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Indeterminacy under Social Constant Returns and Costs of Adjusting Capital

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Abstract

It has been shown that, in a small open economy with traded and nontraded goods, indeterminacy occurs under constant returns to scale for the social technology with an arbitrarily small degree of externalities. This paper claims that costs of adjusting capital increase the required degree of externalities for indeterminacy to arise. Under empirically plausible levels of adjustment costs and externalities, indeterminacy does not arise in a model with social constant returns.

Keywords Indeterminacy, Constant Returns to Scale, Costs of Adjusting Capital

JEL Classification E00, E22, E32

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1. INTRODUCTION

Neoclassical growth models with externalities may feature indeterminate dynamics, in the sense that there is a continuum of equilibrium paths which converge to the unique steady state. The presence of indeterminacy is an interesting phenomenon since models with indeterminacy can generate self-fulfilling business cycle fluctuations. That is, in response to a shock or even without the presence of a shock, the economy can move in an arbitrary way and this movement is self-fulfilling in the sense that the economy is on the stable equilibrium path. See Benhabib and Farmer (1999) for an early survey on this topic, and Chen and Zhang (2011) for recent references.

This literature started with one-sector models, which need a high degree of increasing returns for indeterminacy to arise (as in Benhabib and Farmer, 1994; and Kim, 2003a). However, several recent papers with multi-sector models have shown that indeterminacy occurs under constant return to scale for the social technology. In particular, they model social constant returns together with private decreasing returns and very small externalities.[1]

The concept of adjustment costs has been widely used in the literature on investment. In a continuous-time partial-equilibrium model without adjustment costs, the choice for the optimal capital stock becomes static and the absolute value of investment is infinity or zero. In general-equilibrium models, various forms of investment adjustment costs have been introduced to improve the properties of such models, e.g. persistence. Kim (2003b) reviews the literature on costs of adjusting capital from a macroeconomic perspective.

In models with private constant returns and social increasing return, it has been shown that such adjustment costs increase the required degree of increasing returns for indeterminacy to arise. Kim (2003a) put costs of adjusting capital into a one-sector model of Benhabib and Farmer (1994), and Herrendorf and Valentinyi (2002, 2003) modified two-sector models such as Benhabib and Farmer (1996).[2] The contribution of this paper is to include costs of adjusting capital into an open-economy model with social constant returns and to analyze

[1]Furthermore, indeterminacy can arise with an arbitrarily small degree of externalities in such models. In a closed-economy model with exogenous growth, Benhabib and Nishimura (1998) proved an occurrence of indeterminacy under social constant returns to scale and arbitrarily small degree of externalities. Benhabib, Meng and Nishimura (2000) and Mino (2001) extended this result into endogenous-growth models, and Meng and Velasco (2003, 2004) showed that this scenario is more likely to happen in a small open economy. Meng (2003) used an open-economy setting with endogenous growth.

[2]See also Georges (2003) as well as the references cited in Herrendorf and Valentinyi (2002, 2003), and Kim (2003a).

its economic implications.

Our conclusions are threefold. Indeterminacy can still arise under small adjustment costs. However, we need more-than-arbitrarily-small degree of externalities for indeterminacy to arise. Finally, for empirically plausible parameter values, indeterminacy does not arise in a model with social constant returns.

2. THE ECONOMY

This section introduces costs of adjusting capital into a small open economy model such as Meng and Velasco (2003, 2004). The economy has two sectors, consumption and investment. For ease of presentation, we assume that the consumption goods (y_T) are tradable and the investment goods (y_N) are nontradable.[3] Production side features decreasing returns to scale from the private perspective and constant returns to scale from the social perspective, due to the externalities of the following form:[4]

$$y_T = l_T^{\alpha_0} k_T^{\alpha_1} \overline{l_T^{a_0} k_T^{a_1}}, \text{ where } \alpha_0 + \alpha_1 + a_0 + a_1 = 1, \tag{1}$$

