

Capital Reallocation and Liquidity with Search Frictions*

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Abstract I construct a model of an asset market subject to search frictions, in an environment where both investment and asset liquidity are determined endogenously. This provides a natural framework to analyze the interaction between capital reallocation and liquidity in response to aggregate shocks, which I assess quantitatively. The search model of capital reallocation exhibits strong internal propagation, and can generate substantial changes in factor utilization following aggregate shocks.

Keywords capital reallocation, liquidity, investment, asset market, search

JEL Classification E22, E32, E44, J64

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1. INTRODUCTION

I construct a model of an asset market subject to search frictions, in an environment where both investment and asset liquidity (the speed of asset sales) are determined endogenously. The model provides a natural framework to analyze the interaction between capital reallocation and liquidity in response to aggregate shocks. The existing literature on capital reallocation is sparse and has emphasized the theoretical role of asymmetric information or credit constraints as a source of liquidity frictions.¹ I complement this work by considering search frictions.²

Eisfeldt and Rampini (2006) provide a quantitative analysis where they model the illiquidity of reallocation using a reduced form adjustment cost function. In this paper I show that using the mechanism of search frictions we can help us understand the source of these adjustment costs in a more microfounded way. A second advantage is the joint determination of capital liquidity and capital utilization rates. This is explicitly permitted by using the search framework of the paper which was developed by the labor unemployment literature. By bringing the theoretical techniques from the search literature on unemployment to the issue of capital reallocation, I highlight the advantage of jointly discussing these issues.³

A third advantage is that using this framework, we can make more direct comparisons between reallocation dynamics in capital and labor to highlight the differences in the propagation dynamics implied by search frictions for each of these major factor inputs. The existing unemployment literature (e.g. Shimer (2005)) has highlighted the fact that search frictions in labor reallocation cannot be a source of much propagation, and I show they can be in the case of capital reallocation.

The reallocation of capital modelled in this analysis is that of property, plant and equipment and acquisitions. Some types of capital such as real estate are immobile and others are, but the reallocation we target is the *transfer of controlling*

¹See the discussion in Eisfeldt and Rampini (2006).

²The dynamics of capital reallocation are typically compared with those of labor reallocation where the canonical models incorporate search frictions. Thus, it seems natural to consider search frictions in the context of capital reallocation also. Despite the wealth of papers analyzing search frictions in the context of labor, there are few attempts at modelling search frictions in the context of capital.

³Greenwood, Hercowitz and Huffman (1988), Bils and Cho (1994), Burnside and Eichenbaum (1996) and Basu and Kimball (1997) consider fluctuations in capital utilization arising from adjustment costs independently of issues of capital liquidity. Here I consider a source of these adjustment costs arising from search frictions.

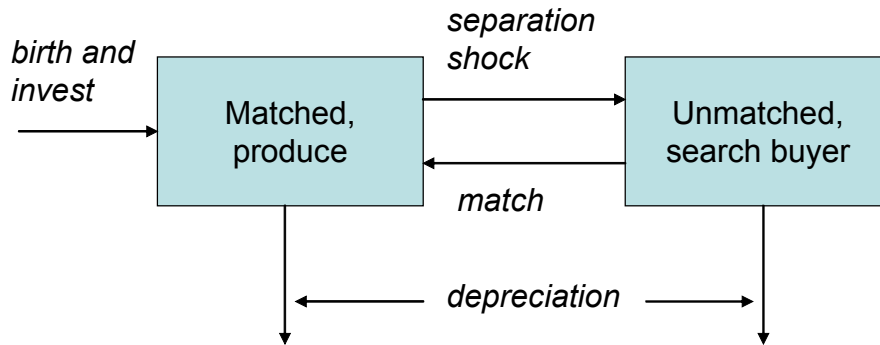


Figure 1: Flow of assets through economy.

ownership. Since the analysis will distinguish between existing capital (through past investment decisions) and investment plans, I use the term “asset” to refer to a business idea or production plan upon which physical capital can be invested in. For instance, the idea to build a house on a particular piece of land can be regarded as an example of such a business idea, and the act of building such a unit is the new investment decision on the “asset”. When the condo complex is sold to somebody who did not make the initial investment decision, we label that as capital reallocation.

Figure 1 summarizes the flow of assets through the economy modelled in the analysis. Assets are either owned by agents who use assets productively, i.e. “matched”, or owned by agents who do not use the asset productively, i.e. “unmatched”. Matched assets become unmatched at some exogenous “separation” rate. Search frictions imply it takes time to re-match assets, where the arrival rate of matches is determined by the endogenous ratio of potential asset buyers and sellers. Both matched and unmatched assets (and their capital stock) can become obsolescent and exit the economy, but are replaced by an exogenous measure of new assets, which are matched and invested in upon entry to the economy. The total population of assets is fixed.

I assess the quantitative implications of the model (section 3) using moments reported by Eisfeldt and Rampini (2006). I consider the impact of two types of exogenous shocks: (i) technology shocks to the productivity of matches, and (ii) shocks to the separation rate of matches. Both predict a positive correlation between reallocation and aggregate output, and a negative correlation between the dispersion in productivity of capital and output as documented by Eisfeldt

and Rampini (2006).

The main findings are as follows. Capital reallocation is associated with low average match rates of existing capital. This implies that changes in these rates lead to gradual adjustments in aggregate capital utilization, and this acts as a source of propagation of shocks on changes to aggregate output. The importance of this propagation mechanism depends on the response of capital utilization (relative to the overall response of output) in the context of the model following the aggregate shocks. The model can generate over half the overall change in output as sourced from changes in capital utilization. In the case of separation shocks, almost all the change in output is sourced from changes in capital utilization.

These findings suggest that a search model of capital reallocation exhibits strong internal propagation, and can generate substantial changes in factor utilization following aggregate shocks. In contrast, in an influential paper, Shimer (2005) has argued that the canonical search model of unemployment exhibits virtually no internal propagation, and has difficulty generating substantial changes in labor utilization. By adapting the canonical framework used to analyze unemployment dynamics to study capital reallocation dynamics, the analysis permits a direct comparison of the differences which give rise to the divergent results (section 4).⁴

The existing search-theoretic literature on asset markets includes Duffie, Gârleanu and Pedersen (2005), Lagos and Rocheteau (2007), Miao (2006), Rust and Hall (2003), Spulber (1996) and Weill (2007). A differentiating feature of my analysis is the interaction of investment decisions with endogenously determined liquidity. In a companion paper, Kim (2008), I analyze the interaction of liquidity and the selection of buyers and sellers in asset markets with search frictions. The analysis extends the canonical search models of unemployment pioneered by Diamond-Mortensen-Pissarides (summarized in Pissarides (2000)) to the context of capital reallocation. To facilitate comparison, I adopt their notation and language whenever it is deemed appropriate.⁵

Section 2 introduces the model. Section 3 conducts numerical simulations

⁴In an independent and related paper, Kurmann and Petrosky-Nadeau (2007) also study the role of search frictions in the context of capital reallocation. After highlighting the potential of search frictions in generating propagation dynamics, they report that these dynamics are quantitatively insignificant. In section 5, I highlight the different modelling strategies (and their justification) used in my formulation which give rise to these divergent results.

⁵In the analysis of unemployment, there are two distinct populations representing each side of the market (firms and workers). In the analysis of asset markets, agents switch from one side of the market to another (today's buyers are tomorrow's sellers).

and discusses results. Section 4 conducts comparisons with the search model of unemployment. Section 5 compares the results to those of the related work of Kurmann and Petrosky-Nadeau (2007). The last section concludes.

2. MODEL

All agents are risk-neutral and infinitely lived, with time preferences determined by a constant discount rate $r > 0$. As discussed in the Introduction, I define an asset as a business idea (or production plan) upon which physical capital can be invested in. The productivity (or earnings) of an owner-asset match is $\pi x f(k) - w, x \in \{0, 1\}$. $\pi > 0$ is the aggregate technology component of the match productivity, and $w \geq 0$ is the exogenous cost of other inputs used in production.⁶ Assume $f(k)$ is differentiable, $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$. k denotes the physical capital stock invested in the asset.

In related research (discussed further in Section 5), Kurmann and Petrosky-Nadeau (2007) model the productivity within matches as the *marginal* product of capital. They equate an asset with a unit of capital rather than a business idea or production plan in my formulation. This amounts to replacing the productivity of a match from $\pi f(k) - w$ in my formulation to $\pi f'(k)$. Thus, in their framework, following a technology shock on π , increased levels of capital accumulation \hat{k} mitigate the effect of the shock on the productivity within matches. In contrast, the capital investment decision in my analysis will increase the productivity within matches following a positive shock to π as discussed in more detail in Section 5.

