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Markets with countably many periods and a countable number of agents without ordered preferences*

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Abstract Peleg-Yaari (1970) provide a theoretical foundation for the intertemporal resource allocation problem over discrete time and prove the existence of a competitive equilibrium with a _nite number of agents. In this paper, we present a generalization that includes non-ordered preferences, no free disposal, and a countable in_nity of agents.

Keywords Non-ordered preferences, properness, a countable number of agents

JEL Classification C62, D51

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1. Introduction

Peleg-Yaari (1970) is one of the earliest papers that deal with infinite dimensional commodity spaces. The authors prove the existence of an equilibrium for an infinite horizon economy with a commodity space of $\mathbb{R}^\infty$ and with a finite number of agents. The striking feature of the paper is that the equilibrium price functional does not assign finite valuation to every commodity in the commodity space. Thus the equilibrium price functional is not an element of the topological dual of the commodity space. As pointed out by Aliprantis-Brown-Burkinshaw (1987), Peleg-Yarri (1970) is the first paper that is outside Debreu's analytical framework which is based on the duality of commodity and price spaces.

Aliprantis-Brown-Burkinshaw (1987) extend Peleg-Yaari (1970) to an economy whose commodity space is a vector lattice or Riesz space but their primary concerns are the existence of Edgeworth equilibria and the equivalence of approximate quasi-equilibrium, extended Walrasian equilibrium and Edgeworth equilibrium. Following Peleg-Yaari (1970), Besada-Estevez-Herves (1988) and Florenzano (1991) study a countably many periods exchange economy with a finite number of agents. One of their contributions is to revive the duality of commodity space and price space.[1] Especially Florenzano (1991) generalizes Besada-Estevez-Herves (1988) by assuming that the economy can have a different Riesz space as its commodity space in differing periods. As preferences are monotonic, transitive, and complete the author is able to apply Mas-Colell's (1986) existence theorem to obtain a quasi-equilibrium for the economy. In this paper, we extend Florenzano (1991) to an economy with non-ordered preferences, no free disposal, and a countable infinity of agents.

The negligibility of individual agents in general equilibrium theory is a way to model

[1] Aliprantis-Brown-Burkinshaw (1987) also consider the economic models where prices lie in the dual space of commodity space.

the idea of perfect competition.[2] Extending the cardinality of the set of agents to countable infinity[3] can be seen as a way to incorporate the negligibility of individual agents into economic models. Bewley (1972) provides the existence of a competitive equilibrium in an economy with a finite number of agents and a commodity space of L^{∞}. Wilson (1981) investigates the existence of a competitive equilibrium in a dynamic exchange economy with a countably many agents. His commodity space is $\mathbb{R}^{\infty}$. Following Bewley (1972), he proves the existence of a general equilibrium by limiting a sequence of competitive equilibria of subeconomies with a finite number of agents. As in Peleg-Yarri (1970), prices in Wilson (1981) do not belong to the dual space of the commodity space. Richard-Srivastava (1988) extend Bewley (1972) to the economy with a countable infinity of agents. Aliprantis-Brown-Burkinshaw (1989) also prove the existence of an equilibrium for an economy with countably many agents. The model studied by Aliprantis-Brown-Burkinshaw has the same structure as overlapping generation models. In some sense, our paper can be seen as an extension of Wilson (1981) to the economy where commodity space is a vector lattice and the duality of commodity space and price space is valid. In addition, this paper also can be regarded as an extension of Richard-Srivastava (1988) and Aliprantis-Brown-Burkinshaw (1989) to the case of non-ordered preferences.

It is well known that infinite dimensional general equilibrium theory suffers from some technical difficulties.[4] Empty-interiority in the better-than-set is one such difficulty. In the finite dimensional model, the price supportability of individual preferred sets is a result of the separating hyperplane theorem. The infinite dimensional version of the theorem requires not only convexity but also interior points in the preferred sets. Since

[2] For the relationship between perfect competition and the negligibility of individual agents, see the entry of perfect competition in Eatwell-Milgate-Newman (1989).

[3] The overlapping generations model is an example of a countably infinite number of agents.

[4] Three main difficulties are as follows: attainable sets may not be compact; preferred sets may not be supportable by prices; budget may not be jointly continuous as a function of prices and quantities. Readers are referred to Mas-Colell-Zame (1991) for more details.

Mas-Colell (1986) introduced the properness of preferences, several types of properness of preferences have been introduced into the literature to overcome the empty-interiority problem. Podczeck (1996) introduces E-proper preferences to prove the existence of a competitive equilibrium with no free-disposal and non-ordered preferences without the assumption that the principal ideal is dense in the entire space. Furthermore, he shows the same result in the case where the ideal is dense, using F-properness as first used by Yannelis-Zame (1986). We will appeal to these properness conditions to obtain the existence of competitive equilibria. However, unlike Podczeck (1996) we assume that the principal ideal is dense in both cases of E-properness and F-properness. These are due to the fact that we deal with a countable number of agents. The existence of the supremum of countably many price functionals requires stringent conditions which are not needed in economies with a finite numbers of agents.

The paper is organized as follows. Section 2 is devoted to mathematical definitions. In section 3, we present a model and some immediate results regarding the properties of the commodity and price spaces. Section 4 contains results on the existence of a competitive equilibrium. Section 5 concludes the paper and some of proofs are placed in the Appendix.

2. Definitions

A partially ordered vector space E is said to be a *Riesz space or vector lattice* if for any $x, y \in E$, the supremum $x \vee y$ and the minimum $x \wedge y$ of the set $\{x, y\}$ exist. For $x \in E$, we define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$. We can write $x = x^+ - x^-$. We denote by E^+ the positive cone of E. For $x, y \in E$, we say $x \geq y$ if $x - y \in E^+$, and $x > y$ if $x \geq y$ and $x \neq y$. For a net $\{x_\alpha\}$ in a Riesz space with $x_\alpha \geq x_\beta$ for any $\alpha \geq \beta$ and $\sup\{x_\alpha\} = x$, we say $x_\alpha \uparrow x$.

A subset A of E is called a *solid set* if $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of E is called an *ideal*. Let $x, y \in E$ satisfy $x \leq y$. Then the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an *order interval* of E. Subsets of order intervals of E are referred to as *order bounded* sets. A Riesz space E is said to be *Dedekind complete* if every nonempty subset that is order bounded from above has a supremum. For an element $u \geq 0$, there exists a smallest (with respect to inclusion) ideal of E that contains u. This ideal is called the ideal generated by u and is the set $A_u = \{x \in E : \exists\lambda > 0 \text{ with } |x| \leq \lambda u\}$. Any ideal of the form A_u is referred to as a *principal ideal*.

A net $\{x_\alpha\}$ in a Riesz space E is *order convergent* to $x \in E$, denoted by $x_\alpha \xrightarrow{o} x$, if there is a net $\{y_\alpha\}$ satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for each α. A linear functional $f : E \to \mathbb{R}$ is said to be *order bounded* if it maps order bounded subsets of E onto order bounded subsets of $\mathbb{R}$. The set of all order bounded linear functionals of E is called the *order dual* of E and denoted by $E^\sim$. For a functional f and an element x in the domain of f, we shall denote the value of $f(x)$ by $f \cdot x$. A linear functional $f \in E^\sim$ is said to be *order continuous* if $x_\alpha \downarrow 0$ in E implies $f \cdot x_\alpha \to 0$ in $\mathbb{R}$. The order continuous dual of E, the set of all order continuous linear functionals of E, is denoted by $E_n^\sim$. A linear functional $f : E \to \mathbb{R}$ is said to be *positive* whenever $f \cdot x \geq 0$ holds for each $x \in E^+$.

A *Riesz dual system* $< E, E' >$ is a dual system such that: (i)E is a Riesz space; (ii)E' is an ideal of the order dual of $E^\sim$; (iii)$< x, x^* >= x^* \cdot x$ holds for all $x \in E$ and all $x^* \in E'$. A Riesz dual system $< E, E' >$ is *symmetric* whenever E is an ideal of E'', the topological dual of E'. A seminorm ρ on a Riesz space E is said to be a *Riesz seminorm* if $|u| \leq |v|$ in E implies $\rho(u) \leq \rho(v)$. A complete normed Riesz space is called a *Banach lattice*.

3. The Model

We deal with a discrete time infinite horizon exchange economy. We first model the commodity and price spaces which are based on Florenzano (1991).

