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A Note on Equilibrium in Constrained Incomplete Markets with Non-Ordered Preferences*

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Abstract We prove the existence of Radner equilibrium of plans, price and price expectations in incomplete markets with non-ordered preferences which are subject to portfolio constraints. In particular, the paper extends the results of Werner (1989) to constrained incomplete markets by providing a unifying framework for a variety of restrictions on the risk-sharing role of redundant assets.

Keywords Radner equilibrium, Portfolio constraints, Redundant assets, Incomplete markets, Arbitrage, Non-ordered preferences

JEL Classification C62, D51

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1. Introduction

When financial asset markets are free from portfolio constraints and other type of market frictions, redundant assets such as options and futures contracts do not contribute to risk-sharing and therefore useless.[1] Thus, they can be deleted from frictionless markets without changing anything real and their prices are determined in equilibrium by linear pricing rule. As discussed in Chen (1995), this is not the case with constrained asset markets because redundant assets may actively participate in risk-sharing under portfolio constraints.

The purpose of this paper is to prove the existence of Radner equilibrium of plans, price and price expectations in incomplete markets with non-ordered preferences which are subject to portfolio constraints. In particular, the paper extends the results of Werner (1989) to the incomplete markets with portfolio constraints and redundant assets in the following ways. First, we provide a new condition which incorporates diverse restrictions of the literature on risk-sharing role of redundant assets in constrained incomplete markets. Hahn and Won (2007, 2008) assume that portfolio constraint sets are conic. However, they need not be conic in this paper. Moreover, the new condition encompasses as a special case many interesting ones which appear in Siconolfi (1986), Balasko, Cass and Siconolfi (1990), and Angeloni and Cornet (2006). Second, no consumption is available in the initial period.[2] This assumption has distinct implications to the literature. This condition corresponds to the case with self-financing trading strategies and is taken for simplicity in the literature of asset pricing such as Harrison and Kreps (1979). It is not a matter of simplicity, however, in the literature of addressing the equilibrium existence issue. As shown in Polemarchakis and Siconolfi (1993) and Hahn and Won (2007), no equilibrium may exist in the economy which allows no consumption in the initial period.

To make sure the existence of equilibrium in constrained asset markets without consumption in the initial period, asset markets must be able to provide an opportunity for agents to transfer income from the next period to the current period. An interesting point is that agents are required to hold positive income in the current period by selling assets though they are not allowed to consume now. This requirement for asset markets is called *survival condition of asset markets*. The survival condition of asset markets is an additional requirement to be fulfilled for the existence of equilibrium in asset markets with no initial consumption. It must be noticed that the results of the

[1] An asset is redundant if its return vector is linearly dependent on the return vectors of other available assets. Thus, the payoff of redundant assets can be freely replicated with a portfolio of other assets in frictionless markets.

[2] This framework is also adopted in Gottardi and Hens (1996) for unconstrained asset markets. Furthermore, it is shown by Chae (1988) that the case where consumption arises in both period is embedded into the current framework.

literature which studies asset markets with initial consumption are not applicable to the economy without consumption in the initial period.

Siconolfi (1986) investigates the existence problem in constrained asset markets with numerically representable preferences by limiting the risk-sharing of redundant assets in an unnecessarily strict way. Angeloni and Cornet (2006) extend the results of Werner (1989) to the case of constrained asset markets by relaxing the restrictions of Siconolfi (1986) on the risk-sharing capability of redundant assets. However, they fail to cover the linear portfolio constraints studied in Balasko, Cass, and Siconolfi (1990). Moreover, they admit consumption in the initial period and thus, may not cover the case without consumption in the initial period. Hahn and Won (2007, 2008) study the case where portfolio constraints are expressed as a closed convex cone with vertex. In contrast, the current paper does not require that the portfolio constraints admit conic representation. Notice that Hahn and Won (2007, 2008) do not cover the case with nonlinear portfolio constraints which are not conic. Geanakoplos and Polemarchakis (1986), Polemarchakis and Siconolfi (1993), and Hahn and Won (2007) assume that no consumption arises in the first period and preferences are numerically representable while Hahn and Won (2008) allow consumption to arise in the first period.

2. The Model

A two-period economy is considered where asset markets are open in the first period (denoted by 0) and markets for consumption goods are open in the second period (denoted by 1). Uncertainty of the future consists of a finite number of events denoted by $\mathcal{S} = \{1, \ldots, S\}$. One of the events in $\mathcal{S}$ is to be revealed in period 1 but, in period 0, no agents know which event will happen in period 1. Assets pay the monetary returns which are contingent on the event to be revealed in the second period.[3] In each state $s \in \mathcal{S}$, there is a market for L commodities. Since consumption is available only in the second period, the number of commodities are $\ell := LS$ and thus, the commodity space can be denoted by $\mathbb{R}^\ell$.

Let $\mathcal{I} = \{1, 2, \ldots, I\}$ denote the set of agents, $\mathcal{J} = \{1, 2, \ldots, J\}$ the set of financial assets, and $\mathcal{L} = \{1, 2, \ldots, L\}$ the set of consumption goods. Each agent $i \in \mathcal{I}$ chooses consumption x_i in the consumption set $X_i \subset \mathbb{R}^\ell$ which contains the initial endowment e_i of goods. Preferences over

[3] Alternatively we can assume that assets pay units of the numeraire good because nominal assets can be converted into real assets and vice versa. For details, see Magill and Shafer (1991).