With exogenous Poisson arrival rate λ , there is a draw of match specific productivity $x = 0$, which motivates the owner to cease production, and list the asset for sale. I refer to this case as unmatched. New matches have the productivity of the buyer-asset match which is constant at π . As in Duffie, Gârleanu and Pedersen (2005), agents can hold either 0 or 1 unit of the asset.

In each period, assets and the capital stock depreciate completely at rate δ , and there is an exogenous entry of new assets of measure δ . New assets are matched upon entry into the economy. New assets motivate investment in the asset at some level k . The population of assets is thus exogenous, and normalized at 1.

In the asset market, I consider outcomes when buyers can *direct* their search to sellers of assets of different capital stocks. Thus, there are potentially different submarkets for every level of k . Following search models of unemployment,

⁶Other inputs are sourced from frictionless spot markets. Later I relate w to the labor share of output.

there is a constant returns to scale match function with the stock of buyers and sellers as arguments. In submarket k , the Poisson arrival rate of matches per buyer is $Aq(\theta(k))$, $q'(\theta(k)) \leq 0$, where $\theta(k)$ is the ratio of buyers to sellers or "market tightness", and A governs the search efficiency. From the assumption of constant returns to scale, the Poisson arrival rate of matches per seller is $Am(\theta(k)) \equiv A\theta(k)q(\theta(k))$.

This determines the asset liquidity defined as the average speed of asset sale. The elasticity of the match function $\eta(\theta(k)) \equiv -\frac{q'(\theta)\theta}{q(\theta)} \in [0, 1]$, where the bounds are implied by the assumption of constant returns.

Let $c > 0$ denote the flow cost of search for a buyer. In submarket k , let $V(k)$ denote the value of being a buyer, $J(k)$ the value of being a matched owner, $U(k)$ the value of being an unmatched owner (an owner who has listed his asset for sale) and $P(k)$ the asset price. There is a free entry of buyers such that in every period and in each submarket $V(k) = 0$.

In steady states, there is a unique level of capital $k = \hat{k}$ invested in every asset. Using this, steady state value equations are given by

$$\begin{aligned} rV(\hat{k}) &= -c + Aq(\theta(\hat{k})) (J(\hat{k}) - P(\hat{k})) = 0, \\ rJ(\hat{k}) &= \pi f(\hat{k}) - w + \lambda (U(\hat{k}) - J(\hat{k})) - \delta J(\hat{k}), \\ rU(\hat{k}) &= Am(\theta(\hat{k})) (P(\hat{k}) - U(\hat{k})) - \delta U(\hat{k}). \end{aligned} \quad (1)$$

The flow to a buyer $rV(\hat{k})$ consists of the per period search cost and the capital gain $(J(\hat{k}) - P(\hat{k}))$ resulting from a match with a seller which occurs at rate $Aq(\theta(\hat{k}))$. The flow to a matched owner at capital stock \hat{k} , $rJ(\hat{k})$, consists of the per period net productivity $\pi f(\hat{k}) - w$, the expected capital gain $(U(\hat{k}) - J(\hat{k}))$ resulting from a separation shock which occurs at rate λ , and the capital loss due to depreciation at rate δ . The flow to an unmatched owner $rU(\hat{k})$, consists of the per period search cost plus the capital gain $(P(\hat{k}) - U(\hat{k}))$ resulting from the asset sale following a match with a buyer which occurs at rate $Am(\theta(\hat{k}))$, and the capital loss due to depreciation at rate δ .

A buyer-seller match determines the sale price $P(\hat{k})$ as the outcome of Nash bargaining. Let $\beta \in (0, 1)$ denote the exogenous bargaining share of sellers. The Nash bargaining rule implies

$$P(\hat{k}) = U(\hat{k}) + \beta (J(\hat{k}) - U(\hat{k})). \quad (2)$$

Let \tilde{k} denote the level of investment for new assets. For new assets, the investment decision in equilibrium is given by

$$\hat{k} = \arg \max J(\tilde{k}) - \tilde{k} = \arg \max \frac{1}{r + \delta} (\pi f(\tilde{k}) - w - \lambda S(\tilde{k})) - \tilde{k}. \quad (3)$$

Investment takes place upon asset entry and I assume it is instantaneous. Thus, I abstract away from time to build delays in new investments. The system of equations (1)–(3) solve for $\{J(\hat{k}), U(\hat{k}), P(\hat{k}), \theta(\hat{k}), \hat{k}\}$ given $\{\pi, w, c, r, \lambda, \delta, \beta\}$, $f(k)$ and the match function $Aq(\theta)$.

2.1. EQUILIBRIUM

From (1) and (2), the buyer "creation margin" is given by⁷

$$(r + \delta + \lambda)S(\hat{k}) = \frac{r + \delta + \lambda}{1 - \beta} \frac{c}{Aq(\theta(\hat{k}))} = \pi f(\hat{k}) - w - \frac{\beta}{1 - \beta} c\theta(\hat{k}). \quad (4)$$

This solves for market tightness $\theta(\hat{k})$, which is increasing in \hat{k} .

Assuming an interior equilibrium, the first order condition for capital accumulation from (3) and (4) is (derived in Appendix)

$$\pi f'(\hat{k}) - (r + \delta) = \frac{\lambda}{1 + \frac{\beta}{\eta} \frac{1}{r + \delta} Am(\theta(\hat{k}))} = \text{Liquidity premium}. \quad (5)$$

Since the left hand side is falling in $\theta(\hat{k})$, \hat{k} is increasing in $\theta(\hat{k})$. Thus, the interaction between liquidity through $\theta(\hat{k})$ and investment \hat{k} imply they are complements. Here, the difference between the marginal productivity of capital and the interest rate plus depreciation rate $\pi f'(\hat{k}) - (r + \delta)$ is the *liquidity premium*. An interior equilibrium with entry of assets requires $J(\hat{k}) - \hat{k} \geq 0$. The following is derived in the Appendix.

Lemma 1 (Interior equilibrium). *An interior equilibrium with entry of assets exists iff given $\theta(\hat{k}) \geq 0$ the capital accumulation decision \hat{k} implied by (5) satisfies*

$$\begin{aligned} \text{Condition 1} : \pi f(\hat{k}) - w - \lambda S(\hat{k}) &\geq (r + \delta)\hat{k} \\ &\Rightarrow \pi f(\hat{k}) - w - \pi f'(\hat{k})\hat{k} > 0. \end{aligned} \quad (6)$$

⁷Using

$$\begin{aligned} (r + \delta)U(\hat{k}) &= \frac{\beta}{1 - \beta} c\theta(\hat{k}), \\ (r + \delta + \lambda)S(\hat{k}) &= \pi f(\hat{k}) - w - \frac{\beta}{1 - \beta} c\theta(\hat{k}). \end{aligned}$$

I assume this condition is always satisfied, and in the quantitative analysis analyze outcomes where this is satisfied. Thus, if the production function is a constant returns to scale function of physical capital, business ideas and other inputs, this condition implies that some share of output must accrue to the business idea input. From (4), $\theta(k)$ is rising in π, A , and falling in c, r, λ . From (5), \hat{k} is rising in π, A and falling in r, δ, λ . The following comparative static results are derived in the Appendix.

Proposition 1 (Investment and liquidity). *The capital stock \hat{k} , and market tightness $\theta(\hat{k})$ are*

- (i) *increasing in match productivity π , search efficiency A and*
- (ii) *decreasing in search costs c , interest rate r , depreciation rate δ , and arrival rate of shocks λ .*

Given $\hat{k} < \infty$ from (5) and $S(\hat{k}) \geq 0$, the left hand side of equation (4) implies $\theta(\hat{k})$ has a finite upper limit. Given this, the expression for the match surplus implies the following:

Lemma 2 (Frictionless limit). *As search frictions disappear $A \rightarrow \infty$, the match surplus $S(\hat{k}) \rightarrow 0$.*

This result is useful when comparing our results to an economy where search frictions are absent. Recall, the difference between the marginal productivity of capital and the interest rate plus depreciation rate $\pi f'(\hat{k}) - (r + \delta)$ is the *liquidity premium*. Using (5) and Proposition 1 we can summarize comparative statics for the liquidity premium.

Proposition 2 (Liquidity premium). *The liquidity premium is*

- (i) *decreasing in match productivity π , search efficiency A and*
- (ii) *increasing in search costs c , interest rate r , depreciation rate δ , and arrival rate of shocks λ .*

Thus, the liquidity premium co-moves with the equilibrium capital stock \hat{k} , and the arrival rate of buyers $m(\theta(\hat{k}))$ following shocks to $\{\pi, A, c, r, \delta, \lambda\}$. From Lemma 2, the premium goes to zero as search frictions disappear, $A \rightarrow \infty$.