3.1. Commodity and Price Spaces

Time is indexed by $t = 1, 2, \cdots$. At each period t, we have a symmetric Riesz dual system $< E_t, E_t' >$.[5] For $x_t \in E_t$ and $p_t \in E_t'$, we denote the evaluation by $p_t \cdot x_t$. There are a countable infinity of agents indexed by $i \in \mathbb{N}$ (the set of natural numbers). Each of them has an endowment $\omega^i = (\omega_1^i, \omega_2^i, \ldots, \omega_t^i, \ldots) \in \Pi_{t=1}^{\infty} E_t^+, \omega^i \neq 0$. We assume that there exists $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_t, \cdots) \in \Pi_{t=1}^{\infty} E_t^+$ such that $\sum_{i=1}^{n} \omega^i \uparrow \bar{\omega}$ and $\bar{\omega} \neq 0$. We call $\bar{\omega}$ the aggregate endowment.

Following the spirit of Peleg-Yaari (1970), the price space is a collection of price functionals which provide the finite valuation to the aggregate endowment. Our price space is given by

$$H = \{p \in \Pi_{t=1}^{\infty} E_t' : \sum_{t=1}^{\infty} |p_t| \cdot \bar{\omega}_t < +\infty\}$$

where $\bar{\omega}_t$ is the t-th element of $\bar{\omega}$. The commodity space is

$$\Lambda(H) = \{x \in \Pi_{t=1}^{\infty} E_t : \sum_{t=1}^{\infty} |p_t| \cdot |x_t| < +\infty,\ \forall p \in H\}.$$

We say that for $x, y \in \Lambda(H)$, $x \geq y$ if $x - y \in \Pi_{t=1}^{\infty} E_t^+$ and also define an order on H in a similar way. It is obvious that $\Lambda(H)$ and H are vector lattices. Since $\omega^i \in \Pi_{t=1}^{\infty} E_t^+$, $\omega^i \in \Lambda(H)^+$. Therefore, aggregate endowment $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_t, \ldots)$ is in the $\Lambda(H)^+$.

We turn to topologies on commodity and price spaces. Topology τ on $\Lambda(H)$ is defined

[5] Symmetric Riesz dual systems include $< \mathbb{R}^n, \mathbb{R}^n >$; $< l_p, l_q >$, $(1 \leq p, q \leq \infty)$, $1/p + 1/q = 1$; $< L_p, L_q >$, $(1 \leq p, q \leq \infty)$, $1/p + 1/q = 1$; $< c_0, l_1 >$; $< \mathbb{R}^\infty, \mathbb{R}^{'\infty} >$; $< ca(\Omega), ca'(\Omega) >$, where $c_0 = \{x \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}$ and $ca(\Omega)$ is the collection of all signed measures of bounded variation.

by a Riesz seminorm

$$\rho_p(x) = \sum_{t=1}^{\infty} |p_t| \cdot |x_t|, \text{where } \ p \in H \text{ and } x \in \Lambda(H)$$

and τ' on H is defined by a Riesz seminorm

$$\rho_x(p) = \sum_{t=1}^{\infty} |p_t| \cdot |x_t|, \text{where } \ x \in \Lambda(H) \text{ and } p \in H.$$

Now we have two topological spaces $(\Lambda(H), \tau)$ and (H, τ'). For $x \in \Lambda(H), p \in H$, the bilinear map is defined by $p \cdot x = \sum_{t=1}^{\infty} p_t \cdot x_t$.

The following two propositions are due to Florenzano (1991). Let $\sigma(\Lambda(H), H)$ be the weak topology on $\Lambda(H)$ and $\sigma(H, \Lambda(H))$ is similarly defined.

Proposition 1 *Every order interval of $\Lambda(H)$ is $\sigma(\Lambda(H), H)$-compact and every order interval of H is $\sigma(H, \Lambda(H))$-compact.*

Proof. See Proposition A.2. in Florenzano (1991). ■

Proposition 2 *$< \Lambda(H), H >$ is a symmetric Riesz dual system. The topologies τ and τ' are Hausdorff locally convex-solid and consistent with the duality. Moreover, $\Lambda(H)$ is Dedekind complete.*

Proof. See Proposition A.1. in Florenzano (1991) and Theorem 2.1 in Aliprantis-Brown-Burkinshaw (1987). ■

We need a principal ideal generated by the aggregate endowment $\bar{\omega}$ to study a reduced economy in the next section. Let $A_{\bar{\omega}}$ be the principal ideal of $\Lambda(H)$ generated by $\bar{\omega}$;

$$A_{\bar{\omega}} = \{x \in \Lambda(H) : \exists \lambda > 0 \text{ such that } |x| \leq \lambda \bar{\omega}\}.$$

On $A_{\bar{\omega}}$, a lattice norm is defined by

$$\|x\|_\infty = \inf\{\lambda > 0 : |x| \leq \lambda\bar{\omega}\}, \text{ where } x \in A_{\bar{\omega}}.$$

$A_{\bar{\omega}}$ with the topology τ_∞ generated by the $\|\cdot\|_\infty$-norm is locally convex-solid Riesz space.[6] Let $A'_{\bar{\omega}}$ be a τ_∞-topological dual of $A_{\bar{\omega}}$.

3.2. Economy

An exchange economy $\mathcal{E}$ is a triple of $(X_i, P_i, \omega^i)_{i\in\mathbb{N}}$ where X_i is a consumption set, and $P_i : X_i \to 2^{X_i}$ is a preference map. We assume $X_i = \Lambda(H)^+,\ \forall i \in \mathbb{N}$. It is clear that $\Lambda(H)^+$ is convex. An *allocation* is $x = (x^1, x^2, \ldots, x^i, \ldots)$ with $x^i \in X_i$ for all $i \in \mathbb{N}$. An allocation x is *feasible* if $\sum_{i=1}^n x^i \uparrow \bar{\omega}$.

We make use of the following assumptions; for all $i \in \mathbb{N}$,

A.1 $x^i \notin P_i(x^i)$ for any $x^i \in X_i$,

A.2 $P_i(x^i)$ is convex for all $x^i \in X_i$,

A.3 P_i has an open graph in $X_i \times X_i$ in the product topology $\sigma(\Lambda(H), H) \times \tau$,

A.4 $P_i(z) \cap A_{\bar{\omega}} \neq \emptyset$ for any $z \in [0, \bar{\omega}]$,

A.5 if $x, y \in \Lambda(H)^+$ and $y > x$, then $y \in P_i(x)$,

A.6 $E'_t = (E_t)^\sim_n$ for all t.

A.7 $\bar{\omega}_t$ is a strictly positive element of E_t for all t.

A.1 shows that preferences are irreflexive. By A.2 the convexity of preferences is assumed. A.3 is about the continuity of preferences. A.4 implies that nonsatiation holds on the feasible sets in $A_{\bar{\omega}}$.[7] A.5 represents strict monotonicity. A.6 implies that the

[6] See Aliprantis-Brown-Burkinshaw (1987) p.1125.

[7] A5 and A.7 implies A.4 but A.4 is used later without A.7. Thus we include A.4 as an assumption.

topological dual of E_t and the order continuous dual of E_t are the same. When E_t is a Banach lattice,[8] A.6 is satisfied.

An *equilibrium* for an exchange economy $\mathcal{E}$ is a pair of (x, p) where x is a feasible allocation and $p \in H, p \neq 0$ such that for all $i \in \mathbb{N}$,

(i) $p \cdot x^i = p \cdot \omega^i$,

(ii) $y \in P_i(x^i)$ implies $p \cdot y > p \cdot \omega^i$.

A *quasi-equilibrium* is a pair of (x, p) where x is feasible and $p \in H, p \neq 0$ such that (i) above and the following hold: (ii') if $y \in P_i(x^i)$, then $p \cdot y \geq p \cdot \omega^i$. A quasi-equilibrium (x, p) is said to be *non-trivial* if there is $i \in \mathbb{N}$ such that $\inf\{p \cdot x : x \in \Lambda(H)^+\} < p \cdot \omega^i$.

Since, as pointed out in Proposition 2, $\Lambda(H)$ is Dedekind complete, so is $A_{\bar{\omega}}$.[9] Then $(A_{\bar{\omega}}, \|\cdot\|_\infty)$ is a Banach lattice. Thus $A'_{\bar{\omega}}$ is equal to $A^{\sim}_{\bar{\omega}}$, the order dual of $A_{\bar{\omega}}$. For each $x \in A_{\bar{\omega}}$ we have $|x| \leq \|x\|_\infty \cdot \bar{\omega}$. Then for all $x \in A_{\bar{\omega}}$, $\rho_p(x) \leq \rho_p(\bar{\omega})\|x\|_\infty$ holds. Thus the Riesz seminorm $\rho_p(\cdot)$ on $\Lambda(H)$ is τ_∞-continuous on $A_{\bar{\omega}}$. Hence, $H \subset A'_{\bar{\omega}}$, i.e., the restrictions of the functionals of H to $A_{\bar{\omega}}$ belongs to $A'_{\bar{\omega}}$. This fact also indicates that $p \in A'_{\bar{\omega}}$ is not always continuous with respect to the topology τ. Clearly, the positive cone $A^+_{\bar{\omega}}$ is τ_∞-closed in $A_{\bar{\omega}}$ and $\bar{\omega}$ is a τ_∞-interior point of $A^+_{\bar{\omega}}$.

