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Evolutionary Stability in Games with Continuous Strategy Space: Recent Developments*

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Abstract The classical evolutionary game theory, pioneered by Maynard Smith and Price (1973), was initially developed in the context of uni-population games with discrete strategy space. Recently some progress has been made in extending the original idea of Maynard Smith and his collaborators into uni-population games with continuous strategy space (Apaloo, 1997; Christiansen, 1991; Cressman, 2006; Eshel, 1983). The current paper reviews them in a unifying framework, and provides some new theorems linking various concepts of evolutionary stability proposed by different authors.

Keywords Nash equilibrium, Local superiority, Local m-stability, Uninvadability, Continuous Stability, Evolutionarily Stable Non-invader Strategy

JEL Classification C67, C72, C73

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1. INTRODUCTION

A key concept in evolutionary game theory is that of an *evolutionarily stable strategy* (ESS), originally proposed by Maynard Smith and Price (1973) as a game-theoretic approach to animal conflicts. An ESS is a strategy such that if most of the members of the population adopt it, there is no alternative strategy that would have a higher expected payoff (reproductive fitness in biology).

For the last forty years since the publication of Maynard Smith and Price (1973), many different meanings have been attached to the *single* word of evolutionary stability. Eshel (1996), for instance, discusses 'eighteen different terms employed in the population dynamics literature that define various concepts of evolutionary stability. What is common in all evolutionary approaches is the idea of *stability of a strategy or a strategy profile against small perturbations*; other details are different.

Standard evolutionary game theory in which randomly matched individuals from an infinite population play a game was initially developed in the context of symmetric uni-population games with discrete strategy space. Bi-population games or games with continuous strategy space are much harder to analyze, and often the obtained results from them do not replicate what are obtained from symmetric uni-population games with discrete strategy space. Most textbooks and surveys on evolutionary game theory, such as Hammerstein and Selten (1994) and Weibull (1995), devote most of their pages to symmetric uni-population games with discrete strategy space.

Given that continuous strategy space models play an important role in all positive sciences, the evolutionary game theory having been largely restricted to games with discrete strategy space is significantly restrictive and quite unsatisfactory. It is also somewhat ironic that evolutionary game theory did not pay proper attention to games with continuous strategy space; indeed a distinguishing feature of *evolution* is that the change of aggregate behavior is *gradual* rather than abrupt due, perhaps, to some inertia, adjustment costs, informational imperfections, and/or bounded rationality.

A fundamental axiom of evolutionary game theory is that any consensus strategy (or a strategy profile) established in the population is never fully fixed, and is always subject to erratic, small deviations from it. Such small deviations from the consensus strategy can occur in at least two ways. First, they may occur due to some substantial change on the part of a certain aberrant minority (a mutant in biology). Second, they may reflect small continuous shifts of the entire population from the consensus strategy. Correspondingly, there are two senses in which a strategy (or a strategy profile) can be said to be *evolutionarily* *stable*. One is that it must be *resistant to invasion* by a rare mutant strategy. This is the condition that motivated the original definition of evolutionary stability offered by Maynard Smith and Price (1973). The second is that a slightly shifted population from an evolutionarily stable state must *actually be able to evolve back to* it.

It is important to note that the first sense of evolutionary stability does not necessarily imply the second sense of stability. Suppose there is no tendency of recovery in the second sense. Then although an x^*-population being resistant to invasion in the first sense would be immune to a mutant, a shifted x-population living near x^* might not be immune to the same mutant; in that case, natural or social selection might amplify any small deviation of the entire population from the original state.

In games with discrete strategy space, small deviations from the consensus strategy can occur only from the first source. In games with continuous strategy space, however, small deviations can occur from both sources. If payoff functions are bi-affine, the condition of resistance to invasion implies the condition of actual recovery from small perturbations and vice versa, but this is not true in general non-linear games with continuous strategy space; see section 2.

That the original definition of Maynard Smith and Price (1973) captures only one notion of evolutionary stability has been appreciated by some authors from early on (see Talyor and Jonker, 1978, for instance), but characterizations of the second condition were not attempted until recently.

Two different proposals for the second sense of evolutionary stability have been made in the literature. Eshel (1983) proposes the following idea: small perturbations of the entire population from an evolutionarily stable strategy should end up with selective advantage to strategies that render the population back to it but not to strategies towards the other side. This condition was later given the name of *local m-stability* by Taylor (1989) and *convergence stability* by Christiansen (1991). Apaloo (1997) proposes another idea, called a *neighborhood invader strategy*, stating that an evolutionarily stable strategy should be able to invade all established communities populated by players using strategies that are sufficiently similar to it. This condition is often called *local superiority*. (See Weibull, 1995, for example.)

These two conditions are different not only from the condition of resistance to invasion by a mutant but also from each other. In contrast to a locally superior strategy, for example, a locally m-stable strategy may not be able to invade some of its near neighbors. Likewise a locally superior strategy may not be able to give selective advantage to strategies that render the population back to it.