Using (1) and (2) the sales price is given by⁸

$$P(\hat{k}) = \frac{1}{r + \delta} [\pi f(\hat{k}) - w] - \left[\frac{\lambda}{r + \delta} + (1 - \beta) \right] S(\hat{k}). \quad (7)$$

Combining this with Lemma 2, we can deduce the presence of search frictions lowers the sale price of assets.

⁸Using $P(\hat{k}) = U(\hat{k}) + \beta (J(\hat{k}) - U(\hat{k})) = J(\hat{k}) - (1 - \beta)S(\hat{k})$.

2.2. CAPITAL UTILIZATION AND AGGREGATE OUTPUT

Let $u(\hat{k})$ denote the utilization rate of assets which coincides with the utilization rate of capital (since each asset has the same capital stock \hat{k}). The evolution of this rate is given by

$$\dot{u}(\hat{k}) = \delta + Am(\theta(\hat{k}))(1 - u(\hat{k})) - \lambda u(\hat{k}) - \delta u(\hat{k}). \quad (8)$$

Inflows into $u(\hat{k})$ occur when new assets enter the economy and when unmatched assets are matched at rate $Am(\theta(\hat{k}))$, while outflows occur with separation shocks or when assets (and their capital) depreciate. Let $u_{ss}(\hat{k})$ denote the steady state utilization rate which is given by

$$u_{ss}(\hat{k}) = \frac{\delta + Am(\theta(\hat{k}))}{\lambda + \delta + Am(\theta(\hat{k}))}. \quad (9)$$

This is less than 1 in the presence of search frictions ($Am(\theta(\hat{k})) < \infty$) and separation shocks ($\lambda > 0$). $1 - u(\hat{k})$ denotes the measure of asset sellers (unmatched owners). Meanwhile, the stock of buyers is given by $\theta(\hat{k})(1 - u(\hat{k}))$. Using Proposition 1:

Proposition 3 (Capital utilization rate). *The steady state capital utilization rate, $u_{ss}(\hat{k})$ is*

- (i) *increasing in match productivity π , search efficiency A and*
- (ii) *decreasing in search costs c , interest rate r , depreciation rate δ , and arrival rate of shocks λ .*

Thus, from Proposition 1 and 2, the utilization rate co-moves positively with investment \hat{k} , and asset liquidity $Am(\theta(\hat{k}))$, and inversely with the liquidity premium. Combining (8) and (9) we can determine the speed by which the utilization rate adjusts to its new steady state level following shocks to $\lambda, \delta, Am(\theta(\hat{k}))$ (induced by exogenous variables specified in Proposition 3). Let $\dot{u}_0(\hat{k})$ denote the change in capital utilization rate *in the period following a shock*. Then the ratio $\frac{\dot{u}_0(\hat{k})}{\Delta u_{ss}(\hat{k})}$ is the share of the overall change in steady state utilization rates realized in the period following the shock. This measure is determined as follows (shown in Appendix).

Proposition 4 (Adjustment of utilization rate). *Following shocks to $\lambda \rightarrow \lambda'$, $\delta \rightarrow \delta'$, or $Am(\theta(\hat{k})) \rightarrow Am(\theta(\hat{k}))'$*

$$\frac{\dot{u}_0(\hat{k})}{\Delta u_{ss}(\hat{k})} = \lambda' + \delta' + Am(\theta(\hat{k}))'. \quad (10)$$

Thus, by this measure, the *level* of the transition rates associated with search frictions determine the *speed* by which adjustment to new steady state utilization rates occur. In particular, gradual adjustment rates which generate serially correlated changes in utilization rates (propagation dynamics) follow when match and separation rates are low. This insight will drive the empirically finding that search frictions associated with capital reallocation can generate strong propagation dynamics.

Asset turnover rates (the share of assets traded per period) is given by the measure of matches occurring per period $Turnover = Am(\theta(\hat{k}))(1 - u(\hat{k}))$. In steady states, this is given by

$$Turnover = Am(\theta(\hat{k})) \frac{\lambda}{\lambda + \delta + Am(\theta(\hat{k}))} \quad (11)$$

which is rising in asset liquidity $Am(\theta(\hat{k}))$. Asset and capital reallocation are measured by the product of the turnover rate and the average sales price $P(\hat{k})$ which is given by (7).

Aggregate output is given by

$$Y = \pi f(\hat{k}) u(\hat{k}). \quad (12)$$

Changes in aggregate output are sourced from changes in productivity within matches and changes in the utilization rate of assets. Aggregate investment I is given by $I = \hat{k}\delta$. The aggregate capital stock K (valued at replacement cost of 1) is given by $\dot{K} = I - \delta K$. In steady states, this is given by $K = \hat{k}$. Combining with (12), the aggregate capital-output ratio in steady state is given by

$$\frac{K}{Y} = \frac{\hat{k}}{\pi f(\hat{k}) u(\hat{k})}. \quad (13)$$

Thus, after correcting for capital utilization $u(\hat{k})$, the aggregate capital-output ratio equals that within matches.

2.3. EFFICIENCY

Given our specification of *directed* search, investment levels are efficient as long as search externalities are internalized.⁹ As is well known, search externalities are internalized under the Hosios (1990) condition $\beta = \eta$. The argument is

⁹In an environment where buyers cannot direct their search to sellers of different k , the investment decision does not internalize the effect on liquidity and is not efficient.

as follows. An externality from investment accrues to future buyers which the owner can be matched with, given by

$$(1 - \beta)m(\theta(\hat{k})) \frac{dS(\hat{k})}{d\hat{k}} = (1 - \beta)\eta \frac{m(\theta(\hat{k}))}{\theta(\hat{k})} S(\hat{k}) \frac{d\theta(\hat{k})}{d\hat{k}}.$$

Given the identity of future potential buyers is not known, contracts cannot be written and this creates an investment hold-up problem. However, under the assumption of directed search, sellers indirectly internalize this externality through a greater likelihood of meeting buyers, given by

$$\beta m'(\theta(\hat{k})) S(\hat{k}) \frac{d\theta(\hat{k})}{d\hat{k}} = \beta(1 - \eta) \frac{m(\theta(\hat{k}))}{\theta(\hat{k})} S(\hat{k}) \frac{d\theta(\hat{k})}{d\hat{k}}.$$

These two expressions are equal under the Hosios condition. From (4), $\theta(k)$ is decreasing in β . The Hosios condition maximizes $\beta m(\theta(\hat{k}))$, and from (5) capital stock \hat{k} is lower when $\beta \neq \eta$. Summarizing:

Proposition 5 (Efficiency). *Under the Hosios condition $\beta = \eta$,*

- (i) *decentralized outcomes are socially efficient,*
- (ii) *when $\beta > \eta$ ($\beta < \eta$), market tightness $\theta(\hat{k})$ is low (high) relative to the social optimum, and*
- (iii) *when $\beta \neq \eta$, \hat{k} is lower than the social optimum.*

Under the Hosios condition, the positive externality of buyers and the negative externality of sellers (to the existing stock of sellers) cancel out. A higher seller bargaining power lowers asset liquidity relative to the social optimum, and investment is always lower than the optimum when the Hosios condition is not satisfied.¹⁰

3. CALIBRATION

I now turn to a quantitative assessment of the model. The production function is set as $\pi f(k) = \pi k^\alpha$. Following the unemployment literature I set $q(\theta) = \theta^{-\eta}$. $\eta \in [0, 1]$ determines the elasticity of the match function. The annual rate of depreciation is set at $\delta = 0.1$, and the output within matches is normalized to $y \equiv \pi \hat{k}^\alpha = 1$. I interpret w as the labor share of output and set this at $w = 0.67$.

¹⁰See Acemoglu and Shimer (1997) for a similar argument in the context of the search unemployment model.