4. Results

We shall show the existence of an equilibrium for $\mathcal{E}$. Infinite dimensional economic models have a well-known empty interior point problem. To avoid this problem, we first consider

[8] A Banach lattice is a complete normed Riesz space. It includes the Euclidean spaces $\mathbb{R}^n$ with their Euclidean norms; $L_p(\mu)$, $(1 \leq p < \infty)$ with their L_p-norms; $C(K)$ the Riesz space of all continuous real functions on a compact space K under the sup norm; $C_b(X)$ the Riesz space of all bounded real continuous functions on a topological function X; $B(X)$ the Riesz space of all bounded real functions on an arbitrary nonempty set X under the lattice norm $\|f\|_\infty = \sup\{|f \cdot x| : x \in X\}$.

[9] For the technical parts in this paragraph, see Aliprantis-Brown-Burkinshaw (1987) p.1116 and p.1125.

a principal ideal by defining a reduced economy. In the reduced economy, the commodity space is restricted to $A_{\bar{\omega}}$. Even though $A_{\bar{\omega}}$ is infinite dimensional, its positive cone $A^+_{\bar{\omega}}$ has non-empty τ_∞-interior points. We start with the case where there are finite number of agents. Let $\|\cdot\|'_\infty$ be the dual norm of $A'_{\bar{\omega}}$.[10]

Proposition 3 *Suppose A.1-A.5 and there are* n *agents. Then there exist an allocation* $\bar{x}_n \in (A^+_{\bar{\omega}})^n$ *with* $\sum_{i=1}^n \bar{x}^i_n = \sum_{i=1}^n \omega^i$*, and a non-zero positive linear functional* $\bar{p}_n \in A'_{\bar{\omega}}$ *with* $\|\bar{p}_n\|'_\infty = 1$*, inf* $\{\bar{p}_n \cdot z : z \in A^+_{\bar{\omega}}\} < \bar{p}_n \cdot \omega^i$ *for some* i*, and such that* $y \in P_i(\bar{x}^i_n) \cap A_{\bar{\omega}}$ *implies* $\bar{p}_n \cdot y \geq \bar{p}_n \cdot \omega^i$ *for all* $i \in \{1, 2, \ldots, n\}$*.*

Proof. With A.1 - A.4, we appeal to Lemma 1 in Podczeck (1996) to assert that there exist $\bar{x}_n \in (A^+_{\bar{\omega}})^n$ and a non-zero linear functional $\bar{p}_n \in A'_{\bar{\omega}}$ such that $\|\bar{p}_n\|'_\infty = 1$, inf $\{\bar{p}_n \cdot z : z \in A^+_{\bar{\omega}}\} < \bar{p}_n \cdot \omega^i$ for some i, and $y \in P_i(\bar{x}^i_n) \cap A_{\bar{\omega}}$ implies $\bar{p}_n \cdot y \geq \bar{p}_n \cdot \omega^i$ for all $i \in \{1, 2, \ldots, n\}$. By A.5, we can conclude that $\bar{p}_n$ is positive. ■

We now extend the result to the case with a countable infinity of agents.

Proposition 4 *Suppose A.1-A.5 and there are a countably infinite number of agents. Then there exists* $(\bar{x}, \bar{p})$ *such that* $\bar{x} \in (A_{\bar{\omega}})^\infty$ *is a feasible allocation, and* $\bar{p}(\neq 0) \in A'_{\bar{\omega}}$ *is a positive functional with inf* $\{\bar{p} \cdot x : x \in A^+_{\bar{\omega}}\} < \bar{p} \cdot \omega^i$ *for some* i*, and for* $y \in P_i(\bar{x}^i) \cap A_{\bar{\omega}}$*,* $\bar{p} \cdot y \geq \bar{p} \cdot \omega^i \; \forall i \in \mathbb{N}$*.*

Proof. See Appendix ■

We are in a position to consider an equilibrium for the entire economy. Using $(\bar{x}, \bar{p})$ of Proposition 4, we are going to derive an equilibrium for $\mathcal{E}$. As discussed in subsection 3.2, $\bar{p}$ is not necessarily τ-continuous. To derive a τ-continuous price functional, we appeal to properness of preferences. We will employ F-properness and E-properness. Definitions of each type of properness are given below. Let X be a consumption set and $P : X \to 2^X$ a preference map.

[10] To be specific, $\|\cdot\|'_\infty$ is defined as follows: for $p \in A'_{\bar{\omega}}$ and $z \in A_{\bar{\omega}}$, $\|p\|'_\infty = \sup_{\|z\|_\infty \leq 1} |p \cdot z|$.

Definition 1 *We say that P is F-proper at $x \in X$ if there exists a vector $v \in \Lambda(H)$ and τ-neighborhood U of zero such that*

1. $x + v \in X$;
2. *if $u \in U$, then $x + \alpha v - \alpha u \in X$ implies that $x + \alpha v - \alpha u \in P(x)$ for every real number $\alpha > 0$ which is sufficiently small.*

The economic meaning of F-properness is as follows. An agent whose consumption bundle x gives up α portion of any sufficiently small u for an additional increment of v measured by α, and the resulting bundle belongs to the better-than set. In this sense v is *extremely desirable* as pointed out by Yannelis-Zame (1986).

Definition 2 *Let $x \in X$ and K be a linear subspace of $\Lambda(H)$ with $x \in K$. P is E-proper at x relative to K if there are some $v \in \Lambda(H)$, some τ-neighborhood of U of zero, and some $A \subset K$ which is radial*[11] *at x (in K) such that*

1. *$x + \alpha v \in P(x)$ for every sufficiently small real number $\alpha > 0$;*
2. *if $\tilde{x} \in A \cap X$ and $\tilde{x} \notin P(x)$, then $u \in U$ implies $\tilde{x} - \alpha v + \alpha u \notin P(x)$ for every real number $\alpha > 0$.*

The meaning of E-properness at x relative to K is as follows. The commodity bundle v is desirable in the sense that adding any sufficiently small amounts of this bundle results in a bundle in the better-than set of x. The set A, radial at x, reflects the idea of a set of sufficiently close points of x. Now consider an agent who starts at a consumption bundle $\tilde{x}$ in K which is not in the better-than set of x but sufficiently close to x. If we take αv out of $\tilde{x}$ and substitute some amount αu of any other sufficiently small commodity bundle u, then the results of the substitution cannot lie in the better-than set of x.[12] In the above two definitions, v is called a properness vector.

[11] Suppose K is a vector space and $A \subset K$. Then A is called *radial* at a point y if for each $z \in K$, there is a real number $\bar{\lambda}, 0 < \bar{\lambda} \leq 1$, such that $(1-\lambda)y + \lambda z \in A$ for every λ with $0 \leq \lambda \leq \bar{\lambda}$.

[12] For detailed discussions of properness, readers are referred to Podczeck (1996).

Theorem 1 *Let $\mathcal{E}$ be an exchange economy and satisfy A.1-A.3 and A.5-A.7, and let $x \in \Pi_{i\in\mathbb{N}}(X_i)$ be a feasible allocation. Suppose either*

- *$P_i(x^i)$ is F-proper at x^i with a properness vector $\bar{\omega}$ for all $i \in \mathbb{N}$,*

  or,

- *P_i is E-proper at x^i relative to $A_{\bar{\omega}}$ with a properness vector v_i satisfying $x^i + v_i \in A^+_{\bar{\omega}}, \forall i \in \mathbb{N}$.*

Then there exist a feasible allocation $\bar{x} \in \Pi_{i\in\mathbb{N}}(X_i)$ and a price functional $\bar{\pi} \in H$ such that $(\bar{x}, \bar{\pi})$ is a non-trivial quasi-equilibrium of $\mathcal{E}$.