The goal of the present paper is to survey these recent developments on the concepts of evolutionary stability in a unifying framework, and provide some novel characterization theorems linking different concepts of evolutionary stability. We also (modestly) extend the recent developments into a general setting, covering games with multidimensional strategy space.

The paper is organized as follows. Section 2 defines various concepts of evolutionary stability and provides characterization theorems. Section 3 discusses two examples. Section 4 concludes.

2. EVOLUTIONARY STABILITIES IN GAMES WITH CONTINUOUS STRATEGY SPACES

Suppose two individuals are repeatedly drawn at random from an infinitely large single population to play a symmetric two-person game.[1] Because the two players are drawn from the same population, they share the same strategy space and the payoff function. The two players do not know whether they are player 1 or 2, and thus cannot condition their strategies upon this information. Let $\pi : S \times S \to \mathbb{R}$ be the payoff function of a player; $\pi(x', x)$ is the payoff of a player who chooses strategy $x' \in S$ against an opponent who chooses $x \in S$.

We assume that strategies take values on a convex subset of the k-dimensional real Euclidean space: $S \subset \mathbb{R}^k$. Our model mainly covers games with continuous pure strategy spaces, but also covers as special cases mixed extensions of discrete strategy games, in which the payoff functions are bi-affine and the strategy spaces are k-dimensional simplexes. We do not study mixed extensions of continuous pure strategy games, where the strategy space is a space of probability measures on the set of Borel subsets of S.

Consider a situation where all individuals are initially programmed to play a certain *consensus strategy* (say, x). We then inject into this x-population a small population share of individuals who are likewise programmed to play some other strategy (say, x'); the latter strategy will be called a *mutant strategy*. Then

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[1] Evolutionary game models derived from a situation with an infinite population are different from those derived from a situation with a finite population. In the former case, where two individuals from an infinite population are repeatedly chosen at random to play a game, the probability that a normal meets a mutant is the same as the probability that a mutant meets another mutant. In the latter case, the probability that a normal meets a mutant is greater than the probability that a mutant meets another mutant because the mutant cannot play against himself in a finite- population game. The present paper does not study the finite population models; see Kandori *et al.* (1993), Riley (1979), Schaffer (1988), Sethi and Somanathan (2001), and Vega-Redondo (1997) for the analysis of models with a finite population assumption.

$\pi(x',x) - \pi(x,x)$ is the net payoff gain of a player playing the mutant strategy x' relative to a player playing the consensus strategy x in the mixed population.

We define $\varphi(x',x;y) \equiv \pi(x',y) - \pi(x,y)$. Then the net payoff gain $\pi(x',x) - \pi(x,x)$ is simply equal to $\varphi(x',x;x)$.

We say x' strongly invades x-population if $\varphi(x',x;x) > 0$. If x' strongly invades x-population, the mutant will ultimately replace the consensus strategy and take over the whole population in the long run. If $\varphi(x',x;x) \geq 0$, on the other hand, we say that x' weakly invades x-population. Under this situation, the mutant strategy may not be able to wipe out the consensus strategy; instead they may coexist. If $\varphi(x',x;x) < 0$, we say no x' can weakly invade x-population, while if $\varphi(x',x;x) \leq 0$, there is no x' which can strongly invade x-population.[2]

We begin with some basic definitions.

Definition 1. Consider a uni-population game where the payoff function is given by $\pi : S \times S \to \mathbb{R}$.

(1) $x^* \in S$ is a *Nash equilibrium strategy* (NES) if $\varphi(x,x^*;x^*) \equiv \pi(x,x^*) - \pi(x^*,x^*) \leq 0$ for all $x \in S$. We call it a *strict NES* if strict inequality holds for all $x \in S \setminus \{x^*\}$.

(2) $x^* \in S$ is a *locally superior strategy* if for all $x \neq x^*$ sufficiently close to x^*, $\varphi(x^*,x;x) \equiv \pi(x^*,x) - \pi(x,x) > 0$.

(3) $x^* \in S$ is a *locally m-stable strategy* if for all $x \neq x^*$ sufficiently close to x^*, $\varphi((1-\lambda)x + \lambda x^*, x; x) > 0$ for small $\lambda > 0$; and $\varphi((1-\lambda)x + \lambda x^*, x; x) < 0$ for small $\lambda < 0$ as long as $(1-\lambda)x + \lambda x^* \in S$.

Some remarks are in order for each of these concepts.

In the context of evolutionary game theory, a Nash equilibrium strategy requires that no mutant strategy x should be able to strongly invade x^*-population, although it does not preclude some mutants who can weakly invade the x^*-population. On the other hand, a strict Nash equilibrium strategy requires that no mutant strategy x should be able to invade x^*-population even weakly. We note that x^* is a Nash equilibrium if and only if it maximizes $\varphi(.,x^*;x^*)$, and x^* is a strict Nash equilibrium if and only if it strictly maximizes $\varphi(.,x^*;x^*)$.