Using (9) and (11) and rearranging pins down the match rate $A\theta (\hat{k})^{1-\eta}$, and separation rate λ , as follows

$$A\theta (\hat{k})^{1-\eta} = \frac{\text{Turnover}}{1 - u(\hat{k})}, \quad (14)$$

$$\lambda = \frac{\text{Turnover}}{u(\hat{k})} + \delta \frac{u(\hat{k})}{1 - u(\hat{k})}.$$

Eisfeldt and Rampini (2006) report annual turnover rates of 1.4%–5.5%. These estimates are consistent with other estimates by Ramey and Shapiro (1998) and Maksimovic and Phillips (2001). I target a turnover rate at the top of this range at $\text{Turnover} = 0.055$. A key finding of this paper is that matches occur rarely in the context of capital reallocation. From (14), using a higher average turnover raises the calibrated match rate and disciplines our results.¹¹

Following Burnside and Eichenbaum (1996), the benchmark capital utilization rate is equated to the long run *capacity* utilization rates calculated by the Federal Reserve of $u(\hat{k}) = 0.81$. The Federal Reserve Board’s capacity indexes attempt to capture the concept of sustainable maximum output – “the greatest level of output a plant can maintain within the framework of a realistic work schedule, after factoring in normal downtime and assuming sufficient availability of inputs to operate the *capital* in place”.¹²

The resulting $A\theta (\hat{k})^{1-\eta} = 0.29, \lambda = 0.09$. Once an asset (and its capital stock) is unmatched, it takes a long time for matches with buyers to occur, of the order of $\frac{1}{0.29} \simeq 3$ years on average. Separation shocks are rare and occur about as frequently than depreciation shocks.¹³

In the search unemployment literature, annual turnover rates are 37%, and utilization rates of the laborforce are 0.93. From Shimer (2005), these imply separation shocks occur at a frequency of 0.4 per year, and matches with firms by unemployed workers occur at a frequency of 5.4 per year on average.¹⁴ In particular, the match rate is an order of magnitude different from that associated with capital reallocation. Note that this is largely the result of the fact that capital turnover rates are an order of magnitude lower than labor turnover rates.

¹¹Results using turnover rates in the midpoint of this range $\text{Turnover} = 0.0345$ are reported in the Appendix.

¹²See <http://www.federalreserve.gov/releases/g17/Current/table11.htm>.

¹³In their study of the aerospace industry, Ramey and Shapiro (1998) report that 70% of capital separations are due to depreciation and 30% are due to reallocation. This is broadly in line with my specification.

¹⁴In the unemployment setting entry and exit is typically ignored so $\delta = 0$.

Table 1: Outcomes for various β, η

β	0.9		0.75		0.5		0.25		0.1	
η	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05
α	0.61	0.53	0.56	0.49	0.46	0.38	0.34	0.25	0.25	0.14
r	15.9%	15.9%	13.2%	13.2%	8.1%	8.1%	1.6%	1.6%	-3.8%	-3.8%
<i>Liqu.prem.</i>	4.3%	0.4%	4.5%	0.5%	4.8%	0.5%	5.4%	0.7%	6.0%	0.9%

$$\delta = 0.1, \pi \hat{k}^\alpha = 1, w = 0.67, \hat{k} = 2.025, \lambda = 0.09, A\theta (\hat{k})^{1-\eta} = 0.29.$$

In the discussion below, I show that this difference implies that search frictions associated with capital reallocation are able to generate a propagation and amplification of shocks on aggregate output in a way that search frictions associated with unemployment dynamics cannot.

The capital-output ratio is targeted at

$$\frac{\hat{k}}{\pi f(\hat{k}) u(\hat{k})} = 2.5. \quad (15)$$

This and the utilization rate pins down $\hat{k} = 2.025$. I target a ratio of reallocation to investment of 0.28 reported by Eisefeldt and Rampini (2006). The ratio of reallocation to investment is

$$\frac{\text{Reallocation}}{\text{Investment}} = \frac{P(\hat{k}) \times \text{Turnover}}{\delta \hat{k}} = 0.28. \quad (16)$$

For a given pair of β, η this last restriction (16), and equilibrium equations (4) and (5), pin down the remaining variables: capital share α , interest rate r , and search costs per seller $c\theta(\hat{k})$.¹⁵ I experiment over a range of values for β and η which satisfy the restrictions specified above. I report statistics of interest for β, η pairs which satisfy these restrictions in Table 1. To save space I report results for very high and low values for η associated with a particular β .

The implied share of capital α is sensitive to the choice of the seller bargaining share β . The interest rate r is also sensitive to the choice of β . The size of the liquidity premium is sensitive to the choice of η , with lower η associated with lower liquidity premia.

Condition 1 ($J(\hat{k}) - \hat{k} \geq 0$ for an equilibrium with asset entry) is satisfied iff

$$y - w - \lambda S(\hat{k}) > (r + \delta)\hat{k}.$$

¹⁵The average level of $\theta(\hat{k})$ is meaningless for the numerical exercise. Given $c\theta(\hat{k})$ we can pick c to target any average buyer to seller ratio $\theta(\hat{k})$.

Since values for $\delta, y = \pi \hat{k}^\alpha, w$ are given, and $\hat{k}, \lambda, A\theta(\hat{k})^{1-\eta}$ are determined by (14) and (15), equations (4), (16) and the choice of β pin down the choice of $r, c\theta(\hat{k})$ which pin down the magnitude of $S(\hat{k})$. Thus, given the parameterization strategy, the choice of η has no effect on whether Condition 1 is satisfied. Moreover, given this strategy there is a critical level of β below which Condition 1 is satisfied, given by $\beta \leq 0.22$. I verify this restriction is satisfied in the final parameter specification.

3.1. OUTPUT DEVIATIONS

For each β, η specification, I consider the percentage deviation in technology π , and resulting percentage deviation in output within matches y (Δy) that is required to generate an increase in steady state aggregate output of 2% (which corresponds to the standard deviation of output from trend for the post-war U.S.). Such changes capture exogenous shocks to technology π only. I hold the labor share w constant in these simulations to mimic a countercyclical labor share of output.

In each case, I also calculate the associated percentage increase in steady state reallocation levels ($\Delta Realloc.$), by expressing the level of reallocation as a fraction of the level of reallocation before the deviation in productivity. I do the same for the percentage change in capital utilization rates ($\Delta u_{ss}(\hat{k})$) also. From (12), the sum of $\Delta y + \Delta u(\hat{k}) = 2\%$ by construction. These statistics are reported in Table 2.

Table 2: Technology shocks which reduce aggregate output by 2%.

β	0.9		0.75		0.5		0.25		0.1	
η	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05
$\Delta\pi$	0.7%	0.5%	0.8%	0.5%	1.0%	0.5%	1.2%	0.6%	1.4%	0.5%
Δy	1.92%	1.1%	1.92%	1.0%	1.92%	1.0%	1.92%	0.9%	1.92%	0.7%
$\Delta Realloc.$	6.1%	6.8%	6.2%	6.7%	6.2%	8.1%	6.4%	11.8%	6.5%	13.0%
$\Delta u_{ss}(\hat{k})$	0.08%	0.9%	0.08%	1.0%	0.08%	1.0%	0.08%	1.1%	0.08%	1.3%

Eisfeldt and Rampini (2006) report a percentage deviation of capital reallocation in trend versus boom of 15% to 23%.¹⁶ Using a variety of different

¹⁶Eisfeldt and Rampini (2006) report a ratio of capital reallocation in booms vs. recessions of 1.6. This implies a target relative reallocation of in trend versus boom of 23%. The contemporaneous correlation of output with reallocation is 0.637. The product of unconditional percentage deviation and correlation with output is $23\% \times 0.637 = 15\%$. This implies a target relative reallocation in trend versus boom of 15% based on contemporaneous correlation.

measures, they also report that the dispersion in capital productivity is *countercyclical* which suggests the costs of reallocation are countercyclical.

The choice of β, η turns out not to affect much the performance of the model to generate changes in reallocation. Lower levels of β, η are associated with a slightly stronger response of reallocation.

However, the choice of η affects the degree of response in capital utilization. In particular, very low values of η are associated with changes in steady state capital utilization of the magnitude observed with changes in aggregate output (empirically, fluctuations in capacity utilization rates are roughly similar and highly correlated with fluctuation in output).

Next I consider the exogenous percentage deviation in match separation rates λ ($\Delta\lambda$) that is required to generate an increase in steady state aggregate output of 2%. Here I hold π constant. Despite this, recall that match productivity y will be affected by changing λ through endogenous changes in \hat{k} . The resulting statistics are reported in Table 3.

Table 3: Separation shocks which reduce aggregate output by 2%.

β	0.9		0.75		0.5		0.25		0.1	
η	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05
$\Delta\lambda$	-7.0%	-9.7%	-7.3%	-9.5%	-6.2%	-8.8%	-7.9%	-4.7%	-7.9%	-2.7%
Δy	1.1%	0.2%	1.0%	0.2%	1.0%	0.2%	0.9%	0.2%	0.9%	0.2%
$\Delta Realloc.$	0.3%	-2.7%	0.2%	-2.1%	0.1%	-0.3%	0.5%	3.8%	1.6%	9.9%
$\Delta u_{ss}(\hat{k})$	0.9%	1.8%	1.0%	1.8%	1.0%	1.8%	1.1%	1.8%	1.1%	1.8%

Again higher levels of deviation in utilization rates result from lower η . A key difference is that now, at low η , the variation in capital utilization approaches that of output.