$$y_N = l_N^{\beta_0} k_N^{\beta_1} \overline{l_N^{b_0} k_N^{b_1}}, \text{ where } \beta_0 + \beta_1 + b_0 + b_1 = 1, \tag{2}$$

where $\overline{l_T^{a_0} k_T^{a_1}}$ and $\overline{l_N^{b_0} k_N^{b_1}}$ represent positive externalities. Both inputs are mobile across sectors and the total labor supply is fixed:[5]

$$k_T + k_N = k, \tag{3}$$

$$l_T + l_N = \bar{l}. \tag{4}$$

The representative agent maximizes

$$\int_0^\infty u(c) e^{-\rho t} dt, \tag{5}$$

where c is the consumption of tradable goods and ρ is the discount rate.[6] The agent's budget constraint is

$$\dot{d} = rd + y_T + py_N - c - pi, \tag{6}$$

[3]We can generalize the model into a setting where both consumption and investment goods can be both tradable and nontradable, as in Meng and Velasco (2004).

[4]Nishimura and Shimomura (2002) constructed an indeterminacy model of both socially and privately constant returns by adopting negative as well as positive externalities. Uzawa-type endogenous discounting was also assumed.

[5]See Meng and Velasco (2003) for a model with elastic labor supply.

[6]In our setup of small open economy, the presence of indeterminacy does not depend on the functional form of the utility function. However, its curvature would matter in a closed-economy setup as in Kim (2005).

where d is the amount of net bond holdings and p is the relative price of investment good to the consumption good. This relative price is exogenous to the agent, but will be determined by a market-clearing condition for the nontraded investment good. A small open economy implies full capital mobility, so the agent can borrow and lend at the world interest rate r. To have a well-defined steady state, we assume that $r = \rho$.

The variable i denotes gross investment, and we adopt costs of adjusting capital into the capital accumulation equation as in Kim (2003a):

$$\frac{\dot{k}}{k} = \Psi\left(\frac{i}{k}\right),$$

where the function satisfies three assumptions,

$$\Psi(\delta) = 0, \Psi'(\delta) = 1, \Psi''(\delta) \leq 0.$$

The first assumption determines the steady-state ratio between investment and capital, and the second assumption makes the steady state of our model not different from that of a model without adjustment costs. We use a parameter ψ to express the degree of adjustment costs by adopting the following functional form:

$$\frac{\dot{k}}{k} = \frac{\delta}{1-\psi}\left[\left(\frac{i}{\delta k}\right)^{1-\psi} - 1\right]. \tag{7}$$

In a model in which firms accumulate capital, this parameter is the inverse of the elasticity of (i/k) with respect to Tobin's q.

The representative agent's problem is to choose c, l_T, l_N, i, k_T, k_N and d to maximize (5) subject to (1), (2), (3), (4), (6) and (7), and given k_0 and d_0. An alternative way to characterize this economy is via a competitive equilibrium by households and firms as in Kim (2003a), where the private decreasing returns would lead to entry and exit decisions due to nonzero profits as in Kim (2004). This setup would introduce input prices such as wages and rental rates, but the dynamics of the real variables would turn out to be identical.

The current-value Hamiltonian is

$$\begin{aligned} H = u(c) &+ \lambda\left(rd + y_T + py_N - c - pi\right) + \lambda qk\delta\frac{\left(\frac{i}{\delta k}\right)^{1-\psi} - 1}{1-\psi} \\ &+ \lambda_1(k - k_T - k_N) + \lambda_2(\bar{l} - l_T - l_N) \end{aligned}$$

where λ is the costate variable for (6) and (λq) is the costate variable for (7). First-order conditions are

$$u'(c) = \lambda \tag{8}$$

$$\lambda_1 = \lambda\,\alpha_1 l_T^{\alpha_0+a_0} k_T^{\alpha_1+a_1-1} = \lambda\,\beta_1 p l_N^{\beta_0+b_0} k_N^{\beta_1+b_1-1} \tag{9}$$