Proof. See Appendix ■

We shall say that the economy $\mathcal{E}$ is *irreducible* if x is any feasible allocation and if N_1 and N_2 is a non-trivial partition of $\mathbb{N}$, then there exists an allocation $\tilde{x}$ such that $\tilde{x}^i \in P_i(x^i)$ for all $i \in N_1$, with $\sum_{N_1} \tilde{x}^i = \sum_{N_1} x^i + \sum_{N_2}(\omega^i - \tilde{x}^i)$. Then with the irreducibility assumption a quasi-equilibrium is a competitive equilibrium by a standard argument (cf. McKenzie (1981)).

Corollary 1 *Suppose that $\mathcal{E}$ satisfies A.1-A.7, and is irreducible. Then there exists an equilibrium $(\bar{x}, \bar{\pi}) \in \Pi_{i\in\mathbb{N}}(X_i) \times H, (\bar{\pi} \neq 0)$.*

5. Conclusion

In this paper, we consider an infinite-horizon exchange economy with a countable number of agents and prove the existence of a competitive equilibrium for the economy. This result provides a theoretical foundation for the intertemporal resource allocation problem.

References

Aliprantis,C.D., D.J. Brown and O. Burkinshaw, 1987, Edgeworth equilibria, Econometrica 55, 1109-1137.

Aliprantis,C.D., D.J. Brown and O. Burkinshaw, 1989, Equilibrium in exchange economies with a countable number of agents, Journal of Mathematical Analysis and Applications 142, 250-299.

Aliprantis,C.D. and O. Burkinshaw, 1985, Positive Operators, Academic Press, NY.

Aliprantis,C.D. and K.C. Border, 1999, Infinite Dimensional Analysis the second edition, Springer, Berlin.

Besada, M., M. Estevez and C. Herves, 1988, Equilibrium in economies with contably many commodities, Economics Letters 26, 203-207.

Bewley, T., 1972, Existence of equilibrium in economies with infinitely many commodities, Journal of Economic Theory 4, 514-540.

Deghdak, M. and M. Florenzano, 1999, Decentralizing Edgeworth equilibrium in economies with many commodities, Economic Theory 14, 297-310.

Eatwell, J., M. Milgate and P. Newman, 1989, The New Palgrave, The Macmillan Press, London.

Florenzano, M., 1991, Markets with countably many periods, Economics Letters 36, 233-238.

McKenzie, L.W., 1981, The classical theorem on existence of competitive equilibrium, Econometrica 49, 819-841.

Mas-Colell, A., 1986, The Price equilibrium in topological vector lattices, Econometrica 54, 1039-1053.

Mas-Colell, A. and W.R. Zame, 1991, Equilibrium theory in infinite dimensional spaces, in Handbook of mathematical economics IV, edited by Hildenbrand and Sonnenschein, 1836-1898.

Peleg, B., and M. Yaari, 1970, Markets with countably many commodities, International Economic Review 11, 369-377.

Podczeck, K., 1996, Equilibria in vector lattice without ordered preferences or uniform properness, Journal of Mathematical Economics 25, 465-485.

Richard, S. and S. Srivastava, 1988, Equilibrium in economies with infinitely many consumers and infinitely many commodities, Journal of Mathematical Economics 17, 9-21.

Wilson, C.A., 1991, Equilibrium in dynamic models with an infinity of agents, Journal of Economic Theory 24, 95-111.

Yannelis, N.C., and W.R. Zame, 1986, Equilibria in Banach lattice without ordered preferences, Journal of Mathematical Economics 15, 85-110.

Appendix: proofs

Proof of Proposition 4. Let $(\bar{x}_n, \bar{p}_n) \in ((A_{\bar{\omega}})^n \times A'_{\bar{\omega}})$ be a pair of an allocation and a price functional that satisfies Proposition 3. Let $\Delta = \{p \in A'_{\bar{\omega}} : \|p\|'_\infty \leq 1\}$. Since $\|\bar{p}_n\|'_\infty = 1$, $\bar{p}_n \in \Delta$ for all n. By Alaoglu's Theorem, Δ is $\sigma(A'_{\bar{\omega}}, A_{\bar{\omega}})$-compact. Observe $\bar{x}^i_n \in [0, \bar{\omega}]$ which is, by Proposition 1, $\sigma(\Lambda(H), H)$-compact. Let $Z = \Delta \times [0, \bar{\omega}]^{\mathbb{N}}$ (with the product topology). For each n, let $z_n = (\bar{p}_n, \bar{x}^1_n, \ldots, \bar{x}^n_n, 0, \ldots, 0, \ldots) \in Z$. By Tychonoff's compactness theorem, Z is compact. Hence, by passing to subnets if necessary, z_n converges to $\bar{z} = (\bar{p}, \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^i, \ldots) \in Z$. It is clear that $\bar{p}_n$ converges to $\bar{p}$ in the $\sigma(A'_{\bar{\omega}}, A_{\bar{\omega}})$ topology and each $\bar{x}^i_n$ converges to $\bar{x}^i$ in the $\sigma(\Lambda(H), H)$ topology.

Let $s^m_n = \sum_{i=1}^m \bar{x}^i_n$. Recall that for each n, $\sum_{i=1}^n \bar{x}^i_n = \sum_{i=1}^n \omega^i \leq \bar{\omega}$. For $m \geq n$, we can write $s^m_n = \bar{x}^1_n + \cdots + \bar{x}^n_n + 0 + \cdots + 0$ where 0 is the zero element of $\Lambda(H)$ and the number of 0's in the summation is $(m - n)$. Then for any $m \geq n$, $s^m_n = \sum_{i=1}^n \omega^i \leq \bar{\omega}$ and since $\sum_{i=1}^n \omega^i \uparrow \bar{\omega}$, we have $s^m_n \uparrow \bar{\omega}$ as $n \to \infty$ and $m \to \infty$. Let $s_n = \lim_{m\to\infty} s^m_n$. Then it is clear that $s_n = \sum_{i=1}^n \omega^i$. It also follows that $\lim_{n\to\infty} s_n \uparrow \bar{\omega}$. Let $s^m = \lim_{n\to\infty} s^m_n$. Then it is clear that $s^m = \sum_{i=1}^m \bar{x}^i$ and that s^m_n converges to s^m in the $\sigma(\Lambda(H), H)$ topology. Since $\bar{x}^i_n$ converges to x^i in the $\sigma(\Lambda(H), H)$ topology for each i and $s^m_n \leq \bar{\omega}$ for every n, we get $s^m \leq \bar{\omega}$ for each m. Thus $\lim_{m\to\infty} s^m$ converges. Notice that $\sigma(\Lambda(H), H)$ topology is order continuous.[13] Thus $s^m_n \uparrow \bar{\omega}$ implies that s^m_n converges to $\bar{\omega}$ in the $\sigma(\Lambda(H), H)$ topology. Hence, for a real number $\varepsilon > 0$ and for any $f \in H$, there is N such that for all $n, m > N$, $|f \cdot s^m_n - f \cdot \bar{\omega}| < \varepsilon/2$ holds. In addition, the weak convergence of s^m_n to s^m implies $|f \cdot s^m - f \cdot s^m_n| < \varepsilon/2$ for sufficiently large n. Hence, $|f \cdot s^m - f \cdot \bar{\omega}| \leq |f \cdot s^m - f \cdot s^m_n| + |f \cdot s^m_n - f \cdot \bar{\omega}| < \varepsilon$ holds. This shows that $f \cdot s^m$ converges to $f \cdot \bar{\omega}$, i.e., s^m converges to $\bar{\omega}$ in the $\sigma(\Lambda(H), H)$ topology. Notice that for

[13] From Proposition 2, $< \Lambda(H), H >$ is symmetric. Then we appeal to Theorem 7.58 in Aliprantis-Border (1999) to assert that $H \subset \Lambda(H)^\sim_n$. Then by Theorem 11.10 in Aliprantis-Burkinshaw (1985), $\sigma(\Lambda(H), H)$ is order continuous.

all i, $\bar{x}^i \geq 0$ which implies that s^m is non-negative and increasing as m increases. Now we should show that $\bar{\omega}$ is the supremum of the net $\{s^m\}$. Let $\hat{x} = \sup_m\{s^m\}$ and let $y^m = \hat{x} - s^m$. Clearly, $y^m \downarrow 0$. Since $\sigma(\Lambda(H), H))$ is order continuous, we have $y^m \to 0$ in the $\sigma(\Lambda(H), H)$ topology. Recall that, from Proposition 2, $(\Lambda(H), \tau)$ is a locally convex-solid Reisz space. Then $y^m \to 0$ in the $\sigma(\Lambda(H), H)$ topology is equivalent to $y^m \to 0$ in the τ topology.[14] Hence, for any $p \in H$, $\rho_p(\hat{x} - s^m)$ goes to zero. From $|f \cdot s^m - f \cdot \hat{x}| \leq |f| \cdot \rho_p(\hat{x} - s^m)$ for any $f \in H$, we can say that $f \cdot s^m$ converges to $f \cdot \hat{x}$. But in the above, we already showed $f \cdot s^m$ converges to $f \cdot \bar{\omega}$. Thus, $f \cdot \hat{x} = f \cdot \bar{\omega}$ holds. Since f is an arbitrary element of H, we can conclude $\hat{x} = \bar{\omega}$ which means that $\bar{\omega}$ is the supremum of the net $\{s^m\}$. Hence, $\sum_{i=1}^m \bar{x}^i \uparrow \bar{\omega}$, i.e., $\bar{x}$ is a feasible allocation.