A mutant strategy cannot invade (even weakly) a population established with a strict Nash equilibrium strategy. But both Nash and strict Nash equilibrium strategies are completely silent about whether x^* can invade x-population. It

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[2] It may happen that both $\varphi(x',x;x) > 0$ and $\varphi(x,x';x') > 0$ hold. In that case, the mutant strategy may be able to invade the consensus strategy for a certain period of time, but the consensus strategy can invade the mutant strategy back when it becomes a minority. Thus neither strategy can eliminate the other, and the population fluctuates between x' and x ; the population becomes essentially dimorphic, and the situation is evolutionarily unstable.

is superiority that captures an idea that x^* should be able to strongly invade x-population for all $x \neq x^*$. Indeed if x^* is a locally superior strategy, there is no nearby x- population which cannot be strongly invaded by x^*.

Local m-stability captures the idea of resistance to small perturbation in a different way than superiority does. A locally m-stable strategy x^* is the one that gives selective advantage to a strategy that lies on the line segment between x^* and $x \neq x^*$ and selective disadvantage to a strategy that lies on the opposite side: $\varphi((1-\lambda)x+\lambda x^*,x;x) > 0$ for small $\lambda > 0$ and $\varphi((1-\lambda)x+\lambda x^*,x;x) < 0$ for small $\lambda < 0$.[3] Thus a strategy x^* is locally m-stable if the answer to the following question is affirmative: if a large enough majority of population was perturbed to choose a strategy $x \neq x^*$, will it be advantageous for each in this population to choose a strategy (x') close to, rather than further apart from, x^*? Local m-stability says that deviations of the entire population from x^* to x will end up with selective advantage to strategy x' that renders the population back to x^* but not to strategies towards the other side.[4]

Figure 1 illustrates the concepts of superiority and m-stability.

Superiority and m-stability are in general independent from each other. There are, however, some cases in which these two concepts are interrelated. We prove:

Theorem 1. *(1) If π is concave in the first argument, then a superior equilibrium is m-stable. (2) If π is convex in the first argument, then m-stablity implies superiority. (3) If π is affine in the first argument, m-stablity is identical to superiority.*

Proof. (1) If π is concave in the first argument, then for all $x \neq x^*$ in the interior, $\pi((1-\lambda)x+\lambda x^*,x) \geq (1-\lambda)\pi(x,x)+\lambda\pi(x^*,x)$ for sufficiently small $\lambda > 0$ and $\pi((1-\lambda)x+\lambda x^*,x) \leq (1-\lambda)\pi(x,x)+\lambda\pi(x^*,x)$ for sufficiently small $\lambda < 0$. Superiority implies that $(1-\lambda)\pi(x,x)+\lambda\pi(x^*,x) > \pi(x,x)$ for $\lambda > 0$ and $(1-\lambda)\pi(x,x)+\lambda\pi(x^*,x) < \pi(x,x)$ for $\lambda < 0$. Thus $\pi((1-\lambda)x+\lambda x^*,x) > \pi(x,x)$ for sufficiently small $\lambda > 0$ and $\pi((1-\lambda)x+\lambda x^*,x) < \pi(x,x)$ for sufficiently small $\lambda < 0$; x^* is m-stable. If $x \neq x^*$ is in the corner, then we only need to prove that $\pi((1-\lambda)x+\lambda x^*,x) > \pi(x,x)$ for $\lambda > 0$. This is straightforward.

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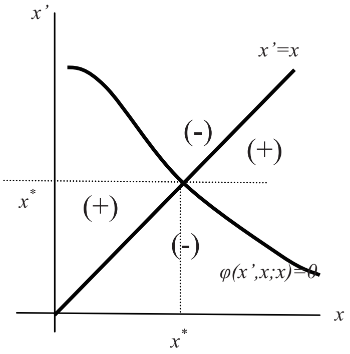
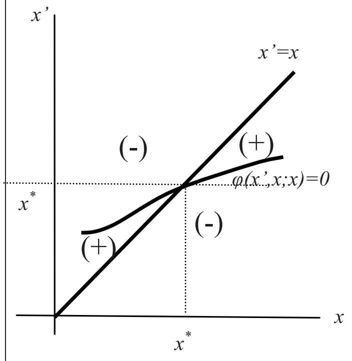
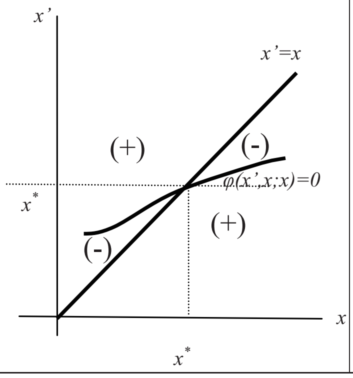
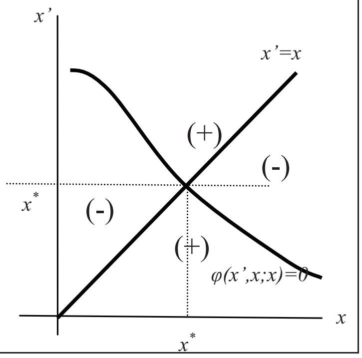
[3] Our definition of m-stability is *weaker* than the one originally proposed by Eshel (1983), although both definitions are identical if the strategy space is one-dimensional. Eshel's (1983) original formulation requires that a locally m-stable strategy give selective advantage to a strategy that is closer *in distance* to x^* than to $x \neq x^*$: for all $x \neq x^*$ sufficiently close to x^* and all $x' \neq x$, $\pi(x',x) > \pi(x,x)$ if and only if $\|x'-x^*\| < \|x-x^*\|$.