X_i are represented by a binary relation $\succ_i$ on $X_i \times X_i$ which is irreflexive but is not necessarily complete and transitive. The preference relation $\succ_i$ induces the correspondence $P_i : X_i \to 2^{X_i}$ such that for each $x_i \in X_i$, $P_i(x_i) = \{x'_i \in X_i : x'_i \succ_i x_i\}$, which is the set of consumption bundles which agent i prefers to x_i. For each $s \in \mathcal{S}$, let us define[4]

$$P_i(x_i, s) = \left\{x'_i \in X_i : x'_i \in P_i(x_i),\ x'_i(-s) = x_i(-s)\right\}.$$

It is assumed that agent i is subject to portfolio constraint which is represented by a set Θ_i in $\mathbb{R}^J$. Agent i chooses portfolio $\theta_i \in \Theta_i$ in the first period to finance his contingent consumption in the second period. We set $X := \prod_{i\in\mathcal{I}} X_i$ and $\Theta := \prod_{i\in\mathcal{I}} \Theta_i$, and let us define the set of attainable consumption-portfolio allocations by $A = \left\{(x,\theta) \in X \times \Theta : \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i,\ \sum_{i\in\mathcal{I}} \theta_i = 0\right\}$. Let $\hat{X}_i$ denote the projection of A on X_i for each $i \in \mathcal{I}$. We also set $\hat{X} := \prod_{i\in\mathcal{I}} \hat{X}_i$.

Each asset $j \in \mathcal{J}$ pays $r_j(s)$ at state s. The vector of asset returns in state s is given by a J-dimensional row vector $r(s) = (r_j(s))_{j\in\mathcal{J}}$ and the return of asset j by a S-dimensional column vector $r_j = (r_j(s))_{s\in\mathcal{S}}$. The asset payoffs are described by an $S \times J$ matrix $R = [(r(s))_{s\in\mathcal{S}}]$. Here either $S \geq J$ or $S < J$ may hold. In particular, redundant assets exist when J is greater than the rank of R. Let $q \in \mathbb{R}^J$ denote asset prices in the first period and $p \in \mathbb{R}^\ell$ denote goods prices in the second period. We introduce the notation

$$p \,\square\, (x_i - e_i) = (p(s) \cdot (x_i(s) - e_i(s)))_{s\in\mathcal{S}}, \quad W(q) = \begin{bmatrix} -q \\ R \end{bmatrix}.$$

Given price vector (p, q), the open budget correspondence $\mathcal{B}_i : \mathbb{R}^\ell \times \mathbb{R}^J \to 2^{X_i\times\Theta_i}$ of agent i is defined by

$$\mathcal{B}_i(p,q) := \left\{(x_i,\theta_i) \in X_i \times \Theta_i : \begin{bmatrix} 0 \\ p \,\square\, (x_i - e_i) \end{bmatrix} \ll W(q) \cdot \theta_i \right\}.^5$$

and the budget correspondence $c\ell\mathcal{B}_i : \mathbb{R}^\ell \times \mathbb{R}^J \to 2^{X_i\times\Theta_i}$ of agent i is defined by $c\ell\mathcal{B}_i(p,q) := c\ell[\mathcal{B}_i(p,q)]$.[6] Note that the zero in the column vector of $\mathcal{B}_i(p,q)$ indicates that no consumption arises in the first period. For a given pair $(p,q) \in \mathbb{R}^\ell_+ \times \mathbb{R}^J$, agent $i \in \mathcal{I}$ chooses a maximal element (x_i, θ_i) in $c\ell\mathcal{B}_i(p,q)$ with respect to the preferences $\succ_i$. This amounts to choosing $(x_i,\theta_i) \in c\ell\mathcal{B}_i(p,q)$ which satisfies $(P_i(x_i) \times \Theta_i) \cap c\ell\mathcal{B}_i(p,q) = \emptyset$. Now we denote our economy by $\mathcal{E} = \{(X_i, P_i, e_i, \Theta_i)_{i\in\mathcal{I}}\}$.

[4]For a collection of points $\{y(1),\ldots,y(S)\}$ in $\mathbb{R}^L$, we write $y = (y(1),\ldots,y(s),\ldots,y(S)) = (y(s), y(-s))$ where $y(-s) = (y(1),\ldots,y(s-1),y(s+1),\ldots y(S))$.

[5]Let v and v' be vectors in an Euclidean space. Then $v \geq v'$ implies that $v - v' \in \mathbb{R}^\ell_+$; $v > v'$ implies that $v \geq v'$ and $v \neq v'$; $v \gg v'$ implies that $v - v' \in \mathbb{R}^\ell_{++}$.

[6]Let A be a nonempty subset of an Euclidean space. We denote the closure of A by $c\ell(A)$, the interior of A by $int(A)$, the boundary of A with respect to the relative topology by ∂A, and the convex hull of A by coA.

DEFINITION 2.1. : A **Radner equilibrium** of the economy $\mathcal{E}$ is a profile $(p^*, q^*, x^*, \theta^*) \in \mathbb{R}^\ell \times \mathbb{R}^J \times X \times \Theta$ such that

(i) $(x_i^*, \theta_i^*) \in c\ell\mathcal{B}_i(p^*, q^*)$, $\forall\, i \in \mathcal{I}$,

(ii) $(P_i(x_i^*) \times \Theta_i) \cap c\ell\mathcal{B}_i(p^*, q^*) = \emptyset$, $\forall\, i \in \mathcal{I}$,

(iii) $(x^*, \theta^*) \in A$.

We make the following assumptions for every $i \in \mathcal{I}$.

(A1) X_i is closed, convex, and bounded from below in $\mathbb{R}^\ell$.

(A2) $x_i \notin coP_i(x_i)$ for every $x_i \in X_i$.

(A3) P_i is lower hemicontinuous with open values on X_i.[7]

(A4) $\forall\, x_i \in \hat{X}_i, \quad x_i \in \partial P_i(x_i, s), \forall\, s \in \mathcal{S}$.

(A5) $e_i \in intX_i$.

(A6) Θ_i is a closed convex set in $\mathbb{R}^J$ with $0 \in \Theta_i$.

