pi}, \omega_2) \cap B_{FI}$, *then* $(\bar{z}_1, \bar{z}_2, \bar{z}_3) \in Z$. *Since* $\bar{\pi} \in \Pi^*$, $(\bar{z}_1, \bar{z}_2) \in A_{1,2}(\omega)$. *Therefore,* $A_{1,2}(\omega) \neq \varnothing$. *Now partition agent 3's preferences into* *two types, type a and type b, that is, partition $\mathcal{R}_3$ into two nonempty subsets $\mathcal{R}_3^a$ and $\mathcal{R}_3^b$. For all $R \in \times_{i\in N}\mathcal{R}_i$, let $\varphi_3(R) \equiv \bar{z}_3$ and let*

$$\varphi_1(R,\omega) \equiv \begin{cases} Max\{z_1 : (z_1,z_2) \in A_{1,2}(\omega)\,,\ \textit{for some } z_2\},\ \textit{if } R_3 \in \mathcal{R}_3^a; \\ Min\{z_1 : (z_1,z_2) \in A_{1,2}(\omega)\,,\ \textit{for some } z_2\},\ \textit{if } R_3 \in \mathcal{R}_3^b. \end{cases}$$

Since agent 3's bundle is fixed and what agents 1 and 2 receive are determined by agent 3's preference, then φ is own preference invariant and is an example of rules characterized in Theorem 2. Clearly, φ is neither constant nor a selection from the Walrasian correspondence.

3.2. Aggregate Uncertainty

In this section, we consider the aggregate uncertainty case. Unlike the previous section, we focus on a simple case with only two states and two agents and establish two contrasting results. When the two agents have *identical* sets of priors and all their priors have *non-zero* probability of the state with the greater aggregate endowment, only dictatorial rules are *efficient* and *strategy-proof* (Theorem 2). However, when a prior has zero probability of the state with the greater aggregate endowment, the result is completely different: any *efficient* and *strategy-proof* rule is a fixed price selection from the Walrasian correspondence after a reallocation of endowments (Theorem 4).

We first study the case in which both agents have the same set of priors Π and all their priors have non-zero probability of the state with the greater aggregate endowment. In this case, we obtain the following result:

Theorem 4. *Assume $|N| = 2$ and $|S| = 2$. Assume that aggregate uncertainty holds and that the two agents have the same set of priors and all their priors have non-zero probability of the state with the greater aggregate endowment. Then a rule is efficient and strategy-proof if and only if it is dictatorial.*

Proof. Without loss of generality suppose $\Omega_1 > \Omega_2$. Let Π be the common set of priors of the two agents. Let $C^* \equiv \{x \in \mathbb{R}_+^2 : x_1 \geq x_2 \text{ and } \Omega_1 - x_1 \geq \Omega_2 - x_2\}$. Let

$\pi^1 \equiv \arg\min\{\pi_1 : \pi \in \Pi\}$ and $\pi^2 \equiv \arg\min\{\pi_2 : \pi \in \Pi\}$. At each allocation in C^*, because both agents receive more at state 1 than at state 2, they apply the same prior π^2 in Π for the evaluation of z. Thus their preferences on C^* coincide with the expected utility preferences associated with π^2. We prove below that the Pareto set is always included in C^*. Once this is proven, we can apply Theorem 3 in Ju (2005, p.231) to draw the desired conclusion.

Let $(R, \omega) \in \mathcal{E}$ and z be an allocation such that $z_1 \equiv (x_1, x_2) \notin C^0$. Then either $x_1 < x_2$ or $\Omega_1 - x_1 < \Omega_2 - x_2$. Consider the former case (we skip the same argument for the latter case). Then $\min_{\pi\in\Pi} \sum_{s\in S} \pi_s u_1 (z_{1s}) = \sum_{s\in S} \pi_s^1 u_1 (z_{1s})$ and $\min_{\pi\in\Pi} \sum_{s\in S} \pi_s u_2 (z_{2s}) = \sum_{s\in S} \pi_s^2 u_2 (z_{2s})$. Let $\bar{z}$ be such that $\bar{z}_1 \equiv (\pi^2 \cdot x, \pi^2 \cdot x)$ and $\bar{z}_2 \equiv (\Omega_1 - \pi^2 \cdot x, \Omega_2 - \pi^2 \cdot x)$. Then by definition of π^2, $\pi^2 \cdot x > \pi^1 \cdot x$ and so $\sum_{s\in S} \pi_s^1 u_1 (\bar{z}_{1s}) > u_i(\pi^1 \cdot x) \geq \sum_{s\in S} \pi_s^1 u_1 (z_{1s})$, which means $\bar{z}_1 \; P_1 \; z_1$. Note that $\bar{z}_2 = \lambda z_2 + (1 - \lambda) (\pi^2 \cdot z_2, \pi^2 \cdot z_2)$ for some $\lambda \in (0, 1)$. Thus by concavity of $u_2 (\cdot)$, $\bar{z}_2 \; R_2 \; z_2$. Therefore, z is not efficient. □