The results of Tables 1-3 suggest that calibrated interest rates r are largely determined by the choice of β , and realistic changes in capital utilization are generated by low η for both technology and separation shocks.

I now restrict the choice of β, η as follows. I match a sum of $r + Liquidity\ premium = 0.04 - 0.0175$ (interest rate minus long term growth rate of output used in Eisfeldt and Rampini (2006)), and a level of $r = 0.02 - 0.0175$ (typical value of riskless rate minus long term growth rate).¹⁷ These imply $\eta = 0.165$,

¹⁷The theoretical analysis assumed no long term growth. The model with steady state growth does not affect any of the equilibrium equations and only implies a reinterpretation of the interest rate as the interest rate net of long term growth, and a reinterpretation of w as the sum of labor and new investment within matches following steady state technology growth. Ignoring growth and

Table 4: Benchmark specification.

β	η	α	r	λ	$A\theta^{1-\eta}$	$c\theta$	w	δ	π
0.21	0.17	0.25	0.0025	0.09	0.29	0.30	0.67	0.1	0.82

and $\beta = 0.207$. The choice of β implies Condition 1 is satisfied since $\beta \leq 0.22$. The associated benchmark outcomes are reported in Table 4.

I assess the model under these restrictions for β, η for the rest of the analysis. The implied elasticity of the match rate with respect to match productivity is $\frac{dAm(\theta)}{dy} \frac{y}{Am(\theta)} = 7.0$. The implied elasticity of the match rate with respect to the separation rate is $\frac{dAm(\theta)}{d\lambda} \frac{\lambda}{Am(\theta)} = -1.7$. From (4) these elasticities are given by the following expressions

$$\frac{dAm(\theta)}{dy} \frac{y}{Am(\theta)} = (1 - \eta) \frac{1}{1 - \frac{w}{y}} \frac{\beta Am(\theta) + (r + \delta + \lambda)}{\beta Am(\theta) + \eta (r + \delta + \lambda)}, \quad (17)$$

$$\begin{aligned} \frac{dAm(\theta)}{d\lambda} \frac{\lambda}{Am(\theta)} &= -(1 - \eta) \frac{\lambda}{\beta Am(\theta) + \eta (r + \delta + \lambda)} \\ &+ \frac{dAm(\theta)}{dy} \frac{y}{Am(\theta)} \times \frac{dy}{d\lambda} \frac{\lambda}{y}. \end{aligned} \quad (18)$$

The absolute value of both elasticities is decreasing in η and $Am(\theta)$. Note that $\frac{dy}{d\lambda} \frac{\lambda}{y} < 0$ from through endogenous changes in \hat{k} .

The percentage change in the rate of capital utilization can be decomposed into changes due to the match rate and changes due to the separation rate from (9) as follows

$$\frac{du_{ss}(\hat{k})}{u_{ss}(\hat{k})} = (1 - u_{ss}(\hat{k})) \left[\frac{Am(\theta(\hat{k}))}{\delta + Am(\theta(\hat{k}))} \frac{dAm(\theta(\hat{k}))}{Am(\theta(\hat{k}))} - \frac{d\lambda}{\lambda} \right]. \quad (19)$$

This percentage change is decreasing in the level of the utilization rate $u_{ss}(\hat{k})$. An increase in technology π raises utilization via changes in the match rate. Meanwhile, a fall in the separation rate λ will have a direct and indirect effect on raising the utilization rate via endogenous changes in the match rate from Proposition 1. Combining these with (17) allows us to assess the quantitative role of technology and separation shocks on utilization rates.

setting the depreciation rate $\delta = 0.1 - 0.0175$ yields the same numerical results.

A 1.0% increase in match productivity (induced from an exogenous increase in π of 0.5%) is associated with a 2% increase in output and a 1.0% increase in capital utilization in steady states. All the change in capital utilization is induced by endogenous changes in the match rate.

An exogenous -3.8% fall in the separation rate λ is associated with a 2% increase in output and a 1.55% increase in capital utilization in steady states. In this case, about half the change in capital utilization is induced by endogenous changes in the match rate.

3.2. PROPAGATION

Here I first consider the transition path of match productivity, output and capital utilization following a sequence of unexpected shocks to technology π only. I then consider a sequence of unexpected shocks to the separation rate λ only.

The simulations I conduct are as follows. For each type of shock I assume the shock is unexpected and once it arrives in period zero, is expected to last forever. After 60 months I introduce a second unexpected shock which reverts the original shock back to its level before period zero. Again I assume agents in

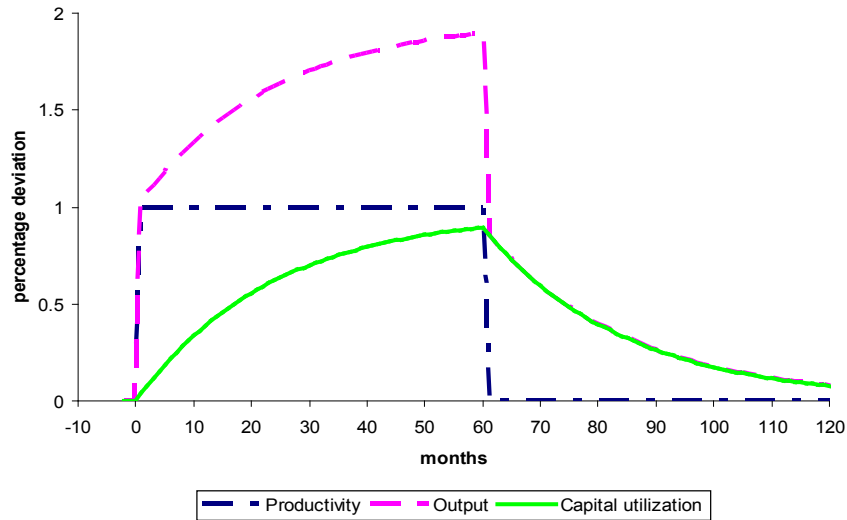


Figure 2: Response of match productivity, output and capital utilization rate to 0.5% increase then decrease in technology π .

the economy expect no future shocks. I analyze shocks associated with a change in steady state output of 2% under the specification of parameters in Table 4. The arrival rate of shocks at annual frequencies analyzed above, are converted to monthly frequencies by dividing through by 12. I assume that following each of these shocks the adjustment of match productivity $y \equiv \pi \hat{k}^\alpha$ is instantaneous.¹⁸

Figure 2 shows the resulting paths of match productivity, output and capital utilization following the sequence of $\pm 0.5\%$ technology shocks to π . Following the associated 1.0% shock to match productivity y , there is a gradual adjustment of capital utilization which results in serially correlated changes in output. Using Proposition 4, it takes about 13 months for capital utilization to adjust to 50% of its overall change to its new steady state level. The full response of output substantially lags that of productivity. Once the shock reverts to its original level in month 60, output and utilization are persistently above the initial level for several years.

Figure 3 shows the resulting paths of match productivity, output and capital utilization following the sequence of $\pm 3.8\%$ separation shocks to λ . In this case, the associated shock to productivity y is only 0.45%. The gradual adjustment of capital utilization dominates the movement of output which is of a similar magnitude. Again, the full response of output substantially lags that of productivity. The separation shock has a persistent effect on output and utilization long after it has disappeared.

I conclude that the search model of capital reallocation is able to generate strong internal propagation following aggregate shocks. Cogley and Nason (1995) document that existing models of capital adjustment costs or time-to-build delays cannot capture sufficient propagation of aggregate shocks at business cycle frequencies. In contrast, modelling capital adjustment costs explicitly as search frictions is able to generate sizable propagation.

¹⁸Because capital accumulation within matches is instantaneous, the *upward* adjustment of match productivity is instantaneous. The *downward* adjustment of match productivity following a reversion of shocks to their original level at 60 months would be gradual since depreciation of capital is gradual. However, in a growth environment negative shocks would be associated with a slower growth of the capital stock rather than an absolute reduction of this stock within matches, and the adjustment of match productivity would then be instantaneous. To mimic this, and to highlight the gradual adjustment of utilization, I assume the downward adjustment of match productivity is instantaneous also.