Assumption A1 is standard. Assumption A2 implies weak convexity of preferences.[8] Assumption A3 imposes weaker continuity on preferences. This condition appears in Gale and Mas-Collel (1979) and implies that coP_i is lower hemicontinuous as well. Assumption A4 is adopted in Werner (1989) to let preferences reveal local nonsatiation at every state $s \in \mathcal{S}$. Assumption A5 indicates a strong form of survival condition in goods markets. Finally, Assumption A6 states that portfolio constraints are represented by a closed convex set with no trading of securities. Note that Assumption A6 can cover market frictions such as short-selling constraints, bid-ask spreads, and proportional transaction costs (see Luttmer, 1996). It also subsumes the case with linear constraints of Balasko, Cass, and Siconolfi (1990), and Polemarchakis and Siconolfi (1997).

To analyze the impact of redundant assets on risk-sharing in constrained asset markets, we need more notations. Let $V = \text{span}\{r(1), r(2), \ldots, r(S)\}$ and $V^\perp = \{\theta \in \mathbb{R}^J : R \cdot \theta = 0\}$. Redundant

[7]We define coP_i by $coP_i(x_i) = co(P_i(x_i))$. Recall that if P_i is lower hemicontinuous with open values, then coP_i is lower hemicontinuous with open values.

[8]Notice that this implies the irreflexivity of preferences.

assets exist if and only if $V^{\perp} \neq \{0\}$. In particular, some assets are redundant if the rank of the return matrix R is less than the minimum of J and S. Portfolios in $V^{\perp}$ are called *zero-income portfolios*, which generate zero income transfer in each state of the second period. In particular, portfolios in $\Theta_i \cap V^{\perp}$ are called *constrained zero-income portfolios* for agent i. A portfolio θ in $\mathbb{R}^J$ has the direct sum $\hat{\theta} + \tilde{\theta}$ where $\hat{\theta} \in V$ and $\tilde{\theta} \in V^{\perp}$. The portfolio $\tilde{\theta}$ does not affect the size of income transfers but may matter to the feasibility of θ under the portfolio constraints.

Let C_i be the recession cone of Θ_i and N_i the lineality space of Θ_i for each $i \in \mathcal{I}$.[9] We set $N = \sum_{i \in \mathcal{I}} (N_i \cap V^{\perp})$ and denote by $N^{\perp}$ the orthogonal complement of N in $\mathbb{R}^J$. The following assumption requires that there is a feasible portfolio in C_i which gives a nonnegative income with a positive income at some state.

(A7) There is $v_i^{\circ} \in C_i$ such that $R \cdot v_i^{\circ} > 0$.

In addition to the convexity of the portfolio constraints, Siconolfi (1986) requires that $C_i \cap V^{\perp} = \{0\}, \forall\, i \in \mathcal{I}$. This extra condition severely restricts the risk-sharing role of redundant assets. We provide a new condition which incorporates different types of extra conditions appearing in the literature.

(A8) If $v_i \in C_i \cap V^{\perp}$ for each $i \in \mathcal{I}$ and $\sum_{i \in \mathcal{I}} v_i = 0$, then $v_i \in N_i \cap V^{\perp}$ for all $i \in \mathcal{I}$.

We note that $v_i \in C_i$ indicates a scale-free feasible portfolio in that any scale of v_i satisfies the portfolio constraint Θ_i. Thus, A8 requires that for each $i \in I$, both a zero-income portfolio v_i and its short position $-v_i$ be scale-free feasible if v_i is scale-free feasible for each i and the portfolio allocation (v_i) is attainable, i.e., $\sum_{i \in I} v_i = 0$. As shown later, portfolios in $N_i \cap V^{\perp}$ have the null price and thus, no impact on the set of income transfers between periods 0 and 1 in equilibrium. Thus, portfolios in a subspace generated by a set of scale-free feasible zero-income portfolios $\{v_i, i \in I\}$ with $\sum_{i \in I} v_i = 0$ will have null value and no risk-sharing role in equilibrium.

It is easy to see that Assumption A8 covers two important classes of portfolio constraints: (i) each Θ_i is a linear subspace in $\mathbb{R}^J$,[10] and (ii) $\{C_i \cap V^{\perp}\}_{i \in \mathcal{I}}$ is positively semi-independent, i.e., if $v_i \in C_i \cap V^{\perp}, \forall\, i \in \mathcal{I}$ and $\sum_{i \in \mathcal{I}} v_i = 0$, then $v_i = 0, \forall\, i \in \mathcal{I}$. Thus, Assumption A8 encompasses as a special case not only the linear constraints of Balasko et al. (1990) but also the convex constraints

[9] Let A be a nonempty convex subset of an Euclidean space. The recession cone of A is the set $\Gamma(A) = \{v \in E : A + v \subset A\}$ and the lineality space is the maximal linear subspace in A. When A is closed, $\Gamma(A)$ is also closed and can be expressed as $\Gamma(A) = \{v \in \mathbb{R}^m : \exists \{x^n\}$ in A and $\{a^n\}$ in $\mathbb{R}$ with $a^n \to 0$ such that $v = \lim_{n \to \infty} a^n x^n\}$.

[10] In this case, $\Theta_i = C_i = N_i$ for all $i \in \mathcal{I}$ and thus, Assumption A7 holds true trivially.

of Siconolfi (1986) and Angeloni and Cornet (2006).[11] Hahn and Won (2007, 2008) assume that each Θ_i is a cone with vertex and in addition, require that the projection of the cone Θ_i onto $N^{\perp}$ be closed. Assumption A8 differs from the conditions of Hahn and Won (2007, 2008) in that Θ_i need not be a cone.

3. The Existence of Equilibrium

Arbitrage opportunities are not admitted in equilibrium for frictionless markets. The notion of arbitrage for frictionless markets, however, is no longer appropriate to equilibrium analysis in constrained incomplete markets because the law of one price fails. The following provides an extension of the no arbitrage condition to constrained incomplete markets.