[4] If x^* is an m-stable strategy, then for all $x \neq x^*$, $x' = (1-\lambda)x+\lambda x^*$, and sufficiently small $\varepsilon > 0, \varepsilon\pi(x',x')+(1-\varepsilon)\pi(x',x) > \varepsilon\pi(x,x')+(1-\varepsilon)\pi(x,x)$ for small $\lambda > 0$. In the literature of adaptive dynamics, an m-stable strategy is an evolutionary attractor, while a strategy that is not m-stable is evolutionary repeller from which an initial population evolves away.

	superiority: yes	superiority: no
m-stability: yes		
m-stability: no		

Note: Consider the set of points (x',x) at which $\varphi(x',x;x)=0$. Since $\varphi(x',x;x)=0$ for $x'=x$, the 45 degree line is included in this set. For x^* to be an equilibrium point, there must be other curves in this set which cross the 45 degree line at x^*. Here we suppose that the set of such points, together with the 45 degree line, divides the space around (x^*,x^*) into four regions. (This may not be true in general.) For x^* to be a strict Nash equilibrium, the vertical line at x* should be in the negative region. For x^* to be superior, the horizontal line at x^* should be in the positive region. For x^* to be m-stable, the sign of $\varphi(x',x;x)$ must be positive above the diagonal on the left of the intersection as well as below the diagonal on the right of the intersection, and negative below the diagonal on the left of the intersection as well as above the diagonal on the right of the intersection. In the literature on adaptive dynamics, these pictures are often called the pairwise invadability plots.

Figure 1: Superiority and m-stability in symmetric uni-population games

(2) If x^* is m-stable, $\pi((1-\lambda)x+\lambda x^*,x) > \pi(x,x)$ for small $\lambda > 0$. Because π is convex in the first argument, $\pi((1-\lambda)x+\lambda x^*,x) \leq (1-\lambda)\pi(x,x)+\lambda\pi(x^*,x)$ for small $\lambda \in (0,1]$. Thus $(1-\lambda)\pi(x,x)+\lambda\pi(x^*,x) > \pi(x,x)$ for small $\lambda > 0$, which implies that $\pi(x^*,x) > \pi(x,x)$; x^* is superior.

(3) The third statement is immediate. □

The definitions provided in Definition 1 are not easy to work with. If π is differentiable, however, calculus characterizations for the above-mentioned definitions can be obtained. We summarize the sufficient conditions in Table 1 for easy reference in the case where the strategy space is one dimensional. (Some of the results in Table 1 are proven in Apaloo, 1997.)

Note that calculus characterizations for the three concepts differ only in the second order conditions; the first order conditions are all identical.

We move on to other definitions.

Definition 2. Consider a uni-population game where the payoff function is given

Table 1: Sufficient conditions for various concepts of evolutionary stability in symmetric uni-population games

Concept	Sufficient conditions for an interior solution
Local strictness	$\partial_1\pi(x^*,x^*)=0$; and $\partial_{11}\pi(x^*,x^*)$ is negative.
Local superiority	$\partial_1\pi(x^*,x^*)=0$; and $\partial_{11}\pi(x^*,x^*)+2\partial_{12}\pi(x^*,x^*)$ is negative.
Local m-stability	$\partial_1\pi(x^*,x^*)=0$; and $(\partial_{11}\pi(x,x)+\partial_{12}\pi(x,x)-\frac{\lambda}{2}\partial_{11}\pi(x,x))\big\rvert_{x=x^*}$ is negative for small $\lambda \in \mathbb{R}\setminus\{0\}$.
Local uninvadability	$\partial_1\pi(x^*,x^*)=0$; and $\partial_{11}\pi(x^*,x^*)+2\varepsilon\partial_{12}\pi(x^*,x^*)$ is negative for sufficiently small $\varepsilon > 0$.
Local ESNIS	$\partial_1\pi(x^*,x^*)=0$; $\partial_{11}\pi(x^*,x^*)$ is negative; and $\partial_{11}\pi(x^*,x^*)+2\partial_{12}\pi(x^*,x^*)$ is negative.
Local CSS	$\partial_1\pi(x^*,x^*)=0$; $\partial_{11}\pi(x^*,x^*)$ is negative; and $(\partial_{11}\pi(x,x)+\partial_{12}\pi(x,x)-\frac{\lambda}{2}\partial_{11}\pi(x,x))\big\rvert_{x=x^*}$ is negative for sufficiently small $\lambda \in \mathbb{R}\setminus\{0\}$.

by $\pi : S \times S \to \mathbb{R}$.