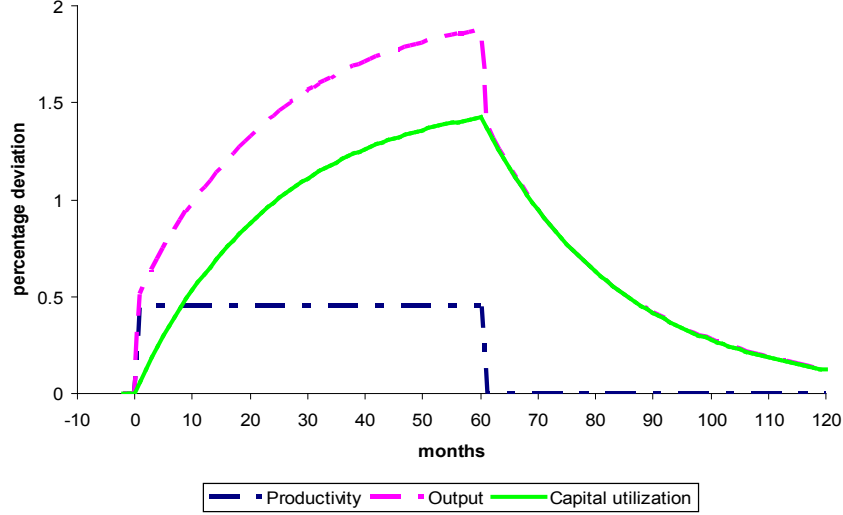


Figure 3: Response of match productivity, output, and capital utilization rate to 3.8% decrease then increase in separation rate λ .

3.3. LABOR INPUT

Let l denote the labor input per matched asset, and ϖ the wage. The labor share $\frac{w}{y} = \frac{\varpi l}{y}$. Output per labor input or aggregate labor productivity is

$$\frac{Y}{L} = \frac{y}{l}. \quad (20)$$

Let $\chi \equiv \frac{d\varpi}{dY/L} \frac{Y/L}{\varpi}$ denote the wage elasticity of wages with respect to labor productivity. This implies

$$\begin{aligned} \frac{\dot{Y}/L}{Y/L} &= \frac{1}{1-\chi} \left(\frac{\dot{y}}{y} - \frac{\dot{w}}{w} \right), \\ \frac{\dot{L}}{L} &= \frac{\dot{y}}{y} - \frac{\dot{Y}/L}{Y/L} + \Delta u(\hat{k}). \end{aligned} \quad (21)$$

In the analysis above we set $\frac{\dot{w}}{w} = 0$, to mimic the countercyclical labor share. Given this, the multiple by which changes in match productivity translate into changes in labor productivity depends on the magnitude of the wage elasticity

χ . When wages are inelastic, $\chi = 0$, innovations to match productivity coincide with innovations to labor productivity $\frac{\dot{y}}{y} = \frac{\dot{Y/L}}{Y/L}$, and the innovation to labor inputs coincide with innovations to capital utilization $\frac{\dot{l}}{l} = \Delta u(\hat{k})$.

The wage elasticity is a subject of ongoing debate with micro-econometric evidence pointing to elasticities in the range of $\chi \in [0, 0.5]$. In data, the variation in labor input is similar in magnitude to the variation in aggregate output and highly correlated at business cycle frequencies. The variation in labor input is closest to that of output when $\chi = 0$. For technology shocks, using $\chi = 0.5$ instead reduces the variation in labor input from 1.0% to 0.1% associated with a change in output of 2%. For separation shocks, using $\chi = 0.5$ instead reduces the variation in labor input from 1.55% to 1.1% associated with a change in output of 2%: a much smaller change. I conclude that overall the search model of capital reallocation is able to generate large changes in the utilization of labor inputs.

4. COMPARISON WITH SEARCH MODEL OF UNEMPLOYMENT

The relevance of these results is highlighted by comparing with similar implications in the context of search frictions associated with unemployment (employment reallocation). Following the search unemployment literature, I take labor productivity p as exogenous, and to facilitate comparison with our simulation of technology shocks above, assume that a 1.0% increase in labor productivity is associated with a 2% increase in steady state aggregate output generated through a 1.0% increase in the employment rate. The change in the employment rate e is given by

$$\dot{e} = \mu(1 - e) - \phi e, \quad (22)$$

where μ is the match rate of the unemployed and ϕ is the separation rate of the employed. Shimer (2005) reports long run annual averages of $\phi = 0.4$, $\mu = 5.4$, which imply long run $e = 0.93$. To generate an increase in e of 1.0% requires an increase in μ of 16.7%.¹⁹

The simulation I conduct is as follows. I assume the labor productivity shock, and resulting immediate increase in match rate μ , is unexpected and once it arrives in period zero, is expected to last forever. After 60 months I introduce a second unexpected shock which reverts the original shock (and resulting match rate μ) back to its level before period zero. Again I assume agents in the economy expect no future shocks.

¹⁹This implies an elasticity of the match rate with respect to labor productivity of 16.7 which is somewhat higher, but similar to what Shimer (2005) deduces from the data.

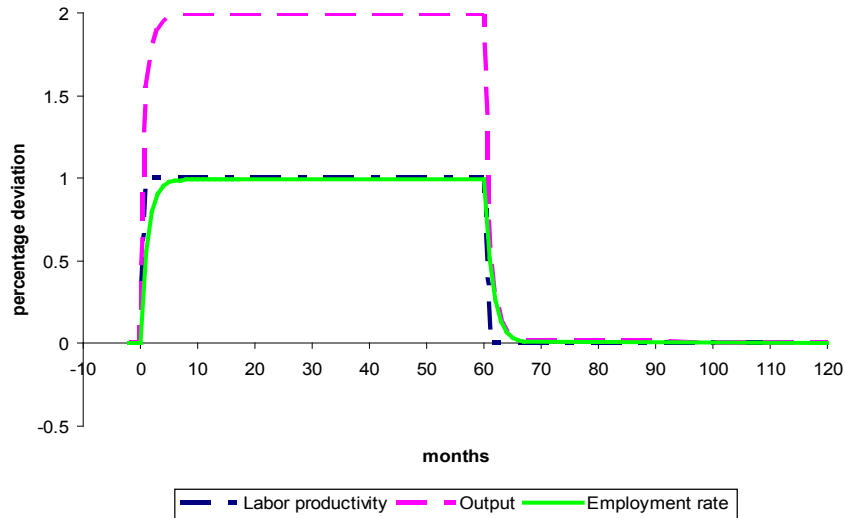


Figure 4: Response of output and employment rate to 1% increase then decrease in labor productivity in search model of unemployment.

Figure 4 shows the resulting paths of labor productivity, output and the employment rate following the shock to labor productivity. As with capital reallocation, the model generates amplification of productivity shocks on output through changes in the utilization of inputs. However, there is virtually no serial correlation in output following shocks, and at quarterly frequencies, output, employment rate and productivity shocks are perfectly correlated. Using Proposition 4, it takes about 1 month for capital utilization to adjust to 50% of its overall change to its new steady state level. This is what leads Shimer (2005) to famously conclude that the canonical Mortensen-Pissarides search model of unemployment cannot generate any internal propagation.²⁰

The absence of propagation is attributed to the speed by which adjustments to the steady state employment rate is made. From Proposition 4, the latter is determined by the sum of the separation rate and match rate. This is much higher in the search unemployment setting than the capital reallocation setting. The

²⁰Shocks to the separation rate are not considered in the unemployment literature for two reasons: (i) the associated change in match rates is too small, and (ii) the prediction that unemployment and vacancies are positively correlated is counterfactual. Both these conclusions do not necessary follow in the analysis of capital reallocation because lower separation rates positively affect the level of match productivity through endogenous changes in investment \hat{k} .

source of this difference is mainly the order of magnitude difference in match rates (5.4 versus 0.29) which were implied by the order of magnitude difference in observed turnover rates (37% versus 5.5%).

In equilibrium equation (4), suppose we set $\delta = 0$ and relabel $\pi f(\hat{k}) - w$ as $p - z$ which we will refer to as the labor productivity net of the utility of unemployment. Suppose further that we think of β as the bargaining share of workers, c as the search cost facing firms who search for unemployed workers. Then this equation coincides with the equilibrium equation for the vacancy to unemployment ratio θ in the canonical search model of unemployment. After this relabelling, we can use expressions (17) and (19) to compare the performance of the unemployment model in generating an amplification of shocks.