Let $C^{\ulcorner} \equiv \{x \in \mathbb{R}_+^2 : x_1 = x_2 \text{ or } x_2 = \Omega_2\}$ and $C^{\lrcorner} \equiv \{x \in \mathbb{R}_+^2 : \Omega_1 - x_1 = \Omega_2 - x_2 \text{ or } x_2 = 0\}$. When one and only one of the two agents has the prior under which the probability of the state with the greater aggregate endowment equals zero, there can be only one type of Pareto set. For example, suppose $\Omega_1 > \Omega_2$ and agent 1 has the prior with the zero probability of state 1. Then agent 1 cares only about the consumption at state 2 over the set of bundles with the greater consumption at state 1 than at state 2. Therefore, when all priors of agent 2 have strictly positive probability of state 1, it is not efficient to make agent 1 to consume more at state 1 than at state 2. Thus the Pareto set is always equal to $C^{\ulcorner}$. Therefore, in this case, any constant selection from $C^{\ulcorner}$ is *efficient* and *strategy-proof.*

A *redistribution scheme* $\rho\colon \mathcal{W} \to \mathcal{W}$ is a function mapping each individual endowments profile into another, possibly the same, profile. The *Walrasian correspondence under redistribution scheme* ρ associates with each economy $(R, \omega) \in \mathcal{E}$ the set of all Walrasian allocations in $(R, \rho(\omega))$.

Theorem 5. *Assume $|N| = 2$ and $|S| = 2$. Assume that aggregate uncertainty holds and that a prior of an agent has zero probability of the state with the greater aggregate endowment. Then a rule is efficient and strategy-proof if and only if there is a redistribution scheme $\rho\colon \mathcal{W} \to \mathcal{W}$ such that the rule is a fixed price selection from the Walrasian correspondence under the redistribution scheme ρ.*

Proof. Without loss of generality suppose $\Omega_1 > \Omega_2$. Suppose that $(0,1) \in \Pi^1$ or $(0,1) \in \Pi^2$.

Case 1: *Either* $(0,1) \in \Pi^1$ *or* $(0,1) \in \Pi^2$ *and not both.*

Suppose $(0,1) \in \Pi^1$ (the same argument for the other case). Then every MEU preference of agent 1 coincides with the risk neutral EU preference associated with the prior $(0,1)$, below the full insurance path. Since $(0,1) \notin \Pi^2$, for all $\pi \in \Pi^2$, $\pi_1 > 0$. Therefore for all $R \in \times_{i\in N}\mathcal{R}_i$, $P_1(R) = C^{\ulcorner}$.

We next show that given $\omega \in \mathcal{W}$, $\varphi(\cdot,\omega)$ is a constant selection from $C^{\ulcorner}$. Let $R, R' \in \times_{i\in N}\mathcal{R}_i$, $z \equiv \varphi(R,\omega)$, and $z' \equiv \varphi(R',\omega)$. By *efficiency*, $z, z' \in C^{\ulcorner}$. Also by *efficiency*, $\varphi_1((R'_1, R_2),\omega) \in C^{\ulcorner}$. If $\varphi_1((R'_1, R_2),\omega) \neq z_1$, then either $\varphi_1((R'_1, R_2),\omega) > z_1$ or $\varphi_1((R'_1, R_2),\omega) < z_1$. In either case, φ violates *strategy-proofness.* Therefore, $\varphi_1((R'_1, R_2),\omega) = z_1$ and so $\varphi((R'_1, R_2),\omega) = z$. Similarly we show that $\varphi((R'_1, R_2),\omega) = z'$. Therefore $z = z'$.

For each $\omega \in \mathcal{W}$, let $\rho(\omega) \equiv \varphi(R,\omega)$ for some $R \in \times_{i\in N}\mathcal{R}_i$. Then ρ is well-defined because $\varphi(\cdot,\omega)$ is constant. And φ is a fixed price selection from the Walrasian correspondence under the redistribution scheme ρ. □

Case 2: *Both* $(0,1) \in \Pi^1$ *and* $(0,1) \in \Pi^2$.

Then for all $R \in \times_{i\in N}\mathcal{R}_i$, $P_1(R) = C^*$ and both R_1 and R_2 coincides with the risk neutral EU preference associated with the prior $(0,1)$. Thus, in this case, each agent cares only about the amount of money at state 2.