We shall show that $(\bar{x}, \bar{p})$ satisfies the desired results. First we will prove $y \in P_i(\bar{x}^i) \cap A_{\bar{\omega}}$ implies that $\bar{p} \cdot y \geq \bar{p} \cdot \omega^i$ for any i and $\bar{p}$ is a non-zero positive functional. Since $\bar{x}_n^i$ converges to $\bar{x}^i$ in the $\sigma(\Lambda(H), H)$ topology and P_i has an open graph in the product topology of $\sigma(\Lambda(H), H) \times \tau$, we can assume that for large enough n, $y \in P_i(\bar{x}_n^i) \cap A_{\bar{\omega}}$ for $i \leq n$. Fix i and then by Proposition 3 we have $\bar{p}_n \cdot y \geq \bar{p}_n \cdot \omega^i$. On taking limits on both sides, we have $\bar{p} \cdot y \geq \bar{p} \cdot \omega^i$.

By A.5, we get $\bar{p} \geq 0$. We need to show $\bar{p} \neq 0$. Note that $A_{\bar{\omega}}$ is an AM-space with order unit $\bar{\omega}$.[15] Then by Lemma 8.26 in Aliprantis-Border (1999), $\|\bar{p}_n\|'_\infty = |\bar{p}_n| \cdot \bar{\omega}$ holds. Thus for each n, we have $\bar{p}_n \cdot \bar{\omega} = 1$ because $\bar{p}_n$ is positive. Since $\bar{p}_n$ weak*-converges to $\bar{p}$ and $\bar{\omega}$ is a τ_∞-interior point of $A_{\bar{\omega}}^+$, taking limits leads to $\bar{p} \cdot \bar{\omega} = 1$. Hence, we infer $\bar{p} \neq 0$ and conclude that $\bar{p}$ is non-zero and positive.

We now prove $\inf\{\bar{p} \cdot x : x \in A_{\bar{\omega}}^+\} < \bar{p} \cdot \omega^i$ for some i. Suppose not, i.e., $\inf\{\bar{p} \cdot x : x \in A_{\bar{\omega}}^+\} \geq \bar{p} \cdot \omega^i$ for every $i \in \mathbb{N}$. Since $\frac{1}{n}\bar{\omega}$ is an interior point of $A_{\bar{\omega}}^+$, there exists $\varepsilon > 0$ such that $\bar{p} \cdot (\frac{\bar{\omega}}{n} + \varepsilon[-\bar{\omega}, \bar{\omega}]) \geq \bar{p} \cdot \omega^i$ holds for every i. By summing over i to n, we obtain

[14] See Theorem 11.8 in Aliprantis-Burkinshaw (1985).

[15] See Aliprantis-Brown-Burkinshaw (1987) p.1118. For the definition of an AM-space, readers are referred to Aliprantis-Border (1999) p.310.

$\bar{p} \cdot (\bar{\omega} - \sum_{i=1}^{n} \omega^i) + \bar{p} \cdot n\varepsilon[-\bar{\omega}, \bar{\omega}] \geq 0$. As n increases, the first term of the left hand decreases and the second term leads to $\bar{p} = 0$. This contradicts to $\bar{p} \neq 0$. ∎

The following results are useful in our proof of Theorem 1.

Lemma 1 *Under A.7, $A_{\bar{\omega}}$ is τ-dense in $\Lambda(H)$.*

Proof. See Proposition A.3 in Florenzano (1991). ∎

Lemma 2 *Let (L, τ) be an ordered topological vector space and M be a vector subspace of L. Let Z be an τ-open and convex subset of L such that $Z \cap M^+ \neq \emptyset$. Let $x_0 \in clZ \cap M^+$ (clZ denotes the τ-closure of Z), f be a linear functional on M. Suppose $f \cdot x_0 \leq f \cdot z$, $\forall z \in Z \cap M^+$, and for any $m \in M^+$ such that $f \cdot m \leq f \cdot x_0$, $(m - M^+) \cap Z = \emptyset$. Then there exists a τ-continuous linear functional ϕ on L such that $f \cdot x_0 = \phi \cdot x_0 \leq \phi \cdot z$, $\forall z \in Z$ and $0 \leq \phi|_M \cdot m \leq f \cdot m$ for all $m \in M^+$ where $\phi|_M$ is the restriction of ϕ to M.*

Proof. Let $S = \{(y, r) | y = z - m, r > f \cdot m, z \in Z, m \in M^+\}$ and $W = \{(y', r') | y' \in -M^+, r' < f \cdot x_0\}$. From the assumptions of the lemma, it follows that S is an open and convex subset and W is a convex subset of $L \times \mathbb{R}$.

Suppose $(\bar{y}, \bar{r}) \in S \cap W$. Then there exist $\bar{z} \in Z$ and $\bar{m} \in M^+$ such that $\bar{y} = \bar{z} - \bar{m} \in -M^+$. It also follows that $f \cdot \bar{m} < \bar{r} < f \cdot x_0$. But $\bar{z} - \bar{m} \in -M^+$ implies $\bar{z} \in (\bar{m} - M^+)$, a contradiction. Hence, $S \cap W = \emptyset$.

We denote by L' the topological dual of L. By the classical separation theorem, there exists $(\phi, t) \neq 0 \in L' \times \mathbb{R}$ such that for all $(y', r') \in W$ and for all $(y, r) \in S$,

$$(\phi, t) \cdot (y', r') < (\phi, t) \cdot (y, r).$$

With $y' = 0$ and $y = 0$, we have $tr' < tr$ which implies $t > 0$. Without loss of generality, we can assume $t = 1$. Then we have

$$(\phi, 1) \cdot (y', r') < (\phi, 1) \cdot (y, r)$$

and thus,

$$(\phi, 1) \cdot (y', f \cdot x_0) \leq (\phi, 1) \cdot (z - m, f \cdot m) \tag{1}$$

$\forall z \in Z, \forall m \in M^+, \forall y' \in -M^+$.

By plugging $y' = 0$ and $m = 0$ into (1), we get

$$f \cdot x_0 \leq \phi \cdot z, \ \ \forall z \in Z. \tag{2}$$

Since $x_0 \in clZ$, there exists a sequence $\{z_n\}$ in Z which converges to x_0. By putting $y' = 0$ and $z = z_n$ into (1), we have

$$f \cdot x_0 - \phi \cdot z_n \leq (f - \phi) \cdot m$$

for all $m \in M^+$ and for each n. Then by taking limits, we obtain

$$(f - \phi) \cdot x_0 \leq (f - \phi) \cdot m, \ \ \forall m \in M^+. \tag{3}$$

When $m = 2x_0$, we get $(f - \phi) \cdot x_0 \geq 0$ from (3). If $m = 0$, we have $(f - \phi) \cdot x_0 \leq 0$. Therefore, $(f - \phi) \cdot x_0 = 0$, i.e., $f \cdot x_0 = \phi \cdot x_0$. Hence, it follows from (2),

$$f \cdot x_0 = \phi \cdot x_0 \leq \phi \cdot z, \ \ \forall z \in Z.$$

From (3), we also obtain $0 \leq (f - \phi|_M) \cdot m$, i.e.,

$$\phi|_M \cdot m \leq f \cdot m, \ \ \forall m \in M^+.$$

With $m = x_0$, (1) becomes $\phi \cdot y' \leq \phi \cdot (z - x_0)$ for all $z \in Z$ and for all $y' \in -M^+$. We can again pick a sequence $\{z_n\}$ from Z converging to x_0. For the sequence $\{z_n\}$,

$\phi \cdot y' \leq \phi \cdot (z_n - x_0)$ holds for all n. By taking limits, we obtain

$$\phi \cdot y' \leq 0, \ \ \forall y' \in -M^+.$$

Hence, we conclude $0 \leq \phi|_M \cdot m$ for all $m \in M^+$, and thus, $0 \leq \phi|_M \cdot m \leq f \cdot m$ for all $m \in M^+$. ■

Lemma 3 *Let (L, τ) be an ordered topological vector space and M be a vector subspace of L. Let Z be an τ-open and convex subset of L such that $Z \cap M^+ \neq \emptyset$. Let $x_0 \in clZ \cap M^+$ (clZ denotes the τ-closure of Z), and f be a linear functional on M. Suppose $f \cdot x_0 \leq f \cdot z$, $\forall z \in Z \cap M^+$. Then there exists a non-zero τ-continuous linear functional ϕ on L such that $\phi|_M \cdot x \leq f \cdot x$ for all $x \in M^+$ and*

$$f \cdot x_0 = \phi \cdot x_0 \leq \phi \cdot z, \ \forall z \in Z.$$

Proof. See Lemma 6.1 in Deghdak-Florenzano (1999). ■

The following result and its proof are from Theorem 2.2 in Aliprantis-Burkinshaw (1985). In the lemma, $\mathcal{L}(E, F)$ denotes the vector space of all operators from E into F.