$$\lambda_2 = \lambda\,\alpha_0 l_T^{\alpha_0+a_0-1} k_T^{\alpha_1+a_1} = \lambda\,\beta_0 p l_N^{\beta_0+b_0-1} k_N^{\beta_1+b_1} \tag{10}$$

$$\dot{\lambda} = \lambda(\rho - r) \tag{11}$$

$$p = q\left(\frac{i}{\delta k}\right)^{-\psi} \tag{12}$$

$$\dot{\lambda} q + \lambda \dot{q} = \lambda q \rho + \lambda q \delta \frac{1-\psi\left(\frac{i}{\delta k}\right)^{1-\psi}}{1-\psi} - \lambda_1 \tag{13}$$

together with the transversality conditions.

3. INDETERMINACY UNDER CONSTANT RETURNS

In this section, we summarize the dynamics of the model and derive a necessary and sufficient condition for indeterminacy. We also give a new intuition for this condition of indeterminacy and discuss the empirical plausibility of indeterminacy. Approximating the model behavior by log-linearization, we reduce the dynamics to the following bivariate first-order system:[7]

$$\begin{bmatrix} 1 & 0 \\ \frac{\psi}{\delta}(\pi_{11}-\rho) & 1+\frac{\psi}{\delta}\pi_{12} \end{bmatrix} \begin{bmatrix} \dot{k}/k^* \\ \dot{p}/p^* \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ 0 & \pi_{22} \end{bmatrix} \begin{bmatrix} (k-k^*)/k^* \\ (p-p^*)/p^* \end{bmatrix}.$$

where

$$\pi_{11} = \frac{\alpha_0 r + \alpha_0\delta(1-\beta_1) + \alpha_1\beta_0\delta}{(\alpha_0\beta_1 - \alpha_1\beta_0)},$$
$$\pi_{12} = \frac{\alpha_0(\beta_0+b_0)(r+\delta) + (\beta_1+b_1)[\alpha_0 r + \alpha_0(1-\beta_1)\delta + \alpha_1\beta_0\delta]}{(\alpha_0\beta_1-\alpha_1\beta_0)[(\alpha_0+a_0)(\beta_1+b_1)-(\alpha_1+a_1)(\beta_0+b_0)]},$$
$$\pi_{22} = \frac{(\beta_0+b_0)(r+\delta)}{-[(\alpha_0+a_0)(\beta_1+b_1)-(\alpha_1+a_1)(\beta_0+b_0)]},$$

and variables with a star represent their steady state.

[7] See the appendix for detailed derivation of this bivariate system.

3.1. CONDITION FOR INDETERMINACY

In a two-sector model without externalities, there is a unique equilibrium path. Locally near the steady state, uniqueness implies that one of the two eigenvalues is negative and the other is positive.[8] Indeterminacy in a model with externalities requires two negative eigenvalues. In a model without adjustment costs ($\psi = 0$), it is easy to see that the necessary and sufficient condition for indeterminacy is that both π_{11} and π_{22} be negative. In other words, the nontraded good sector is labor intensive from the private perspective (i.e. $\alpha_0\beta_1 - \alpha_1\beta_0 < 0$), but capital intensive from the social perspective (i.e. $(\alpha_0 + a_0)(\beta_1 + b_1) - (\alpha_1 + a_1)(\beta_0 + b_0) > 0$). This scenario can happen with an arbitrarily small degree of externalities.[9] The plausibility of this condition can be motivated from the observation that capital goods—especially for human capital—are nontraded and also that human capital production needs much labor and creates positive externalities.[10]