Specifically, we assumed above that a 1.0% increase in labor productivity p generates a 1.0% increase in e (implying an elasticity of $\frac{de}{dp} \frac{p}{e} = 1.0$) to mimic the performance of the capital reallocation model. Combining (17) and (19), the elasticity of the employment rate with respect to labor productivity is

$$\begin{aligned} \frac{de}{dp} \frac{p}{e} &= (1-e)(1-\eta) \frac{1}{1-\frac{z}{p}} \frac{\beta\mu + (r+\phi)}{\beta\mu + \eta(r+\phi)} \\ &= 0.07(1-\eta) \frac{1}{1-\frac{z}{p}} \frac{\beta 5.4 + (0.04 + 0.44)}{\beta 5.4 + \eta(0.04 + 0.44)}. \end{aligned}$$

Using the same $\beta = 0.21$, $\eta = 0.17$ and setting $\frac{z}{p} = \frac{w}{y} = 0.67$ as in our specification for capital reallocation, implies an elasticity of $\frac{de}{dp} \frac{p}{e} = 0.08$ which is much lower. $\frac{z}{p} = 0.67$ coincides with commonly used values in the unemployment literature (see Hall (2007)). Examining (17) and (19), this weaker performance is sourced from empirically higher match rates for unemployed workers than the match rate for unmatched capital (5.4 versus 0.29), and average employment rates which are higher than average capital utilization rates (0.93 versus 0.81).²¹

For the separation rate, combining (17) and (19), for the elasticity of the

²¹The calibration of the search unemployment model is the subject of ongoing debate. Hagedorn and Manovskii (2008) calibrate a ratio of $\frac{z}{p}$ close to 1 to generate larger amplification in the search unemployment model. Likewise, were we to introduce some benefit $b > 0$ accruing to unmatched asset owners (such as an option value of holding onto unmatched assets), equation (17) is modified to

$$\frac{dAm(\theta)}{dy} \frac{y}{Am(\theta)} = (1-\eta) \frac{1}{1-\frac{w}{y}-b} \frac{\beta Am(\theta) + (r+\delta+\lambda)}{\beta Am(\theta) + \eta(r+\delta+\lambda)},$$

which could potentially increase amplification substantially further.

employment rate with respect to the separation rate

$$\begin{aligned} \frac{de}{d\lambda} \frac{\lambda}{e} &= -(1-e) \left[(1-\eta) \frac{\lambda}{\beta\mu + \eta(r+\phi)} + 1 \right] \\ &= -0.07 \left[(1-\eta) \frac{0.4}{\beta 5.4 + \eta(0.04 + 0.4)} + 1 \right]. \end{aligned}$$

Using the same $\beta = 0.21$, $\eta = 0.17$ as in our specification for capital reallocation implies an elasticity of $\frac{de}{d\lambda} \frac{\lambda}{e} = -0.09$. In contrast, the elasticity of the utilization rate implied by the capital reallocation dynamics is -0.41 . Overall, these results show that a search model of capital reallocation can generate much larger amplification following aggregate shocks.

5. KURMANN AND PETROSKY-NADEAU (2007)

It is instructive to compare results with the modelling of search frictions and capital reallocation of Kurmann and Petrosky-Nadeau (2007). Kurmann and Petrosky-Nadeau (2007) model the productivity within matches as the *marginal* product of capital. They equate an asset with a unit of capital rather than a business idea or production plan in my formulation. This amounts to replacing the productivity of a match from $\pi f(\hat{k}) - w$ in my formulation to $\pi f'(\hat{k})$. Thus, in their framework, following a technology shock on π , increased levels of capital accumulation \hat{k} mitigate the effect of the shock on the productivity within matches.

Using the Cobb-Douglas production function $y = \pi \hat{k}^\alpha$, they and I both adopt, the elasticity of productivity within matches with respect to technology shocks in their framework is²²

$$\begin{aligned} \frac{d\pi f'(\hat{k})}{d\pi} \frac{\pi}{\pi f'(\hat{k})} &= \frac{\alpha \hat{k}^{\alpha-1} + \pi \alpha (\alpha - 1) \hat{k}^{\alpha-2} \frac{d\hat{k}}{d\pi}}{\alpha \hat{k}^{\alpha-1}} \\ &= \left(1 - (1 - \alpha) \frac{d\hat{k}}{d\pi} \frac{\pi}{\hat{k}} \right) = \frac{dy}{d\pi} \frac{\pi}{y} - \frac{d\hat{k}}{d\pi} \frac{\pi}{\hat{k}}. \end{aligned}$$

y has the interpretation of output per worker in Kurmann and Petrosky-Nadeau (2007). In practice, this elasticity is small since $\frac{dy}{d\pi} \frac{\pi}{y} \simeq \frac{d\hat{k}}{d\pi} \frac{\pi}{\hat{k}}$. As a result, Kurmann and Petrosky-Nadeau (2007) fail to find an important quantitative role of search frictions as a propagation mechanism. In particular, they find this conclusion is robust to the choice of β, η in their analysis.

²²Using $\frac{dy}{d\pi} \frac{\pi}{y} = 1 + \alpha \frac{d\hat{k}}{d\pi} \frac{\pi}{\hat{k}}$ from the Cobb-Douglas specification.

In my formulation, the elasticity of productivity within matches is given by

$$\frac{d(\pi \hat{k}^\alpha - w)}{d\pi} \frac{\pi}{\pi \hat{k}^\alpha - w} = \frac{\hat{k}^\alpha + \alpha \pi \hat{k}^{\alpha-1} \frac{d\hat{k}}{d\pi}}{\pi \hat{k}^\alpha - w} \pi = \frac{1}{\left(1 - \frac{w}{\pi \hat{k}^\alpha}\right)} \frac{dy}{d\pi} \frac{\pi}{y}.$$

Since the labor share $\frac{w}{\pi \hat{k}^\alpha} = 0.67$, this elasticity is about three times the elasticity $\frac{dy}{d\pi} \frac{\pi}{y}$. This is the key source of the difference in quantitative results with respect to changes in match productivity.

A second difference is the way Kurmann and Petrosky-Nadeau (2007) model the entry of assets. Since they model an asset as a unit of capital, it is natural they model asset entry as the outcome of household decisions to forego consumption. In equilibrium, the price of new assets have to equal 1 (the price of consumption). A crucial assumption of their analysis is that new capital is initially *unmatched*. Given this, the value of unmatched assets U , is equal to 1 which implies

$$r + \delta = \frac{\beta}{1 - \beta} c \theta \Rightarrow \frac{d\theta}{dr} \frac{r}{\theta} = \frac{r}{r + \delta} > 0. \quad (23)$$

Their qualitative result that technology shocks are correlated with the match rate are driven by this comparative static in the business cycle environment they construct (where r is procyclical). Given θ is pinned down by this expression, the level of capital accumulation \hat{k} is given by equation (4) after replacing $\pi f(\hat{k}) - w$ in my formulation to $\pi f'(\hat{k})$ (the marginal product of capital).²³

Given their formulation of asset entry in (23), shocks to the separation rate λ do not affect θ in their benchmark analysis. To hold θ constant, lower λ leads to higher equilibrium \hat{k} (and lower $\pi f'(\hat{k})$) which increases output. The latter quantitatively dominates the change in output arising from changes in capital utilization following lower λ , and this leads to their conclusion that separation shocks cannot generate significant internal propagation. In my framework, lower λ leads to higher equilibrium \hat{k} , via a lower liquidity premium, which increases output. From (4), these both act to increase θ , such that lower λ and higher $Am(\theta)$ both act to raise capital utilization. These changes generate substantial internal propagation dynamics.

Following Eisfeldt and Rampini (2006), the reallocation of capital modeled in this analysis is that of property, plant and equipment and acquisitions. In

²³Alternatively, they could have assumed that new capital is *matched*, in which case $J = 1$ and θ is then pinned down by $c = Aq(\theta)$ and independent of r . The empirical discussion on capital reallocation focuses on the frictions associated with reallocating existing capital which has been put to some previous use (e.g. Eisfeldt and Rampini (2006)). These discussions explicitly assume that new capital is not subject to such frictions.

these instances, the incidence of new capital investment and capital reallocation is typically lumpy. For instance, a production plant is typically built at once, and when it needs to be resold to another owner (another company), it is sold in whole. Then, the assumption that capital reallocation decisions are not made on marginal units of investment as in Kurmann and Petrosky-Nadeau (2007) seems more relevant for an analysis of capital reallocation. Meanwhile, the modeling of search frictions focusing on lumpy capital reallocation in this paper does a better job of addressing the stylized facts of reallocation documented by Eisfeldt and Rampini (2006), and a better job of generating propagation mechanisms to exogenous shocks which are a topic of ongoing research debate.

6. CONCLUSION

This paper considered the role of search frictions in explaining the amount of capital reallocation and utilization, and their change at business cycle frequencies. In contrast to search models of labor reallocation, search models of capital reallocation can deliver strong internal propagation dynamics and large responses in factor utilization following aggregate shocks.

APPENDIX

Derivation of investment rule.