(1) $x^* \in S$ is an *uninvadable strategy* (UIS) if it is a Nash equilibrium strategy for which $\varphi(x,x^*;x^*) = 0$ implies $\varphi(x^*,x;x) > 0$ for all $x \in S \setminus x^*$.

(2) An *evolutionarily stable neighborhood invader strategy* (ESNIS) is a strict Nash equilibrium strategy that is locally superior.

(3) A *continuously stable strategy* (CSS) is a strict Nash equilibrium strategy that is locally m-stable.

An uninvadable strategy is what Maynard Smith and Price (1973) call an *evolutionarily stable strategy* (ESS).[5] From the definition, it is obvious that an UIS is a stronger concept than a Nash equilibrium strategy and weaker than a strict Nash equilibrium strategy.

An evolutionarily stable neighborhood invader strategy (ESNIS) was first proposed by Apaloo (1997). Roughly put, x^* is an ESNIS if x cannot weakly invade x^*-population and x^* can strongly invade x-population if $x \neq x^*$ is sufficiently close to x^*.

A continuously stable strategy was first proposed by Eshel (1983). Intuitively, x^* is a CSS if x cannot weakly invade x^*-population and x' can strongly invade x-population if $x' \neq x$ is closer to x^* than x.

We remark that our definitions of an ESNIS and a CSS are stronger than original ones; the original definitions of Apaloo (1997) and Eshel (1983) require local strictness rather than global strictness. We call the original ones a local ESNIS and a local CSS, respectively.

If π is differentiable, we may also obtain sufficient conditions for UIS, ESNIS, and CSS. Sufficient conditions for an interior UIS, an interior ESNIS, and an interior CSS are also summarized in Table 1.

It is easy to see that Definition 2-(1) is identical to the following definition:

Definition (ESS-I). $x^* \in S$ is an *uninvadable strategy* (UIS) if for all $x \in S \setminus \{x^*\}$ and sufficiently small $\varepsilon > 0$, $(1-\varepsilon)\varphi(x,x^*;x^*) - \varepsilon\varphi(x^*,x;x) < 0$.[6]

Taylor and Jonker (1978) define an ESS somewhat differently from Maynard Smith and Price (1973):

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[5] Bomze and Pötscher (1989) call a strategy uninvadable if it is protected by a positive 'global' invasion barrier. Their notion of uninvadability is thus stronger than ours. We may call Bomze and Pötscher's definition strong uninvadability.

[6] This is equivalent to: $\varepsilon\pi(x^*,x) + (1-\varepsilon)\pi(x^*,x^*) > \varepsilon\pi(x,x) + (1-\varepsilon)\pi(x,x^*)$. Thus if a small population using strategy x is entered into an x^*-population, then the expected payoff to the player who chooses the strategy chosen by the majority (x^*) should be strictly higher than the expected payoff to the player who chooses the strategy chosen by the minority (x).

Definition (ESS-II). $x^* \in S$ is an *evolutionarily stable strategy* if for all $x \in S \setminus \{x^*\}$ and sufficiently small $\varepsilon > 0$, $\varphi(x, x^*; \bar{x}_\varepsilon) < 0 \Leftrightarrow \pi(x^*, \bar{x}_\varepsilon) > \pi(x, \bar{x}_\varepsilon)$, where $\bar{x}_\varepsilon = \varepsilon x + (1-\varepsilon)x^*$.

An ESS-II is identical to our definition of an UIS only when π is affine in the second argument. In general, however, they are different. Also an ESS-II is not necessarily a Nash equilibrium strategy, although an UIS is always a Nash equilibrium strategy.[7]

Theorem 2 establishes the relationship between local superiority, local m-stability, and uninvadability.

Theorem 2. *(1) If π is concave in the first argument and affine in the second argument, a locally superior strategy is an uninvadable strategy. If π is strictly quasiconvex in the first argument and affine in the second argument, then an uninvadable strategy is locally superior.*

(2) If π is bi-affine, a locally m-stable strategy is an uninvadable strategy. If π is concave and strictly quasi-convex in the first argument and affine in the second argument, then an uninvadable strategy is locally m-stable.