Since the trace and the determinant of the Jacobian are the sum and the product of the eigenvalues, the necessary and sufficient condition for indeterminacy is that the trace be negative and the determinant be positive. In our model with adjustment costs ($\psi > 0$), the trace and the determinant of the Jacobian are

$$\begin{aligned}
\mathrm{Tr} &= \left(1 + \frac{\psi}{\delta}\pi_{12}\right)^{-1}\left(\pi_{11} + \pi_{22} + \frac{\psi}{\delta}\rho\pi_{12}\right), \\
\mathrm{Det} &= \left(1 + \frac{\psi}{\delta}\pi_{12}\right)^{-1}\pi_{11}\pi_{22}.
\end{aligned}$$

Therefore, the condition for indeterminacy is that both π_{11} and π_{22} be negative and

$$\pi_{12} > -\frac{\delta}{\psi}.$$

Note that the last inequality is not binding in a model without adjustment costs ($\psi = 0$).

[8] It should be emphasized that local uniqueness does not imply global uniqueness. In Guo and Lansing (2002), a set of parameters implying local uniqueness gives rise to a flip bifurcation, which allows for cycles or sunspots away from the steady state.

[9] This point was emphasized by Benhabib and Nishimura (1998), Meng (2003), Meng and Velasco (2004).

[10] See Lahiri (2001) for more on the former observation.

3.2. INDETERMINACY WITHOUT ADJUSTMENT COSTS

To gain some intuition for the condition of indeterminacy, we construct the following example:[11]

$$\alpha_0 = 0.66, \alpha_1 = 0.34, a_0 = 0, a_1 = 0$$

for the traded good sector and

$$\beta_0 = 0.66 - \varepsilon, \beta_1 = 0.34 + \varepsilon - \xi, b_0 = 0, b_1 = \xi$$

for the non-traded good sector. Two parameters, ε and ξ, are defined as the degree of heterogeneity and the degree of externalities, respectively. We call ε the degree of heterogeneity since the capital intensity of the two sectors from the social perspective is related to $[(\alpha_0 + a_0)(\beta_1 + b_1) - (\alpha_1 + a_1)(\beta_0 + b_0)]$ which is equal to ε in our example. For our model to be well-defined, both β_0 and β_1 need be between 0 and 1. We also assume that externalities are positive $(\xi > 0)$. The region for ε and ξ satisfying these conditions is the triangle ABC in Figure 1.

In a model without adjustment costs, the condition for indeterminacy is

$$\pi_{11} < 0 \Leftrightarrow \alpha_0\beta_1 - \alpha_1\beta_0 = \varepsilon - 0.66\xi < 0$$

and

$$\pi_{22} < 0 \Leftrightarrow (\alpha_0 + a_0)(\beta_1 + b_1) - (\alpha_1 + a_1)(\beta_0 + b_0) = \varepsilon > 0.$$

That is, the degree of heterogeneity should be positive $(\varepsilon > 0)$ so that the non-traded good sector is capital intensive from the social perspective, and the required degree of externalities increases as the degree of heterogeneity increases $(\xi > \varepsilon/0.66)$ so that externalities conducive to indeterminacy offset heterogeneity adverse to indeterminacy.[12] Herrendorf, Valentinyi and Waldmann (2000) and Kim (1998) showed that heterogeneity in technologies makes it difficult for indeterminacy to arise. The region of (ε, ξ) for indeterminacy in a model without adjustment costs is represented by the triangle OCD in Figure 1. Therefore, if we choose a pair of parameter values very close to the origin within this triangle, this pair result in the property that indeterminacy can arise with an arbitrarily small degree of externalities.[13]

[11] This parameterized example is motivated by numerical examples in Meng (2003) and Meng and Velasco (2004).

[12] This intuition complements the intuition involving the Rybczynski theorem and the Stopler-Samuelson theorem, as in Benhabib and Nishimura (1998), Meng (2003), and Meng and Velasco (2004).

[13] The values calibrated in Meng and Velasco (2004) are $\varepsilon = 0.01$ and $\xi = 0.05$, which satisfies the condition for indeterminacy and lies within the triangle of OCD in Figure 1.