Differentiating (4)

$$\eta \frac{r + \delta + \lambda}{1 - \beta} \frac{c}{Aq(\theta(\hat{k}))} \frac{\theta'(\hat{k})}{\theta(\hat{k})} = \pi f'(\hat{k}) - \frac{\beta}{1 - \beta} c \theta(\hat{k}) \frac{\theta'(\hat{k})}{\theta(\hat{k})}$$

$$\frac{\theta'(\hat{k})}{\theta(\hat{k})} = \frac{\pi f'(\hat{k})}{(r + \delta + \lambda) \eta S(\hat{k}) + \frac{\beta}{1 - \beta} c \theta(\hat{k})}.$$

Differentiating (3) and multiplying through by $(r + \delta)$, then substituting in for $\frac{\theta'(\hat{k})}{\theta(\hat{k})}$ implies

$$r + \delta = \pi f'(\hat{k}) - \lambda \eta S(\hat{k}) \frac{\theta'(\hat{k})}{\theta(\hat{k})}$$

$$= \pi f'(\hat{k}) \left[1 - \frac{\lambda \eta S(\hat{k})}{(r + \delta + \lambda) \eta S(\hat{k}) + \frac{\beta}{1 - \beta} c \theta(\hat{k})} \right]$$

$$\begin{aligned}
\pi f'(\hat{k}) &= \frac{(r + \delta + \lambda) \eta S(\hat{k}) + \frac{\beta}{1-\beta} c \theta(\hat{k})}{(r + \delta) \eta S(\hat{k}) + \frac{\beta}{1-\beta} c \theta(\hat{k})} (r + \delta) \\
&= (r + \delta) + \frac{\lambda}{(r + \delta) + \frac{\beta}{1-\beta} \frac{c \theta(\hat{k})}{\eta S(\hat{k})}} (r + \delta) \\
&= (r + \delta) + \frac{\lambda}{1 + \frac{1}{r + \delta} \frac{\beta}{\eta} Am(\theta(\hat{k}))}.
\end{aligned}$$

Derivation of Lemma 1.

From (4) and (5) we have

$$\begin{aligned}
\pi f(\hat{k}) - w - \lambda S(\hat{k}) &\geq (r + \delta) \hat{k}, \\
\pi f'(\hat{k}) k - \lambda \eta S(\hat{k}) \frac{\theta'(\hat{k}) \hat{k}}{\theta(\hat{k})} &= (r + \delta) \hat{k}, \\
\frac{\hat{k} \pi f'(\hat{k})}{\eta (r + \delta + \lambda) S(\hat{k}) + \frac{\beta}{1-\beta} c \theta(\hat{k})} &= \frac{\theta'(\hat{k}) \hat{k}}{\theta(\hat{k})},
\end{aligned}$$

these imply

$$\begin{aligned}
\pi f(\hat{k}) - w - \pi f'(\hat{k}) \hat{k} &\geq \lambda S(\hat{k}) \left(1 - \eta \frac{\theta'(\hat{k}) \hat{k}}{\theta(\hat{k})} \right) \\
&\geq \lambda S(\hat{k}) \frac{\eta (\pi f(\hat{k}) - w - \pi f'(\hat{k}) k) + (1 - \eta) \frac{\beta}{1-\beta} c \theta(\hat{k})}{\eta (r + \delta + \lambda) S(\hat{k}) + \frac{\beta}{1-\beta} c \theta(\hat{k})} \\
&\geq \lambda \frac{(\pi f(\hat{k}) - w - \pi f'(\hat{k}) k) + \frac{(1-\eta)}{\eta} \frac{\beta}{1-\beta} c \theta(\hat{k})}{(r + \delta + \lambda) + \frac{\beta}{\eta} m(\theta(\hat{k}))}, \\
&\geq \frac{\lambda}{r + \delta + \frac{\beta}{\eta} m(\theta(\hat{k}))} \frac{(1-\eta)}{\eta} \frac{\beta}{1-\beta} c \theta(\hat{k}) > 0.
\end{aligned}$$

Thus, $J(\hat{k}) - \hat{k} \geq 0$ iff this condition is satisfied.

The existence of equilibrium is shown as follows. Let $\hat{k}_{\min,1}, \hat{k}_{\max,1}$ denote the the lowest and highest values of \hat{k} implied by the accumulation decision (5) respectively. Note that (5) implies that \hat{k} is bounded above and below by

$$\begin{aligned}
\pi f'(\hat{k}_{\max,1}) - (r + \delta) &= 0 \text{ as } \theta \rightarrow \infty, \\
\pi f'(\hat{k}_{\min,1}) - (r + \delta) &= \lambda \text{ as } \theta \rightarrow 0.
\end{aligned}$$

Let $\hat{k}_{\min,2}, \hat{k}_{\max,2}$ denote the the lowest and highest values of \hat{k} implied by the creation margin (4) respectively. (4) implies that $\hat{k}_{\max,2}$ is unbounded as $\theta \rightarrow \infty$. Meanwhile, as $\theta \rightarrow 0$, the lower bound for \hat{k} is implicitly given by

$$\pi f(\hat{k}_{\min,2}) = w.$$

Then a sufficient condition for existence is the condition that $\hat{k}_{\min,2} \leq \hat{k}_{\min,1}$. To confirm this note that as $\theta \rightarrow 0$, Condition 1 implies that this is the case since $\hat{k}_{\min,1} > 0$ and $\pi f(\hat{k}_{\min,1}) - w > 0$.

Proof of Proposition 1. The equilibrium characterized above in the derivation of Lemma 1 implies that the relation between capital and tightness implied by (5), $\hat{k}_1(\theta)$ crosses that implied by (4), $\hat{k}_2(\theta)$ at a point where $\frac{d\hat{k}_1(\theta)}{d\theta} < \frac{d\hat{k}_2(\theta)}{d\theta}$. An increase in π increases $\hat{k}_1(\theta)$ for every θ from (5), and reduces $\hat{k}_2(\theta)$ for every θ from (4) implying an increase in equilibrium \hat{k}, θ . Other comparative statics for A, c, r, λ can be shown in a similar way. ■

Proof of Proposition 4.

$$\begin{aligned} \dot{u}_0 &= (\delta' + Am(\theta)')(1 - u_{ss}) - \lambda' u_{ss}, \\ \Delta u_{ss} &= u'_{ss} - u_{ss} = \frac{\delta' + Am(\theta)'}{\lambda' + \delta' + Am(\theta)'} - \frac{\delta + Am(\theta)}{\lambda + \delta + Am(\theta)}, \\ \frac{\dot{u}_0}{\Delta u_{ss}} &= \frac{(\delta' + Am(\theta)') \frac{\lambda}{\lambda + \delta + Am(\theta)} - \lambda' \frac{\delta + Am(\theta)}{\lambda + \delta + Am(\theta)}}{\frac{\delta' + Am(\theta)'}{\lambda' + \delta' + Am(\theta)'} - \frac{\delta + Am(\theta)}{\lambda + \delta + Am(\theta)}} \\ &= (\lambda' + \delta' + Am(\theta)') \times \\ &\quad \frac{(\delta' + Am(\theta)')\lambda - \lambda'(\delta + Am(\theta))}{(\lambda + \delta + Am(\theta))(\delta' + Am(\theta)') - (\delta + Am(\theta))(\lambda' + \delta' + Am(\theta)')} \\ &= \lambda' + \delta' + Am(\theta)'. \quad \blacksquare \end{aligned}$$

Outcomes using $Turnover = 0.0345$

Table A.1: Outcomes for various β, η .

β	0.9	0.75	0.5	0.25	0.1					
η	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05
α	0.38	0.33	0.35	0.29	0.29	0.23	0.22	0.15	0.16	0.09
r	5.8%	5.8%	4.2%	4.2%	1.0%	1.0%	-3.0%	-3.0%	-6.4%	-6.4%
Liqu.prem.	3.1%	0.3%	3.2%	0.3%	3.4%	0.4%	3.8%	0.5%	4.2%	0.6%

$$\delta = 0.1, \pi \hat{k}^\alpha = 1, w = 0.67, \hat{k} = 2.025, \lambda = 0.066, A\theta(\hat{k})^{1-\eta} = 0.18.$$

Table A.2: Technology shocks which reduce aggregate output by 2%.

β	0.9		0.75		0.5		0.25		0.1	
η	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05	0.9	0.05
$\Delta\pi$	1.2%	0.7%	1.2%	0.7%	1.3%	0.7%	1.5%	0.7%	1.5%	0.5%
Δy	1.93%	1.1%	1.93%	1.1%	1.93%	1.0%	1.92%	0.9%	1.92%	0.7%
$\Delta Realloc.$	6.2%	7.8%	6.2%	8.4%	6.3%	8.9%	6.4%	11.7%	6.6%	15.6%
$\Delta Utiliz.$	0.07%	0.9%	0.07%	0.9%	0.07%	1.0%	0.08%	1.1%	0.08%	1.3%

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