Lemma 4 *Let $T : E \to F$ be a positive linear operator between two Riesz spaces with F Dedekind complete. Assume also that G is a Riesz subspace of E and that $S : G \to F$ is a linear operator satisfying $0 \leq S \cdot x \leq T \cdot x$ for all $x \in G^+$. Then S can be extended to a positive linear operator (again denoted by S) from E into F such that for all $x \in E^+$, $0 \leq S \cdot x \leq T \cdot x$ and for all $x \in E$, $S \cdot x \leq T \cdot x^+$ hold in $\mathcal{L}(E, F)$.*

Proof. Let $\rho : E \to F$ be defined by $\rho \cdot x = T \cdot x^+$, and note that ρ is sublinear and that $S \cdot x \leq \rho \cdot x$ holds for all $x \in G$. By the Hahn-Banach Theorem, there exists an extension of S to all of E (which we denote by S again) satisfying $S \cdot x \leq \rho \cdot x$ for all

$x \in E$. Hence if $x \in E^+$, then

$$-S \cdot x = S \cdot (-x) \leq \rho \cdot (-x) = T \cdot (-x)^+ = T \cdot 0 = 0.$$

and so $0 \leq S \cdot x \leq \rho \cdot x = T \cdot x$ holds. ■

Proof of Theorem 1. First of all, A.5 and A.7 implies A.4. Hence, we can apply the results based on A.4 in the following proof.

Step 1: By Proposition 4, there exists $(\bar{x}, \bar{p}) \in ((A_{\bar{\omega}})^{\infty} \times A'_{\bar{\omega}})$ such that $y \in P_i(\bar{x}^i) \cap A_{\bar{\omega}}$ implies $\bar{p} \cdot y \geq \bar{p} \cdot \omega^i \ \ \forall i \in \mathbb{N}$, $\bar{p} \geq 0$ $(\bar{p} \neq 0)$ and $\inf \{\bar{p} \cdot x : x \in A^+_{\bar{\omega}}\} < \bar{p} \cdot \omega^i$ for some i.

Since $\Lambda(H)$ is Dedekind complete, $A_{\bar{\omega}}$ is also Dedekind complete and thus $A_{\bar{\omega}}$ is a Banach lattice and then, its norm dual coincides with its order dual, $A'_{\bar{\omega}} = A^{\sim}_{\bar{\omega}}$[16] and $A^{\sim}_{\bar{\omega}}$ can be written as $A^{\sim}_{\bar{\omega}} = (A_{\bar{\omega}})^{\sim}_n \oplus ((A_{\bar{\omega}})^{\sim}_n)^d$ where $((A_{\bar{\omega}})^{\sim}_n)^d$ is the disjoint complement of $(A_{\bar{\omega}})^{\sim}_n$.[17] Then there is a unique decomposition $\bar{p} = q + s$ where $q(\geq 0) \in (A_{\bar{\omega}})^{\sim}_n$ is the order continuous component and $s(\geq 0) \in ((A_{\bar{\omega}})^{\sim}_n)^d$ is the singular component. Let $N_s = \{x \in A_{\bar{\omega}} : s \cdot |x| = 0\}$. Then N_s is the null ideal of s and Theorem 2.2 in Aliprantis-Brown-Burkinshaw (1989) shows that N_s is order dense. Hence for $y \in P_i(\bar{x}^i) \cap A^+_{\bar{\omega}}$ there exists a net $\{y^{\alpha}\}$ in N_s such that $y^{\alpha} \xrightarrow{o} y$. Since $< \Lambda(H), H >$ is a symmetric Riesz dual system, τ is an order continuous topology.[18] Thus for large enough α, $y^{\alpha} \in P_i(\bar{x}^i) \cap A^+_{\bar{\omega}}$ and $q \cdot y^{\alpha} = \bar{p} \cdot y^{\alpha} \geq (q + s) \cdot \omega^i \geq q \cdot \omega^i$. From the order continuity of q, taking limits leads to $q \cdot y \geq q \cdot \omega^i$ for all i.

We are going to prove $q \cdot \bar{x}^i = q \cdot \omega^i, \forall i$. Since $\bar{\omega}$ is a τ_{∞}-interior point of $A^+_{\bar{\omega}}$, A.5 implies $\bar{x}^i + \alpha\bar{\omega} \in P_i(\bar{x}^i) \cap A^+_{\bar{\omega}}$ for all $\alpha > 0$. Thus, $q \cdot (\bar{x}^i + \alpha\bar{\omega}) \geq q \cdot \omega^i$.[19] Note that from $A'_{\bar{\omega}} = A^{\sim}_{\bar{\omega}}$, q is τ_{∞}-continuous. By the continuity of q, we have $q \cdot \bar{x}^i \geq q \cdot \omega^i$ for all i. By

[16] See Aliprantis-Brown-Burkinshaw (1987) p.1125.

[17] See Aliprantis-Border (1999) p.282.

[18] As mentioned in footnote 13, from the symmetry of $< \Lambda(H), H >$, $H \subset \Lambda(H)^{\sim}_n$ holds. Then the result follows from Theorem 11.10 in Aliprantis-Burkinshaw (1985).

[19] This part benefits from a suggestion of an anonymous referee.

summing over i from 1 to n, $q\cdot(\sum_{i=1}^{n}\bar{x}^i)\geq q\cdot(\sum_{i=1}^{n}\omega^i)$ holds. Then from the feasibility of $\bar{x}$ with $\sum_{i=1}^{n}\omega^i\uparrow\bar{\omega}$ as well as from the order continuity of q, we have $q\cdot\bar{x}^i=q\cdot\omega^i,\forall i$.

We are now going to show $q\neq 0$. We know that there is some i with $\inf\{\bar{p}\cdot x: x\in A_{\bar{\omega}}^{+}\}<\bar{p}\cdot\omega^i$. Fix i. Since $\bar{p}\geq 0$, it follows $\inf\{\bar{p}\cdot x: x\in A_{\bar{\omega}}^{+}\}=0$ which implies $\bar{p}\cdot\omega_i>0$. Again pick $y\in P_i(\bar{x}^i)\cap A_{\bar{\omega}}^{+}$ and since N_s is order dense, there is a net $\{y^\alpha\}$ in N_s such that $y^\alpha\xrightarrow{o}y\in P_i(\bar{x}^i)\cap A_{\bar{\omega}}^{+}$ for the fixed i. For large enough α, $y^\alpha\in P_i(\bar{x}^i)\cap A_{\bar{\omega}}^{+}$ and $q\cdot y^\alpha=\bar{p}\cdot y^\alpha\geq\bar{p}\cdot\omega^i>0$. Since $y^\alpha\in A_{\bar{\omega}}^{+}$, we have $q\neq 0$.[20]

Suppose $\inf\{q\cdot x: x\in A_{\bar{\omega}}^{+}\}\geq q\cdot\omega^i$ for every $i\in\mathbb{N}$. Since $\frac{1}{n}\bar{\omega}$ is an interior point of $A_{\bar{\omega}}^{+}$, there exists $\varepsilon>0$ such that $q\cdot(\frac{\bar{\omega}}{n}+\varepsilon[-\bar{\omega},\bar{\omega}])\geq q\cdot\omega^i$ holds for every i. By summing over i to n, we obtain $q\cdot(\bar{\omega}-\sum_{i=1}^{n}\omega^i)+q\cdot n\varepsilon[-\bar{\omega},\bar{\omega}]\geq 0$. As n increases, the first term of the left hand disappears and the second term leads to $q=0$. This contradicts to $q\neq 0$. Therefore, $\inf\{q\cdot x: x\in A_{\bar{\omega}}^{+}\}<q\cdot\omega^i$ holds for some i.