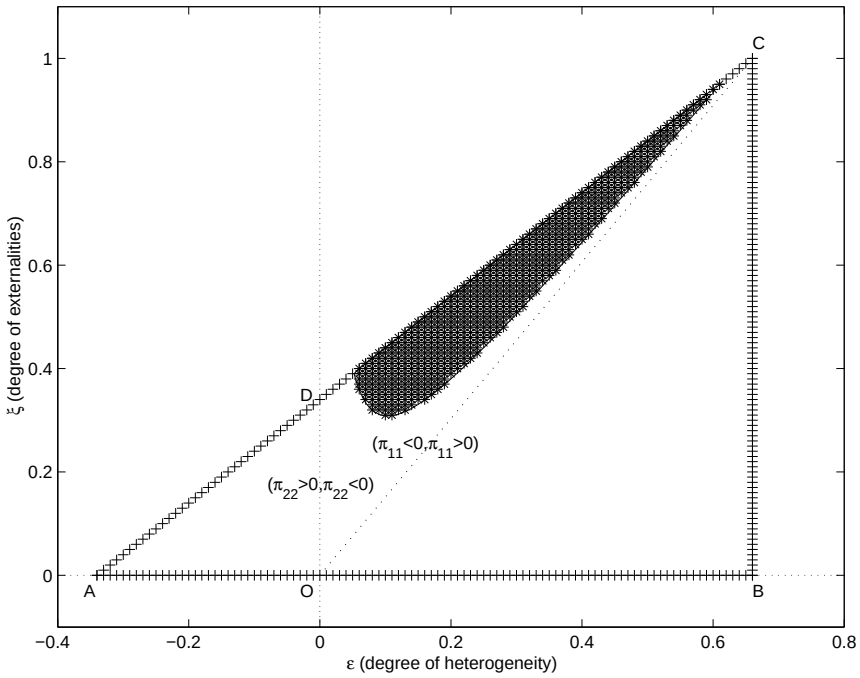

Figure 1: Indeterminacy and Adjustment Costs

3.3. PLAUSIBILITY OF INDETERMINACY WITH ADJUSTMENT COSTS

In a model with adjustment costs, the region for indeterminacy shrinks away from the lines OC and OD, since π_{12} diverges to $-\infty$ as we move towards these two lines within the triangle. For example, when we introduce very small adjustment costs of $\psi = 0.01$, the region for indeterminacy is represented by the shaded area in Figure 1. This is based on the calibration of $\rho = 0.05$ and $\delta = 0.1$, consistent with the annual data. It is not true any more that indeterminacy arises with an arbitrarily small degree of externalities. With $\psi = 0.01$, the degree of externalities (ξ) required for indeterminacy is higher than 0.3 which is an upper bound for the empirical estimates. See, for example, the summary of the empirical literature on ψ and ξ in Herrendorf and Valentinyi (2002, 2003), and Kim (2003a).[14]

[14]The empirical values for ψ as surveyed in Kim (2003a) are between 0 and 0.2. Hence our calibrated value of 0.01 is a conservative estimate for the degree of adjustment costs. Under this value, the calibrated values in the preceding footnote lies outside the shaded range of Figure 1.

As we increases the degree for adjustment costs, the region for indeterminacy shrinks and then disappears. With ψ as low as 0.04, indeterminacy cannot arise at all in our model. We conclude that indeterminacy would not arise in a model with plausible degree of adjustment costs.[15]

4. CONCLUSION

In this paper, we have investigated the conditions under which indeterminacy arises near the steady state in a model with sector-specific externalities and social constant returns. We have two findings. First, though indeterminacy can still arise with adjustment costs for capital, the property that indeterminacy occurs with an arbitrarily small degree of externalities does not hold in a model with an arbitrarily small degree of adjustment costs. Second, for empirically relevant parameter values, indeterminacy cannot arise in a model with social constant returns.

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APPENDIX

This appendix provides detailed derivation of the model's dynamics.