DEFINITION 3.1. : A price vector $q \in \mathbb{R}^J$ admits *no constrained arbitrage* for agent i in the economy $\mathcal{E}$ if there is no $\theta_i \in C_i$ such that $q \cdot \theta_i \leq 0$ and $R \cdot \theta_i > 0$. A price vector $q \in \mathbb{R}^J$ admits *no constrained arbitrage* for the economy $\mathcal{E}$ if it admits no constrained arbitrage for every agent $i \in \mathcal{I}$.

When $\Theta_i = \mathbb{R}^J$ for each $i \in \mathcal{I}$, then $C_i = \mathbb{R}^J$. In this case, the absence of constrained arbitrage coincides with the absence of arbitrage in frictionless markets. The notion of constrained arbitrage is used in Jouini and Kallal (1995, 1999) and Luttmer (1996) among others. Let Q_i denote the set of prices which admit no constrained arbitrage for agent i. We set $Q = \bigcap_{i \in \mathcal{I}} Q_i$. The set Q denotes the set of prices which admit no constrained arbitrage for the economy $\mathcal{E}$.

As in the case with complete markets, survival conditions are needed to guarantee the existence of a Radner equilibrium in constrained incomplete markets. Since no consumption is allowed in the first period, agents are required to survive asset markets in the first period. We impose the following form of asset market survival condition for each $i \in \mathcal{I}$.

(A9) For each $q \in c\ell(Q_i)$, there is $\theta_i^{\circ} \in \Theta_i$ such that $q \cdot \theta_i^{\circ} < 0$.

Assumption A9 requires that agent i be able to transfer positive income at each $q \in c\ell(Q_i)$ from the second period to the first period through asset markets.

PROPOSITION 3.1. : Under Assumption A4 and A7, equilibrium asset prices belongs to $Q \cap N^{\perp}$.

[11]Angeloni and Cornet (2006) directly impose the condition that A is bounded. This condition is equivalent to (ii) in a two-period economy.

PROOF : To show this, suppose that (p, q, x, θ) is an equilibrium. If $q \notin Q_i$ for some $i \in \mathcal{I}$, then there exists $v_i \in C_i$ which satisfies $q \cdot v_i \leq 0$ and $R \cdot v_i > 0$. Clearly, we have $\theta_i + v_i \in \Theta_i$, $q \cdot (\theta_i + v_i) \leq 0$ and $R \cdot \theta_i < R \cdot (\theta_i + v_i)$. By the state-wise local nonsatiation of P_i (Assumption A4), we can choose $x_i' \in X_i$ such that $x_i' \in P_i(x_i)$ and $(x_i', \theta_i + v_i) \in c\ell\mathcal{B}_i(p, q)$, which contradicts the optimality of (x_i, θ_i) in $c\ell\mathcal{B}_i(p, q)$. Thus, q must be in Q.

Now we show that $q \in N^\perp$, i.e., $q \cdot v = 0$ for all $v \in N$. Suppose otherwise. Then there exists $v_i \in N_i \cap V^\perp$, $\forall\, i \in \mathcal{I}$ such that $v = \sum_{i \in \mathcal{I}} v_i$ and $q \cdot v < 0$, without loss of generality. This implies that there exists $v_i \in N_i \cap V^\perp$ for some $i \in \mathcal{I}$ such that $q \cdot v_i < 0$. By Assumption 7, we can pick v_i° in C_i with $R \cdot v_i^\circ > 0$. Since we can take a sufficiently large v_i in $N_i \cap V^\perp$, without loss of generality, we can assume that $q \cdot (v_i + v_i^\circ) < 0$. Clearly, $v_i + v_i^\circ \in C_i$ and $R \cdot (v_i + v_i^\circ) > 0$. It follows that $\theta_i' := \theta_i + (v_i + v_i^\circ) \in \Theta_i$, $q \cdot \theta_i' < 0$, and $R \cdot \theta_i' > 0$. By Assumption A4, we can choose $x_i' \in X_i$ such that $x_i' \in P_i(x_i)$ and $(x_i', \theta_i') \in c\ell\mathcal{B}_i(p, q)$, which contradicts the optimality of (x_i, θ_i) in $c\ell\mathcal{B}_i(p, q)$. Thus we conclude that $q \in Q \cap N^\perp$. ■

By Proposition 3.1, equilibrium asset prices would be in $Q \cap N^\perp$. We define the sets of normalized prices: $\Delta = \Delta_0 \times \Delta_1$ where $\Delta_0 = \left\{q \in c\ell(Q) \cap N^\perp : \|q\| \leq 1\right\}$ and $\Delta_1 = \prod_{s \in \mathcal{S}} \Delta_s$ with $\Delta_s = \left\{p(s) \in \mathbb{R}^L : \|p(s)\| \leq 1\right\}$.[12] Now we are ready to provide the main existence theorem for the economy $\mathcal{E}$.

THEOREM 3.1. : The economy $\mathcal{E}$ has a Radner equilibrium under Assumptions A1–A9.

PROOF : We build a sequence of truncated economies in the following way. We take an increasing sequence $\{(K_n, M_n)\}$ of compact convex cube pairs with center 0 such that $K_n \subset \mathbb{R}^\ell$ with $\{e_1, \ldots, e_m\} \subset intK_1$ and $M_n \subset \mathbb{R}^J$ with $0 \in M_1$ satisfying $\bigcup_n K_n = \mathbb{R}^\ell$ and $\bigcup_n M_n = \mathbb{R}^J$. For each n and $i \in \mathcal{I}$, we set $X_i^n := X_i \cap K_n$, $\Theta_i^n := \Theta_i \cap M_n$, $X^n := \prod_{i \in \mathcal{I}} X_i^n$ and $\Theta^n := \prod_{i \in \mathcal{I}} \Theta_i^n$. We denote by $\mathcal{E}^n$ the truncated economy $\{(X_i^n, coP_i, e_i, \Theta_i^n)_{i \in \mathcal{I}}\}$. In the economy $\mathcal{E}^n$, each agent i has a nonempty compact convex choice set $X_i^n \times \Theta_i^n$. We define $\gamma : \Delta \to \mathbb{R}^{S+1}$ by $\gamma(p, q) = (\gamma_s(p, q))_{s \in \{0\} \cup \mathcal{S}}$, where

$$\gamma_s(p, q) = \begin{cases} 1 - \|q\|, & \text{if } s = 0, \\ 1 - \|p(s)\|, & \text{if } s \in \mathcal{S}. \end{cases}$$