Step 2: In this step, we are going to obtain an equilibrium price functional in H. We start with the first case where $\mathcal{E}$ satisfies P_i is F-proper at $\bar{x}^i$ with a properness vector $\bar{\omega}$. By F-properness at $\bar{x}^i$, there exists a τ-neighborhood of zero U_i such that a τ-open and convex cone with vertex zero Γ_i is generated by $\bar{\omega}+U_i$, i.e., $\Gamma_i=\{\alpha(\bar{\omega}-u_i): u_i\in U_i,\alpha\in\mathbb{R}_{++}\}$. It follows that $\bar{x}^i+\Gamma_i$ is also τ-open and convex. Obviously $\bar{x}^i\in\mathrm{cl}(\bar{x}^i+\Gamma_i)$ where 'cl' denotes τ-closure.

It is clear that $(\bar{x}^i+\Gamma_i)\cap\Lambda(H)^+\neq\emptyset$. Since $A_{\bar{\omega}}$ is τ-dense in $\Lambda(H)$ under A.7, $A_{\bar{\omega}}^{+}$ is τ-dense in $\Lambda(H)^+$ (by Lemma 3 in Podczeck (1996)). Moreover, $\bar{x}^i$ and $\bar{\omega}$ belong to $A_{\bar{\omega}}^{+}$ as well as to $\Lambda(H)^+$. Thus we have

$$(\bar{x}^i+\Gamma_i)\cap A_{\bar{\omega}}^{+}\neq\emptyset.$$

Let $y\in\big(\bar{x}^i+\Gamma_i\big)\cap A_{\bar{\omega}}^{+}$. For a small enough real number $\alpha>0$, $\bar{x}^i+\alpha(y-\bar{x}^i)\in P_i(\bar{x}^i)\cap A_{\bar{\omega}}^{+}$ and thus $q\cdot\big(\bar{x}^i+\alpha(y-\bar{x}^i)\big)\geq q\cdot\bar{x}^i$. Therefore, $q\cdot y\geq q\cdot\bar{x}^i$.

[20] The referee provides the idea of this proof.

We now assert that $(\bar{x}^i + \Gamma_i) \cap (m - A_{\bar{\omega}}^+) = \emptyset$ for $m \in A_{\bar{\omega}}^+$ such that $q \cdot m \leq q \cdot \bar{x}^i$. We prove it by way of contradiction. Suppose $y \in (\bar{x}^i + \Gamma_i) \cap (m - A_{\bar{\omega}}^+)$. Then there exist $\bar{\alpha} > 0$ and $u_i \in U_i$ such that $y = \bar{x}^i + \bar{\alpha}(\bar{\omega} - u_i)$. Since $A_{\bar{\omega}}$ is τ-dense in $\Lambda(H)$, we may assume that $u_i \in A_{\bar{\omega}}$. Note that for all $z \in m - A_{\bar{\omega}}^+$, $q \cdot z \leq q \cdot m \leq q \cdot \bar{x}^i$ because q is positive. Hence, $q \cdot y \leq q \cdot \bar{x}^i$ and thus, $q \cdot \bar{\omega} \leq q \cdot u_i$. It follows that $\|u_i\|_\infty \geq 1$. If $\|u_i\|_\infty < 1$, there exists λ such that $0 < \lambda < 1$ with $u_i \in \lambda[-\bar{\omega}, \bar{\omega}]$ and then, from $-\lambda\bar{\omega} \leq u_i \leq \lambda\bar{\omega}$, $q \cdot u_i \leq |q \cdot u_i| \leq \lambda(q \cdot \bar{\omega}) < q \cdot \bar{\omega}$. As shown by Propostion 2, τ is Hausdorff locally convex-solid. Without loss of generality, we can assume that U_i is convex, solid and symmetric. Notice that $\|u_i\|_\infty \geq 1$ implies $|u_i| \geq \bar{\omega}(= |\bar{\omega}|)$. Since U_i is solid, $\bar{\omega} \in U_i$. It follows that $\bar{\omega} + U_i$ contains zero and then, Γ_i is not a τ-open cone with vertex zero, which is impossible.

In order to apply Lemma 2, let $Z = \bar{x}^i + \Gamma_i, M^+ = A_{\bar{\omega}}^+, x_0 = \bar{x}^i$, $f = q$. Then, Lemma 2 implies that, for each i, there exists a τ-continuous linear functional $\phi_i \in H$ such that $\phi_i \cdot \bar{x}^i = q \cdot \bar{x}^i$, $\phi_i \cdot y \geq \phi_i \cdot \bar{x}^i$ for $y \in \bar{x}^i + \Gamma_i$, and $0 \leq \phi_i|_{A_{\bar{\omega}}} \cdot x \leq q \cdot x$ for all $x \in A_{\bar{\omega}}^+$ where $\phi_i|_{A_{\bar{\omega}}}$ denotes the restriction of ϕ_i to $A_{\bar{\omega}}$. Under A.6, $H = (A_{\bar{\omega}})_n^\sim$ holds; see Proposition A.3 in Florenzano (1991). Hence, q is also an element of H. Let $\rho : \Lambda(H) \to \mathbb{R}$ be defined by $\rho \cdot x = q \cdot x^+$ for all $x \in \Lambda(H)$. It follows that ρ is sublinear. Since $0 \leq \phi_i|_{A_{\bar{\omega}}} \cdot x \leq q \cdot x$ for any $x \in A_{\bar{\omega}}^+$, $\phi_i|_{A_{\bar{\omega}}} \cdot x = \phi_i|_{A_{\bar{\omega}}} \cdot (x^+ - x^-) \leq \phi_i|_{A_{\bar{\omega}}} \cdot x^+ \leq q \cdot x^+$ for all $x \in A_{\bar{\omega}}$. Therefore, $\phi_i|_{A_{\bar{\omega}}} \cdot x \leq \rho \cdot x$ for all $x \in A_{\bar{\omega}}$. By Lemma 4, we have a positive extension π_i of $\phi_i|_{A_{\bar{\omega}}}$ on $\Lambda(H)$ such that $0 \leq \pi_i \cdot x \leq q \cdot x$ for all $x \in \Lambda(H)^+$ and thus, $\pi_i \cdot x = \pi_i \cdot (x^+ - x^-) \leq \pi_i \cdot x^+ \leq q \cdot x^+ = \rho \cdot x$ for any $x \in \Lambda(H)$ for all i. Since ϕ_i is τ-continuous and $\pi_i = \phi_i$ on $A_{\bar{\omega}}$ for all i, π_i is τ-continuous and so $\pi_i \in H$ for all i. Moreover, $\pi_i \cdot \bar{x}^i (= \phi_i \cdot \bar{x}^i) = q \cdot \bar{x}^i$ holds for all i. Hence, the set $\{\pi_1, \pi_2, \ldots, \pi_i, \ldots\}$ is order bounded from above. Since $< \Lambda(H), H >$ is symmetric, $< H, \Lambda(H) >$ is also a Riesz dual system.[21] From Proposition 1, any order interval of H is $\sigma(H, \Lambda(H))$-compact. Then by

[21] See Aliprantis-Border (1999) p.296.

Theorem 2.1 in Aliprantis-Brown-Burkinshaw (1987), H is Dedekind complete. Thus, the set $\{\pi_1, \ldots, \pi_i, \ldots\}$ has a supremum $\bar{\pi}$ in H. As pointed out earlier,[22] $H \subset \Lambda(H)^{\sim}_{n}$ holds. Therefore, $\bar{\pi}$ is also order continuous.