Proof. (1) We first note that when π is affine in the second argument an UIS is identical to the ESS-II of Taylor and Jonker (1978). To prove the first statement, suppose π is concave in the first argument. Then for all $\varepsilon \in (0,1]$, $\pi(\bar{x}_\varepsilon, \bar{x}_\varepsilon) \geq \varepsilon\pi(x, \bar{x}_\varepsilon) + (1-\varepsilon)\pi(x^*, \bar{x}_\varepsilon)$. Because of local superiority $\pi(x^*, \bar{x}_\varepsilon) > \pi(\bar{x}_\varepsilon, \bar{x}_\varepsilon)$ for small $\varepsilon > 0$; thus $\pi(x^*, \bar{x}_\varepsilon) > \varepsilon\pi(x, \bar{x}_\varepsilon) + (1-\varepsilon)\pi(x^*, \bar{x}_\varepsilon)$ for small $\varepsilon > 0$, which implies that $\pi(x^*, \bar{x}_\varepsilon) > \pi(x, \bar{x}_\varepsilon)$ for small $\varepsilon > 0$. Thus x^* is an ESS-II. To prove the second statement, we note that x^* being an ESS-II implies that $\pi(x^*, \bar{x}_\varepsilon) > \pi(x, \bar{x}_\varepsilon)$ for small $\varepsilon > 0$. Because π is strictly quasi-convex in the first argument, $\pi(x^*, \bar{x}_\varepsilon) > \pi(x, \bar{x}_\varepsilon)$ implies $\pi(x^*, \bar{x}_\varepsilon) > \pi(\bar{x}_\varepsilon, \bar{x}_\varepsilon)$ for small $\varepsilon > 0$. Thus $\pi(x^*, x) > \pi(x, x)$ for all $x \neq x^*$ sufficiently close to x^*.

(2) If π is bi-affine, it is both convex and concave in the first argument. By part (2) of Theorem 1, local m-stability implies local superiority; by part (1) a locally m-stable strategy is an UIS. If π is concave and strictly quasi-convex in the first argument and affine in the second argument, then uninvadability implies

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[7] If the payoff function is continuous, however, a strict Nash equilibrium strategy is an ESS-II and an ESS-II is a Nash equilibrium strategy. To prove the first statement, suppose $\pi(x^*, x^*) > \pi(x, x^*)$ for all $x \neq x^*$. Because π is continuous in the first argument, $\pi(x^*, \bar{x}_\varepsilon) > \pi(x, \bar{x}_\varepsilon)$ for small $\varepsilon > 0$. To prove the second statement, suppose x^* is not a NES. Then there exists $x \neq x^*$ such that $\pi(x, x^*) > \pi(x^*, x^*)$. Because π is continuous in the first argument, $\pi(x, \bar{x}_\varepsilon) > \pi(x^*, \bar{x}_\varepsilon)$ for small $\varepsilon > 0$. Thus x^* is not an ESS-II. Bomze and Pötscher (1989) show with examples that continuity cannot be dispensed with.

local superiority by part (1) and local superiority implies local m-stability by part (1) of Theorem 1. ☐

Thus if π is bi-affine, local m-stability, local superiority, and uninvadability are all identical.

Corollary 1 relates the Nash equilibrium condition to local superiority and local m-stability. Its proof is straightforward.

Corollary 1. *(1) If π is concave in the first argument and affine in the second argument, a locally superior strategy is a Nash equilibrium strategy. If π is strictly quasi-convex in the first argument and affine in the second argument, then a strict Nash equilibrium strategy is locally superior.*

(2) If π is bi-affine, a locally m-stable strategy is a Nash equilibrium strategy. If π is concave and strictly quasi-convex in the first argument and affine in the second argument, then a strict Nash equilibrium strategy is locally m-stable.

The relationship between various equilibrium concepts in uni-population games is summarized in Figure 2.

3. EXAMPLES

We now provide some examples. For each example, we provide the graph of $\varphi(x',x;x) = 0$ explained in Figure 1. They are collected in Figure 2.

Hotelling's price competition model

Consider two stores separately located in the two end points of a unit interval. There is a continuum of consumers distributed by a distribution function F in the unit interval. We assume that F is absolutely continuous and differentiable with density f. The median of F will be denoted by m; $F(m) = \frac{1}{2}$.

Each consumer has a unit demand and incurs travel cost t per unit of distance from their location. Let $S = [c, \infty)$, where c is the marginal cost of the two stores.

If p and q are the prices set by the stores, then $\pi(p,q) = (p-c)F\left(\frac{1}{2} + \frac{q-p}{2t}\right)$ and $\frac{\partial \pi}{\partial p} = F\left(\frac{1}{2} + \frac{q-p}{2t}\right) - \frac{(p-c)}{2t} f\left(\frac{1}{2} + \frac{q-p}{2t}\right)$. If the payoff function is strictly concave, the unique pure strategy Nash equilibrium strategy is $p^* = c + 2mt/f(0.5)$, which is in the interior.