For every $i \in \mathcal{I}$, correspondences $\mathcal{B}_i^n : \Delta \to 2^{X_i^n \times \Theta_i^n}$ and $c\ell\mathcal{B}_i^n : \Delta \to 2^{X_i^n \times \Theta_i^n}$ are built as follows,

[12] $\|\cdot\|$ is the Euclidean norm.

respectively.

$$\mathcal{B}_i^n(p,q) = \left\{ (x_i,\theta_i) \in X_i^n \times \Theta_i^n : \begin{bmatrix} 0 \\ p \,\square\, (x_i - e_i) \end{bmatrix} \ll W(q)\cdot\theta_i + \gamma(p,q) \right\}$$

$$c\ell\mathcal{B}_i^n(p,q) = \left\{ (x_i,\theta_i) \in X_i^n \times \Theta_i^n : \begin{bmatrix} 0 \\ p \,\square\, (x_i - e_i) \end{bmatrix} \leq W(q)\cdot\theta_i + \gamma(p,q) \right\},$$

where $\gamma(p,q)$ has $\gamma_s(p,q)$ as the sth element. Based on the above correspondences, we define

$$\mathfrak{B}_i^n(p,q) = \begin{cases} \{(e_i,0)\}, & \text{if } \mathcal{B}_i^n(p,q) = \emptyset \\ c\ell\mathcal{B}_i^n(p,q), & \text{if } \mathcal{B}_i^n(p,q) \neq \emptyset. \end{cases}$$

We construct the following correspondences $\varphi_0^n : \Delta \times X^n \times \Theta^n \to 2^{\Delta}$ and $\varphi_i^n : \Delta \times X^n \times \Theta^n \to 2^{X_i^n \times \Theta_i^n}$ for every $i \in \mathcal{I}$:

$$\begin{aligned} \varphi_0^n(p,q,x,\theta) &= \{(\hat{p},\hat{q}) \in \Delta : \textstyle\sum_{s\in\mathcal{S}}[\hat{p}(s) - p(s)]\cdot z(s) + (\hat{q} - q)\cdot \textstyle\sum_{i\in\mathcal{I}}\theta_i > 0\}, \\ \varphi_i^n(p,q,x,\theta) &= \begin{cases} \mathfrak{B}_i^n(p,q), & \text{if } (x_i,\theta_i) \notin c\ell\mathcal{B}_i^n(p,q), \\ (coP_i(x_i)\times\Theta_i^n)\cap\mathcal{B}_i^n(p,q), & \text{if } (x_i,\theta_i) \in c\ell\mathcal{B}_i^n(p,q), \end{cases} \end{aligned}$$

where $z(s) = \sum_{i\in\mathcal{I}}[x_i(s) - e_i(s)]$ for each $s \in \mathcal{S}$. It is observed that, for each n, the economy $\mathcal{E}^n$ satisfies Assumptions A1–A6.

Claim 1 : For each n, there is a profile $(p^n, q^n, x^n, \theta^n) \in \Delta \times X^n \times \Theta^n$ such that, for every $i \in \mathcal{I}$,

(1) $(x_i^n, \theta_i^n) \in c\ell\mathcal{B}_i^n(p^n, q^n)$,

(2) $(coP_i(x_i^n) \times \Theta_i^n) \cap \mathcal{B}_i^n(p^n, q^n) = \emptyset$,

(3) $\sum_{s\in\mathcal{S}} p^n(s)\cdot z^n(s) + q^n \cdot \sum_{i\in\mathcal{I}}\theta_i^n \geq \sum_{s\in\mathcal{S}} p(s)\cdot z^n(s) + q\cdot\sum_{i\in\mathcal{I}}\theta_i^n$, $\forall\,(p,q) \in \Delta$,

(4) $z^n = 0$ and $\sum_{i\in\mathcal{I}}\theta_i^n = 0$,

where $z^n(s) := \sum_{i\in\mathcal{I}}(x_i^n(s) - e_i(s))$ for every $s \in \mathcal{S}$.

Proof : By Lemma 3.5 of Hahn and Won (2008), φ_i^n is lower hemicontinuous and convex-valued for each $i \in \mathcal{I}_0 := \{0\} \cup \mathcal{I}$ and each n. By applying the fixed point theorem of Gale and Mas-Colell (1975, 1979) to φ_i^n's, we obtain $(p^n, q^n, x^n, \theta^n) \in \Delta \times X^n \times \Theta^n$ satisfying (1), (2), and (3) for every $i \in \mathcal{I}$.

To prove (4), suppose that $\sum_{i\in\mathcal{I}}\theta_i^n \neq 0$. Then (3) implies $\|q^n\| = 1$ and $q^n \cdot \sum_{i\in\mathcal{I}}\theta_i^n > 0$. However, (1) implies that $q^n \cdot \sum_{i\in\mathcal{I}}\theta_i^n \leq I \cdot (1 - \|q^n\|) = 0$, which is a contradiction. If $z^n(s') \neq 0$

for some $s' \in \mathcal{S}$, then (3) implies $\|p^n(s')\| = 1$ and $p^n(s') \cdot z^n(s') > 0$. However, (1) implies that $p^n(s) \cdot z^n(s) \leq r(s) \cdot \sum_{i \in \mathcal{I}} \theta_i^n = 0. \forall\, s \in \mathcal{S}$, which is a contradiction. □

For each n, let $\{(p^n, q^n, x^n, \theta^n)\}$ denote the profile obtained in Claim 1. Notice that, since each X_i is closed and bounded from below, $\hat{X}_i$ is compact and so $\hat{X}$ is compact. Since $\{(p^n, q^n, x^n)\}$ are in $\Delta \times \hat{X}$, without loss of generality, we can assume that $\{(p^n, q^n, x^n)\}$ converges to a point (p^*, q^*, x^*). Observe that (4) implies that $\sum_{i \in \mathcal{I}}(x_i^* - e_i) = 0$.