A. THE ECONOMY

As is standard in open economy macroeconomics, we impose $\rho = r$, a condition that ensures a well-defined steady-state with constant bond-holdings. This assumption will also imply, by (11), that marginal utility remains constant over all time –that is,

$$\lambda = \bar{\lambda} \text{ or } \dot{\lambda} = 0 \tag{14}$$

Substituting $\lambda = \bar{\lambda}$ into all other first-order conditions, by (8), we have

$$c = c(\bar{\lambda}) = \bar{c} \tag{15}$$

which means that consumption is completely smooth.

Dividing (9) by (10) yields

$$\frac{\alpha_1}{\alpha_0}\frac{l_T}{k_T} = \frac{\beta_1}{\beta_0}\frac{l_N}{k_N} \tag{16}$$

Using (9) and (16) to solve for $\frac{l_N}{k_N}$ we have

$$\frac{l_N}{k_N} = \xi p^{\frac{1}{\alpha_0+a_0-\beta_0-b_0}} = \xi p^{\frac{1}{(\alpha_0+a_0)(\beta_1+b_1)-(\alpha_1+a_1)(\beta_0+b_0)}} \equiv g(p) \tag{17}$$

where $\xi = \frac{\beta_1}{\alpha_1}\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{\alpha_0+a_0} > 0$. Substituting (3), (16) and (17) into (4), we can solve for k_N

$$k_N = \frac{\alpha_0\beta_1}{\alpha_0\beta_1 - \alpha_1\beta_0}k - \frac{\alpha_1\beta_0}{\alpha_0\beta_1 - \alpha_1\beta_0}\frac{\bar{l}}{g(p)} \tag{18}$$

In addition, the market-clearing condition for the investment (nontraded) good is

$$i = y_N = l_N^{\beta_0+b_0} k_N^{\beta_1+b_1} = \frac{\alpha_0\beta_1 g(p)^{\beta_0+b_0}}{\alpha_0\beta_1 - \alpha_1\beta_0} k - \frac{\bar{l}\alpha_1\beta_0 g(p)^{-(\beta_1+b_1)}}{\alpha_0\beta_1 - \alpha_1\beta_0}. \tag{19}$$

From (9), (12), (13) and (14), we have

$$\dot{q} = q\left[\rho + \delta\frac{1-\psi\left(\frac{i}{\delta k}\right)^{1-\psi}}{1-\psi}\right] - \beta_1 p l_N^{\beta_0+b_0} k_N^{\beta_1+b_1-1} =$$
$$q\left[\rho + \delta\frac{1-\psi\left(\frac{i}{\delta k}\right)^{1-\psi}}{1-\psi}\right] - \beta_1 p g(p)^{\beta_0+b_0} \tag{20}$$

Four equations describing the dynamics are (7), (12), (19) and (20). The steady states satisfy

$$\beta_1 g(p^*)^{\beta_0+b_0} = \rho + \delta,$$
$$\left[\alpha_0(\rho+\delta) - (\alpha_0\beta_1 - \alpha_1\beta_0)\right]k^* = \bar{l}\alpha_1\beta_0 g(p^*)^{-(\beta_1+b_1)}.$$

B. ANALYSIS OF THE DYNAMICS

Combining (7) and (19), we have

$$\dot{k} \approx i - \delta k = \frac{\alpha_0\beta_1 g(p)^{\beta_0+b_0}}{\alpha_0\beta_1 - \alpha_1\beta_0} k - \frac{\bar{l}\alpha_1\beta_0 g(p)^{-(\beta_1+b_1)}}{\alpha_0\beta_1 - \alpha_1\beta_0} - \delta k, \tag{21}$$

whose linearized version is

$$\dot{k} \approx a_{22}(k-k^*) + a_{21}(p-p^*), \tag{22}$$

where

$$a_{22} = \frac{\alpha_0\beta_1}{\alpha_0\beta_1 - \alpha_1\beta_0} g(p^*)^{\beta_0+b_0} - \delta$$
$$= \frac{\alpha_0 r + \alpha_0\delta(1-\beta_1) + \alpha_1\beta_0\delta}{(\alpha_0\beta_1 - \alpha_1\beta_0)},$$