It is easy to compute that

$$\left.\frac{\partial^2 \pi(p,q)}{\partial p^2}\right|_{(p,q)=(p^*,p^*)} = \frac{-2f(0.5)^2 + mf'(0.5)}{2tf(0.5)}$$

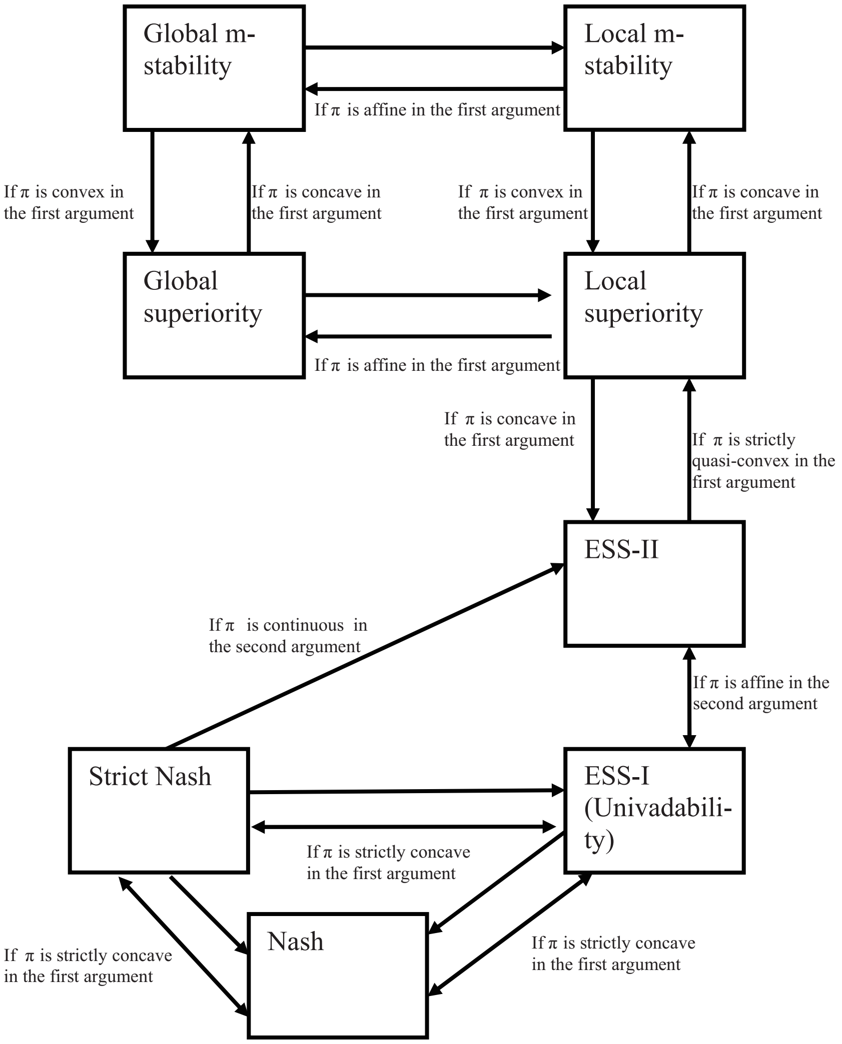


Figure 2: Relationship between various equilibrium concepts in symmetric unipopulation games

and

$$\frac{\partial^2\pi(p,q)}{\partial p\partial q}\Big|_{(p,q)=(p^*,p^*)} = \frac{f(0.5)^2 - mf'(0.5)}{2tf(0.5)},$$

where $f'(x)$ is the derivative of f at x. Thus the condition for p^* to be a locally strict NES (or a locally uninvadable strategy) is: $mf'(0.5) < 2f(0.5)^2$.

p^* is locally m-stable because

$$\left(\frac{\partial^2\pi(p,q)}{\partial p^2} + \frac{\partial^2\pi(p,q)}{\partial p\partial q}\right)\Big|_{(p,q)=(p^*,p^*)} = -\frac{f(0.5)}{2t} < 0.$$

On the other hand, $\left(\frac{\partial^2\pi(p,q)}{\partial p^2} + 2\frac{\partial^2\pi(p,q)}{\partial p\partial q}\right)\Big|_{(p,q)=(p^*,p^*)} = -\frac{mf'(0.5)}{2tf(0.5)}$; thus it is locally superior only when $f'(0.5) > 0$.