Let $\omega \in \mathcal{W}$. We show below that $\varphi(\cdot,\omega)$ is a selection from C^* such that φ chooses a constant amount of money at state 2 for each agent, that is, for all $R, R' \in \times_{i\in N}\mathcal{R}_i$ and all $i = 1, 2$, $\varphi_{i2}(R,\omega) = \varphi_{i2}(R',\omega)$.

Let $R, R' \in \times_{i \in N} \mathcal{R}_i$, $z \equiv \varphi(R, \omega)$, and $z' \equiv \varphi(R', \omega)$. By *efficiency*, $z, z' \in C^0$. Also by *efficiency*, $\varphi_1((R'_1, R_2), \omega) \in C^0$. If $\varphi_{12}((R'_1, R_2), \omega) \neq z_{12}$, then either $\varphi_{12}((R'_1, R_2), \omega) > z_{12}$ or $\varphi_{12}((R'_1, R_2), \omega) < z_{12}$. In either case, since agent 1 does not care about the amount of money at state 1, then φ violates *strategy-proofness*. Therefore, $\varphi_{12}((R'_1, R_2), \omega) = z_{12}$ and so $(\varphi_{i2}((R'_1, R_2), \omega))_{i=1,2} = (z_{i2})_{i=1,2}$. Similarly we show that $(\varphi_{i2}((R'_1, R_2), \omega))_{i=1,2} = (z'_{i2})_{i=1,2}$. Therefore $(z_{i2})_{i=1,2} = (z'_{i2})_{i=1,2}$.

Now defining $\rho(\cdot)$ as in Case 1, we conclude that φ is the fixed price selection, associated with the price given by the common prior $(0, 1)$, from the Walrasian correspondence under ρ. □ □

Adding *individual rationality*, we establish the following result:

Corollary 6. *Under the same assumption as in Theorem 5, a rule is efficient, individually rational, and strategy-proof if and only if it is a fixed price selection from the Walrasian correspondence.*

Note that Theorems 4 and 5 do not cover all possible cases. For example, two agents may have different sets of multiple priors and all their priors have non-zero probability of the state with the greater aggregate endowment. In addition, the two results are applicable only for the state space with two elements. Investigation of these remaining cases is left for future research.

References

Billot, A., A. Chateauneuf, I. Gilboa, and J.-M. Tallon, 2000, Sharing Beliefs: Between Agreeing and Disagreeing, Econometrica, Vol. 68, 685-694.

Chateauneuf, A., R.-A. Dana, J.-M. Tallon, 2000, Optimal risk-sharing rules and equilibria with Choquet-expected-utility, Journal of Mathematical Economics, Vol. 34, 191-214.

Dasgupta, P., P. Hammond, and E. Maskin, 1979, The implementation of social choice rules, Review of Economic Studies, Vol. 46, 153-170.

Gibbard, A., 1973, Manipulation of voting scheme: a general result, Econometrica, Vol. 41, 587-602.

Gilboa, I., and D. Schmeidler, 1989, Maximin expected utility with non-unique prior, Journal of Mathematical Economics, Vol. 18, 141-153.

Hurwicz, L., 1972, On informationally decentralized systems, in McGuire and Radner (eds), Decision and Organization, North-Holland, Amsterdam, 297-336.

Ju, B.-G., 2003, Strategy-proofness versus efficiency in exchange economies: general domain properties and applications, Social Choice and Welfare, Vol. 21, 73-93.

Ju, B.-G., 2005, Strategy-proof risk sharing, Games and Economic Behavior, Vol. 50, 225-254.

Mas-Colell, A., M.D. Whinston, and J.R. Green, 1995, Microeconomic Theory, New York: Oxford University Press.

Satterthwaite, M., 1975, Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions, Journal of Economic Theory, Vol. 10, 187-217.

Satterthwaite, M. and H. Sonnenschein, 1981, Strategy-proof allocation mechanisms at differentiable points, Review of Economic Studies, Vol. 48, 587-597.

Schummer, J., 1997, Strategy-proofness versus efficiency on restricted domains of exchange economies, Social Choice and Welfare, Vol. 14, 47-56.

Serizawa, S., 2002, Inefficiency of strategy-proof rules for pure exchange economies, J. Econ. Theory, Vol. 106, 219–241.

Serizawa, S. and J.A. Weymark, 2003, Efficient strategy-proof exchange and minimum consumption guarantees, J. Econ. Theory, Vol. 109, 246–263.

Zhou, L., 1991, Inefficiency of strategy-proof allocation mechanisms in pure exchange economies, Social Choice and Welfare, Vol. 8, 247-254.