We are going to show that $\bar{\pi}$ is a quasi-equilibrium price functional. Recall that $H = (A_{\bar{\omega}})^{\sim}_{n}$ and $H \subset \Lambda(H)^{\sim}_{n}$ and thus q and $\bar{\pi}$ are order continuous on both $A_{\bar{\omega}}$ and $\Lambda(H)$. Note that for all i, $q \cdot \bar{x}^i = \pi_i \cdot \bar{x}^i \leq \bar{\pi} \cdot \bar{x}^i$. Since $\sum_{i=1}^{n} \bar{x}^i \uparrow \bar{\omega}$, we have $q \cdot \bar{\omega} \leq \bar{\pi} \cdot \bar{\omega}$. But on $A^{+}_{\bar{\omega}}$, $q \geq \pi_i$ for all i because $\pi_i = \phi_i$ on $A_{\bar{\omega}}$. Since $\bar{\pi}$ is the supremum of π_i's, $q \geq \bar{\pi}$ on $A^{+}_{\bar{\omega}}$ and, therefore, $q \cdot \bar{\omega} \geq \bar{\pi} \cdot \bar{\omega}$. Hence, $q \cdot \bar{\omega} = \bar{\pi} \cdot \bar{\omega}$. But we know that $(q - \bar{\pi})$ is a positive functional on $A^{+}_{\bar{\omega}}$ and $(q - \bar{\pi}) \cdot \bar{\omega} = 0$. Since $\bar{\omega}$ is an interior point of $A^{+}_{\bar{\omega}}$, we obtain $q = \bar{\pi}$ on $A^{+}_{\bar{\omega}}$. Thus $q \cdot \bar{x}^i = \bar{\pi} \cdot \bar{x}^i$ and $q \cdot \omega^i = \bar{\pi} \cdot \omega^i$. This in turn means $\bar{\pi} \cdot \bar{x}^i = \bar{\pi} \cdot \omega^i$ because $q \cdot \bar{x}^i = q \cdot \omega^i$. Since $q \neq 0$, $\bar{\pi}$ is non-zero. We can also see that if $y \in P_i(\bar{x}^i) \cap A^{+}_{\bar{\omega}}$, then $\bar{\pi} \cdot y \geq \bar{\pi} \cdot \bar{x}^i$. Recall that $A^{+}_{\bar{\omega}}$ is τ-dense in $\Lambda(H)^{+}$. Since $P_i(\bar{x}^i)$ is τ-open in $\Lambda(H)^{+}$ and $\bar{\pi}$ is τ-continuous, we can see that for $y \in P_i(\bar{x}^i)$, $\bar{\pi} \cdot y \geq \bar{\pi} \cdot \bar{x}^i (= \bar{\pi} \cdot \omega^i)$ for all $i \in \mathbb{N}$. Hence, $(\bar{x}, \bar{\pi})$ is a quasi-equilibrium of $\mathcal{E}$. Since $\bar{\pi}$ coincides with q on $A_{\bar{\omega}}$, it is clear that $(\bar{x}, \bar{\pi})$ is non-trivial.

Next, we turn to the second case where P_i is E-proper at $\bar{x}^i$ relative to $A_{\bar{\omega}}$. with a properness vector v_i such that $\bar{x}^i + v_i \in A^{+}_{\bar{\omega}}$ and $\bar{x}^i + \alpha v_i \in P_i(\bar{x}^i)$ where α is a small enough positive real number. Then we can have a τ-open convex cone Γ_i such that $v_i \in \Gamma_i$ for each i. As above, $A_{\bar{\omega}}$ and $A^{+}_{\bar{\omega}}$ are τ-dense in $\Lambda(H)$ and in $\Lambda(H)^{+}$, repectively. We are going to show that $P_i(\bar{x}^i) + \Gamma_i$ can be a set Z in Lemma 3. It is clear that $\bar{x}^i + v_i \in \big(P_i(\bar{x}^i) + \Gamma_i\big)$. Then it follows immediately that

$$(P_i(\bar{x}^i) + \Gamma_i) \cap A^{+}_{\bar{\omega}} \neq \emptyset.$$

Clearly $\bar{x}^i \in \mathrm{cl}(P_i(\bar{x}^i) + \Gamma_i) \cap A^{+}_{\bar{\omega}}$. To apply Lemma 3, we need to verify the following

[22]See footnote 13.

condition:

$$q \cdot \bar{x}^i \leq q \cdot z, \ \forall z \in (P_i(\bar{x}^i) + \Gamma_i) \cap A_{\bar{\omega}}^+.$$

We choose $z \in (P_i(\bar{x}^i) + \Gamma_i) \cap A_{\bar{\omega}}^+$. Note that $\bar{x}^i \in A_{\bar{\omega}}^+$ and that by E-properness there is A_i which is radial at $\bar{x}^i$. Thus there exists $\lambda (0 < \lambda \leq 1)$ such that $z_\lambda = (1-\lambda)\bar{x}^i + \lambda z \in A_i \cap A_{\bar{\omega}}^+$. Since z is also in $P_i(\bar{x}^i) + \Gamma_i$, it can be decomposed into $z_1 + \gamma$ where $z_1 \in P_i(\bar{x}^i)$ and $\gamma \in \Gamma_i$. Thus $z_\lambda = (1-\lambda)\bar{x}^i + \lambda(z_1 + \gamma)$. From the convexity of $P_i(\bar{x}^i)$ along with $\bar{x}^i \in \mathrm{cl}P_i(\bar{x}^i)$, we have $(1-\lambda)\bar{x}^i + \lambda z_1 = z_\lambda - \lambda\gamma \in \mathrm{cl}P_i(\bar{x}^i)$. But $z_\lambda - \lambda\gamma$ also belongs to the set $\{z_\lambda\} - \Gamma_i$. This implies that

$$(z_\lambda - \Gamma_i) \cap \mathrm{cl}P_i(\bar{x}^i) \neq \emptyset. \tag{4}$$

From E-properness, we know that $y \in A_i \cap \Lambda(H)^+$ but $y \notin P_i(\bar{x}^i)$ implies $(\{y\} - \Gamma_i) \cap P_i(\bar{x}^i) = \emptyset$. Since Γ_i is open, this condition can be written as

$$y \in A_i \cap \Lambda(H)^+ \text{but } y \notin P_i(\bar{x}^i) \text{ implies } (y - \Gamma_i) \cap \mathrm{cl}P_i(\bar{x}^i) = \emptyset. \tag{5}$$

Considering (5), we can say that (4) implies that $z_\lambda \in P_i(\bar{x}^i)$ and therefore $z_\lambda \in P_i(\bar{x}^i) \cap A_{\bar{\omega}}^+$. Then as proved in Step 1, we have $q \cdot \bar{x}^i \leq q \cdot z_\lambda$ which implies $q \cdot \bar{x}^i \leq q \cdot z$.[23] This shows the desired result.

We are now ready to apply Lemma 3. Let $Z = P_i(\bar{x}^i) + \Gamma_i, M^+ = A_{\bar{\omega}}^+, x_0 = \bar{x}^i, f = q$. Then Lemma 3 implies that there exists a non-zero linear functional $\phi_i \in H$ such that $\phi_i \cdot \bar{x}^i = q \cdot \bar{x}^i$, $\phi_i \cdot y \geq \phi_i \cdot \bar{x}^i$ for $y \in P_i(\bar{x}^i) + \Gamma_i$ and $\phi_i|_{A_{\bar{\omega}}} \cdot x \leq q \cdot x$ for all $x \in A_{\bar{\omega}}^+$ for each i. By A.5, it follows $\phi_i \geq 0$ [24] and thus, $0 \leq \phi_i|_{A_{\bar{\omega}}} \cdot x \leq q \cdot x$ for all $x \in A_{\bar{\omega}}^+$. We appeal to Lemma 4 to assert that there exists an extension $\pi_i \in H$ of $\phi_i|_{A_{\bar{\omega}}}$ such that $\pi_i \leq \rho$ on $\Lambda(H)$ for each i. As above, we can have a supremum $\bar{\pi} \in H$ of π_i's such

[23] This part benefits from a suggestion of the referee.
[24] This was drawn to our attention by the referee.

that $\bar{\pi} = q$ on A_{ω}^{+}. From the property of q, it follows that for each i, $\bar{\pi} \cdot \bar{x}^i = \bar{\pi} \cdot \omega^i$ and $\bar{\pi} \cdot y \geq \bar{\pi} \cdot \bar{x}^i,\ \forall y \in (P_i(\bar{x}^i) + \Gamma_i) \cap A_{\omega}^{+}$. Recall that A_{ω}^{+} is also τ-dense in $\Lambda(H)^{+}$. Since $P_i(\bar{x}^i) + \Gamma_i$ is τ-open and $\bar{\pi}$ is τ-continuous, we can show $\bar{\pi} \cdot y \geq \bar{\pi} \cdot \bar{x}^i$ for all $y \in (P_i(\bar{x}^i) + \Gamma_i)$. Since Γ_i is a cone, $\bar{\pi} \cdot y \geq \bar{\pi} \cdot \bar{x}^i$, $\forall y \in P_i(\bar{x}^i)$ for each i. Finally, we can also show that $(\bar{x}, \bar{\pi})$ is non-trivial as above. This shows that $(\bar{x}, \bar{\pi})$ is a quasi-equilibrium for $\mathcal{E}$. ■