CLAIM 2 : There exists $\theta_i^* \in \Theta_i$ for each $i \in \mathcal{I}$ such that $\sum_{i \in \mathcal{I}} \theta_i^* = 0$ and $\lim_{n \to \infty} R \cdot \theta_i^n = R \cdot \theta_i^*$ for all $i \in \mathcal{I}$.

PROOF : For each $i \in \mathcal{I}$ and each n, we decompose θ_i^n as $\theta_i^n = \phi_i^n + \eta_i^n$ where $\eta_i^n \in N_i \cap V^\perp$ and $\phi_i^n \in (N_i \cap V^\perp)^\perp$, where $(N_i \cap V^\perp)^\perp$ is the orthogonal complement of $N_i \cap V^\perp$ in $\mathbb{R}^J$. Moreover, by Claim 1, $\sum_{i \in \mathcal{I}} \theta_i^n = 0$ and thus, $\sum_{i \in \mathcal{I}} \phi_i^n + \sum_{i \in \mathcal{I}} \eta_i^n = 0$ for all n. We claim that $\{\phi_i^n\}$ for each $i \in \mathcal{I}$ and $\{\sum_{i \in \mathcal{I}} \eta_i^n\}$ are bounded. Suppose otherwise. For each n, we set $a_n^{-1} = 1 + \sum_{i \in \mathcal{I}} \|\phi_i^n\| + \|\sum_{i \in \mathcal{I}} \eta_i^n\|$. Then it follows that $a_n \to 0$. Clearly, $\{a_n \phi_i^n\}$ for all $i \in \mathcal{I}$ and $\{a_n \sum_{i \in \mathcal{I}} \eta_i^n\}$ are bounded. Thus, they have subsequences convergent to $\hat{\phi}_i$ for each $i \in \mathcal{I}$ and $\hat{\eta}$, respectively. Since $\sum_{i \in \mathcal{I}} a_n \phi_i^n + a_n \sum_{i \in \mathcal{I}} \eta_i^n = 0$ for all n, it holds that $\sum_{i \in \mathcal{I}} \hat{\phi}_i + \hat{\eta} = 0$ and $\sum_{i \in \mathcal{I}} \|\hat{\phi}_i\| + \|\hat{\eta}\| = 1$. In particular, the two equalities imply that $\hat{\phi}_i \neq 0$ for some $i \in \mathcal{I}$. Since $\sum_{i \in \mathcal{I}} \eta_i^n \in N$, $\hat{\eta}$ is in N. Thus, there exists $\hat{\eta}_i \in N_i \cap V^\perp$ for all $i \in \mathcal{I}$ such that $\hat{\eta} = \sum_{i \in \mathcal{I}} \hat{\eta}_i$. Clearly, we have $\sum_{i \in \mathcal{I}}(\hat{\phi}_i + \hat{\eta}_i) = 0$. Since $-\eta_i^n \in N_i$, we also have $\phi_i^n = \theta_i^n - \eta_i^n \in \Theta_i$ and thus,

$$\phi_i^n \in \Theta_i \cap (N_i \cap V^\perp)^\perp. \tag{1}$$

By Corollary 8.3.3 of Rockafellar (1970), the recession cone of $\Theta_i \cap (N_i \cap V^\perp)^\perp$ is $C_i \cap (N_i \cap V^\perp)^\perp$. Thus, by (1) we see that $\hat{\phi}_i$ is in $C_i \cap (N_i \cap V^\perp)^\perp$ for all $i \in \mathcal{I}$.

On the other hand, $p^n(s) \cdot (x_i^n(s) - e_i(s)) \leq r(s) \cdot \theta_i^n + \gamma_s(p^n, q^n)$ for all n and $s \in \mathcal{S}$. Since $\eta_i^n \in V^\perp$, it holds that $p^n(s) \cdot (x_i^n(s) - e_i(s)) \leq r(s) \cdot \phi_i^n + \gamma_s(p^n, q^n)$ for all n and $s \in \mathcal{S}$. By multiplying both sides of the inequalities by a^n and passing to the limit, we have $R \cdot \hat{\phi}_i \geq 0$. Recalling that $\sum_{i \in \mathcal{I}} \hat{\phi}_i + \hat{\eta} = 0$ and $\hat{\eta} \in V^\perp$, we see that $R \cdot \left(\sum_{i \in \mathcal{I}} \hat{\phi}_i\right) = 0$ and thus, $R \cdot \hat{\phi}_i = 0$ for all $i \in \mathcal{I}$. This implies $\hat{\phi}_i \in V^\perp$. Consequently, it follows that for all $i \in \mathcal{I}$,

$$\hat{\phi}_i \in V^\perp \cap \left[C_i \cap (N_i \cap V^\perp)^\perp\right]. \tag{2}$$

Since $\sum_{i \in \mathcal{I}}(\hat{\phi}_i + \hat{\eta}_i) = 0$ and $\hat{\phi}_i + \hat{\eta}_i \in C_i \cap V^\perp$ for all $i \in \mathcal{I}$, Assumption A8 requires that $\hat{\phi}_i + \hat{\eta}_i \in N_i \cap V^\perp$ for all $i \in \mathcal{I}$. Recalling that $\hat{\eta}_i \in N_i \cap V^\perp$, this implies that $\hat{\phi}_i \in N_i \cap V^\perp$ for all $i \in \mathcal{I}$. By

(2), we also have $\hat{\phi}_i \in (N_i \cap V^{\perp})^{\perp}$ for all $i \in \mathcal{I}$. Consequently, it holds that $\hat{\phi}_i = 0$ for all $i \in \mathcal{I}$, which leads to a contradiction. Therefore, we conclude that $\{\phi_i^n\}$ for all $i \in \mathcal{I}$ and $\{\sum_{i\in\mathcal{I}} \eta_i^n\}$ are bounded.