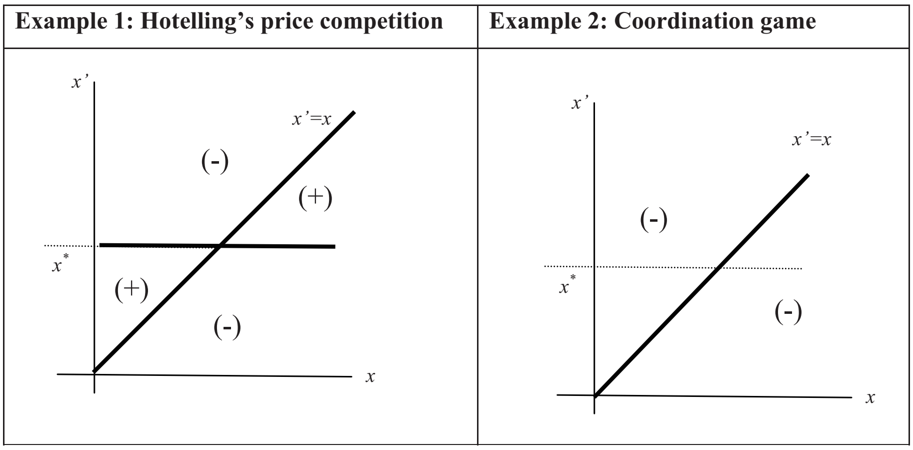
If F is a uniform distribution, for instance, the pure strategy Nash equilibrium is $(p^*,p^*) = (c+t, c+t)$. It is strict, uninvadable, and m-stable, but not superior because for all $q \in [c,\infty)$, $\pi_i(p^*,q) = \frac{q-c}{2} = \pi_i(q,q)$. Because $\varphi(p',p;p) = \frac{1}{2t}(p'-p)(t+c-p')$, the set of points at which $\varphi(p',p;p) = 0$ consists of two curves; $p' = p$ (the 45 degree line) and $p' = p^*$. The graph of $\varphi(p',p;p) = 0$ is illustrated in the first panel of Figure 3. We see that the graph of $\varphi(p',p;p) = 0$ in this example is of the type shown in panel 2 of Figure 1.

Coordination game

This is a game possessing infinitely many strict Nash equilibria (and thus infinitely many uninvadable strategies) but none of them survive the test of superiority or m-stability. That strict Nash equilibria are neither superior nor m-stable implies that although coordination may be *resistant to invasion* by an alternative strategy, a slightly perturbed population from the coordination may not *actually evolve back to* it.

Consider a two player game in which each person chooses an effort level $e \in [0,\bar{e}]$ simultaneously. The payoff of an individual choosing e and facing an opponent choosing f is given by $\pi(e,f) = \alpha\min[e,f] - \beta e$ where $\alpha > \beta > 0$. This game possesses a continuum of pure strategy Nash equilibria; any (e^*,e^*) where $e^* \in [0,\bar{e}]$ is a Nash equilibrium.

Because $\pi(e^*,e^*) = (\alpha-\beta)e^* > \pi(e,e^*) = \alpha\min[e,e^*] - \beta e$ for all $e \neq e^*$, all of these Nash equilibria are strict and thus uninvadable. But none of them are superior, because $\pi(e^*,e) = \alpha\min[e^*,e] - \beta e^* < \pi(e,e) = (\alpha-\beta)e$ for all $e \neq e^*$. To check if they are m-stable, we compare $\pi(e',e)$ and $\pi(e,e)$ for all $e \neq e^*$



Note: The sign of $\varphi(x',x;x)$ is shown in each region.

Figure 3: Graphs of $\varphi(x',x;x) = 0$ for the examples in section 3

and all e' between e and e^*. Suppose $e < e^*$. Then $\pi(e',e) = \alpha e - \beta e' < \pi(e,e) = \alpha e - \beta e$ for all $e' > e$. If $e > e^*$, $\pi(e',e) = \alpha e' - \beta e' < \pi(e,e) = \alpha e - \beta e$ for all $e' < e$. We thus conclude that none of the Nash equilibria are m-stable.

Note that

$$\varphi(e',e;e) = \begin{cases} (\alpha - \beta)(e' - e) & \text{if } e' < e \\ 0 & \text{if } e' = e \\ -\beta(e' - e) & \text{if } e' > e. \end{cases}$$

Thus the set of points at which $\varphi(x',x;x) = 0$ consists of only one curve; $e' = e$ (the 45 degree line); see the second panel of Figure 3.

4. CONCLUSION

In the present paper, we reviewed various concepts of evolutionary stability in uni-population games with continuous strategy space. Many of these concepts we examined in the present paper are either vacuous or coincide each other when strategy space is discrete. In continuous strategy games, however, they are all different.

Various characterizations provided in the current paper are based upon static concepts of evolutionary stability; we did not study dynamic properties of these various characterizations. Our characterizations are, however, largely compatible with the stability conditions studied in the literature on adaptive dynamics.

Complete studies on dynamics require examination of measure dynamics. For studies on measure dynamics using strong topology, see Bomze (1991), Bomze and Bürger (1995), and Seymour (2000). For studies on the measure dynamics using weak topology, see Cressman *et al.* (2006), Eshel *et al.* (1997), and Oechssler and Riedel (2002).

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