Thus, $\{((\phi_i^n)_{i\in\mathcal{I}}, \sum_{i\in\mathcal{I}} \eta_i^n)\}$ has a subsequence convergent to a point $((\phi_i)_{i\in\mathcal{I}}, \eta)$. Then we have $\sum_{i\in\mathcal{I}} \phi_i = -\eta$, $\eta \in N$ and by (1), $\phi_i \in \Theta_i$ for all $i \in \mathcal{I}$. Since $\eta \in N$, we can choose $\eta_i \in N_i \cap V^{\perp}$ such that $\eta = \sum_{i\in\mathcal{I}} \eta_i$. Then, we see that $\phi_i + \eta_i \in \Theta_i$ for all $i \in \mathcal{I}$ and $\sum_{i\in\mathcal{I}}(\phi_i + \eta_i) = 0$. For each $i \in \mathcal{I}$, we set $\theta_i^* = \phi_i + \eta_i$. Then it holds that $\sum_{i\in\mathcal{I}} \theta_i^* = 0$ and $R \cdot \theta_i^* = R \cdot \phi_i = \lim_{n\to\infty} R \cdot \theta_i^n$. □

To show that $(p^*, q^*, x^*, \theta_i^*)$ is an equilibrium for $\mathcal{E}$, we define the intermediate open budget correspondence $\hat{\mathcal{B}}_i : \Delta \to 2^{X_i \times \Theta_i}$ by

$$\hat{\mathcal{B}}_i(p,q) = \left\{ (x_i, \theta_i) \in X_i \times \Theta_i : \begin{bmatrix} 0 \\ p \,\square\, (x_i - e_i) \end{bmatrix} \ll W(q) \cdot \theta_i + \gamma(p,q) \right\}.$$

Note that $\hat{\mathcal{B}}_i(p^*, q^*) = \mathcal{B}_i(p^*, q^*)$ if $\gamma(p^*, q^*) = 0$.

CLAIM 3 : The profile $(p^*, q^*, x^*, \theta^*)$ satisfies the following: for every $i \in \mathcal{I}$,

(1) $(x_i^*, \theta_i^*) \in c\ell\hat{\mathcal{B}}_i(p^*, q^*)$,

(2) $(P_i(x_i^*) \times \Theta_i) \cap \hat{\mathcal{B}}_i(p^*, q^*) = \emptyset$,

PROOF : Recalling that $q^n \in c\ell(Q) \cap N^{\perp}$ for all n and thus, $q^* \in c\ell(Q) \cap N^{\perp}$, we see that $q^n \cdot \theta_i^n = q^n \cdot \phi_i^n$ for all n and $i \in \mathcal{I}$, and $q^* \cdot \phi_i = q^* \cdot \theta_i^*$ for all $i \in \mathcal{I}$. Thus, it is straightforward to verify (1). The proof of the second result is referred to the proof of Claim 2 of Theorem 3.2 in Hahn and Won (2008). □

CLAIM 4 : For every $i \in \mathcal{I}$, $\hat{\mathcal{B}}_i(p^*, q^*) \neq \emptyset$.

PROOF : Fix $i \in \mathcal{I}$. Since $q^* \in c\ell(Q)$, Assumption A9 allows us to choose $\theta_i^\circ \in \Theta_i$ such that $q^* \cdot \theta_i^\circ < 0$ for every $i \in \mathcal{I}$. Now let us define $\mathcal{S}_0 = \{s \in \mathcal{S} : p^*(s) = 0\}$ and $\mathcal{S}_1 = \{s \in \mathcal{S} : p^*(s) \neq 0\}$. Then $\{\mathcal{S}_0, \mathcal{S}_1\}$ is a partition of $\mathcal{S}$, i.e., $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ with $\mathcal{S}_0 \cap \mathcal{S}_1 = \emptyset$. It is observed that, by Assumption A5, there exists $x_i^\circ \in X_i$ such that $p^*(s) \cdot [x_i^\circ(s) - e_i(s)] < 0, \forall\, s \in \mathcal{S}_1$. When $\mathcal{S}_0 = \emptyset$, since $p^* \,\square\, (x_i^\circ - e_i) \ll 0$, we can pick a sufficiently small $\alpha > 0$ which satisfies $p^* \,\square\, (x_i^\circ - e_i) \ll R \cdot (\alpha\theta_i^\circ)$. This implies that

$$\begin{bmatrix} 0 \\ p^* \,\square\, (x_i^\circ - e_i) \end{bmatrix} \ll W(q^*) \cdot (\alpha\theta_i^\circ) + \gamma(p^*, q^*),$$

i.e., $(x_i^\circ, \alpha\theta_i^\circ) \in \hat{\mathcal{B}}_i(p^*, q^*)$.

When $\mathcal{S}_0 \neq \emptyset$, let us define $\hat{x}_i \in X_i$ by

$$\hat{x}_i(s) = \begin{cases} e_i(s), & \text{if } s \in \mathcal{S}_0, \\ x_i^\circ(s), & \text{if } s \in \mathcal{S}_1. \end{cases}$$

Noting that $\gamma_s(p^*, q^*) = 1, \forall\, s \in \mathcal{S}_0$, we can choose a sufficiently small $\alpha > 0$ such that $p^*(s) \cdot [\hat{x}_i(s) - e_i(s)] = p^*(s) \cdot [e_i(s) - e_i(s)] = 0 < r(s) \cdot (\alpha\theta_i^\circ) + 1, \forall\, s \in \mathcal{S}_0$, and $p^*(s) \cdot [\hat{x}_i(s) - e_i(s)] = p^*(s) \cdot [x_i^\circ(s) - e_i(s)] < r(s) \cdot (\alpha\theta_i^\circ), \forall\, s \in \mathcal{S}_1$. This implies that

$$\begin{bmatrix} 0 \\ p^* \,\square\, (\hat{x}_i - e_i) \end{bmatrix} \ll W(q^*) \cdot (\alpha\theta_i^\circ) + \gamma(p^*, q^*),$$

i.e., $(\hat{x}_i, \alpha\theta_i^\circ) \in \hat{\mathcal{B}}_i(p^*, q^*)$. Hence, it is concluded that $\hat{\mathcal{B}}_i(p^*, q^*) \neq \emptyset$. □

CLAIM 5 : For every $i \in \mathcal{I}$, $(P_i(x_i^*) \times \Theta_i) \cap c\ell\hat{\mathcal{B}}_i(p^*, q^*) = \emptyset$.

PROOF : Suppose to the contrary that, for some $i \in \mathcal{I}$, there exists $(x_i, \theta_i) \in (P_i(x_i^*) \times \Theta_i) \cap c\ell\hat{\mathcal{B}}_i(p^*, q^*)$. Due to Claim 4, one can take $(x_i', \theta_i') \in \hat{\mathcal{B}}_i(p^*, q^*)$. Then, for $\alpha \in (0, 1)$ sufficiently close to 1, it holds that $\alpha(x_i, \theta_i) + (1 - \alpha)(x_i', \theta_i') \in P_i(x_i^*) \cap \hat{\mathcal{B}}_i(p^*, q^*)$. This means that $(P_i(x_i^*) \times \Theta_i) \cap \hat{\mathcal{B}}_i(p^*, q^*) \neq \emptyset$, a contradiction to Claim 3. □

CLAIM 6 : For every $i \in \mathcal{I}$, $(x_i^*, \theta_i^*) \in c\ell\mathcal{B}_i(p^*, q^*)$ and $(P_i(x_i^*) \times \Theta_i) \cap c\ell\mathcal{B}_i(p^*, q^*) = \emptyset$.

PROOF : We need to show that $\gamma(p^*, q^*) = 0$, i.e., $\|q^*\| = 1$ and $\|p^*(s)\| = 1$ for all $s \in \mathcal{S}$. First, we will prove that

$$\begin{bmatrix} 0 \\ p^* \,\square\, (x_i^* - e_i) \end{bmatrix} = W(q^*) \cdot \theta_i^* + \gamma(p^*, q^*), \forall\, i \in \mathcal{I}, \tag{3}$$

To see this, suppose otherwise. Due to (1) of Claim 3, there exists an agent $i \in \mathcal{I}$ such that (i) $q^* \cdot \theta_i^* < 0$ or (ii) $p^*(s) \cdot [x_i^*(s) - e_i^*(s)] < r(s) \cdot \theta_i^* + \gamma_s(p^*, q^*)$ at some state $s \in \mathcal{S}$. If (i) occurs, by Assumption A7, we can take a sufficiently small $\alpha > 0$ and a portfolio $v_i^\circ \in C_i$ with $R \cdot v_i^\circ > 0$ such that $q^* \cdot (\theta_i^* + \alpha v_i^\circ) < 0$ and $R \cdot \theta_i^* < R \cdot (\theta_i^* + \alpha v_i^\circ)$. It is noted that $\theta_i^* + \alpha v_i^\circ \in \Theta_i$. By Assumption A4, we can find x_i satisfying $x_i \in P_i(x_i^*)$ and $(x_i, \theta_i^* + \alpha v_i^\circ) \in c\ell\hat{\mathcal{B}}_i(p^*, q^*)$. This is a contradiction to Claim 5. If (ii) occurs, by Assumption A4 again, we we can find x_i which satisfies $x_i \in P_i(x_i^*)$ and $(x_i, \theta_i^*) \in c\ell\hat{\mathcal{B}}_i(p^*, q^*)$, a contradiction to Claim 5. Therefore (3) holds.

Summing (3) over $i \in \mathcal{I}$, we have $I \cdot \gamma(p^*, q^*) = 0$, giving $\gamma(p^*, q^*) = 0$. This implies that, for every $i \in \mathcal{I}$, $(x_i^*, \theta_i^*) \in c\ell\mathcal{B}_i(p^*, q^*)$ by Claim 3 and $(P_i(x_i^*) \times \Theta_i) \cap c\ell\mathcal{B}_i(p^*, q^*) = \emptyset$ by Claim 5. □

Hence, we conclude that $(p^*, q^*, x^*, \theta^*)$ is an equilibrium for $\mathcal{E}$.[13] ■

4. Conclusion

We have shown the existence of equilibrium in the asset markets where portfolio constraints are expressed as a convex set. In particular, Assumption A8 incorporates special conditions of the literature on the income spanning capability of asset markets with redundant assets. Portfolio constraints raise the asset market survival problem in the initial period where no consumption is available. To address it, we provide Assumption A9 which specifies a survival condition for constrained asset markets.

This paper covers only the case with convex portfolio constraints which satisfy Assumption A8. This condition is more general than the counterparts of the literature mentioned earlier, but still limited in explaining the risk-sharing role of redundant assets. It will be a challenging problem to provide a unifying framework for Assumption A8 and the cone conditions of Hahn and Won (2007, 2008). Another interesting theme is to extend the consequence of the paper to the case that portfolio constraints are endogenously determined.

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[13] It is remarked that $q^* \in Q \cap N^{\perp}$ due to Proposition 3.1.

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