and

$$a_{21} = \frac{\alpha_0\beta_1(\beta_0+b_0)k^* g(p^*)^{\beta_0+b_0} + \alpha_1\beta_0\bar{l}(\beta_1+b_1)g(p^*)^{-(\beta_1+b_1)}}{p^*(\alpha_0\beta_1 - \alpha_1\beta_0)[(\alpha_0+a_0)(\beta_1+b_1) - (\alpha_1+a_1)(\beta_0+b_0)]}$$
$$= \frac{k^*}{p^*}\frac{\alpha_0(\beta_0+b_0)(r+\delta) + (\beta_1+b_1)[\alpha_0 r + \alpha_0(1-\beta_1)\delta + \alpha_1\beta_0\delta]}{(\alpha_0\beta_1 - \alpha_1\beta_0)[(\alpha_0+a_0)(\beta_1+b_1) - (\alpha_1+a_1)(\beta_0+b_0)]}.$$

We use the notations a_{ij}'s for ease of comparison with the results of Meng and Velasco (2003, 2004).

We transform (12) as

$$\frac{q}{p} = \left(\frac{i}{\delta k}\right)^{\psi} \approx 1 + \psi\left(\frac{i}{\delta k} - 1\right) \approx 1 + \frac{\psi}{\delta}\frac{\dot{k}}{k}$$

and transform (20) as

$$\begin{aligned}\dot{q} &= q\left[\rho + \delta\frac{1-\psi\left(\frac{i}{\delta k}\right)^{1-\psi}}{1-\psi}\right] - \beta_1 p g(p)^{\beta_0+b_0} \\ &\approx q(\rho+\delta) - q\delta\left(\frac{q}{p} - 1\right) - \beta_1 p g(p)^{\beta_0+b_0}\end{aligned}$$

Now we have

$$\begin{aligned}\frac{\dot{q}}{p} &\approx \frac{q}{p}(\rho+\delta) - \delta\left(\frac{q}{p}\right)\left(\frac{q}{p} - 1\right) - \beta_1 g(p)^{\beta_0+b_0} \\ &\approx \left(1 + \frac{\psi}{\delta}\frac{\dot{k}}{k}\right)(\rho+\delta) - \psi\frac{\dot{k}}{k} - \beta_1 g(p)^{\beta_0+b_0} \\ &\approx (\rho+\delta) + \frac{\psi}{\delta}\rho\frac{\dot{k}}{k} - \beta_1 g(p)^{\beta_0+b_0},\end{aligned}$$

whose left hand side is

$$\frac{\dot{q}}{p} \approx \frac{\frac{d}{dt}\left[p\left(1+\frac{\psi}{\delta}\frac{\dot{k}}{k}\right)\right]}{p} \approx \frac{1}{p}\left(\dot{p} + p\frac{\psi}{\delta}\frac{\ddot{k}}{k}\right) = \frac{\dot{p}}{p} + \frac{\psi}{\delta}\frac{\ddot{k}}{k} \approx \frac{\dot{p}}{p} + \frac{\psi}{\delta}\frac{a_{21}\dot{p} + a_{22}\dot{k}}{k}$$

Note that the last equality comes from (22). Therefore, we have

$$\begin{aligned}\frac{\dot{p}}{p} &\approx (\rho+\delta) + \frac{\psi}{\delta}\rho\frac{\dot{k}}{k} - \beta_1 g(p)^{\beta_0+b_0} - \frac{\psi}{\delta}\frac{a_{21}\dot{p} + a_{22}\dot{k}}{k} \\ &\approx (\rho+\delta-\beta_1 g(p)^{\beta_0+b_0}) - \frac{\psi}{\delta k}\left[a_{21}\dot{p} + (a_{22}-\rho)\dot{k}\right],\end{aligned}$$

whose linearization is

$$\left(1 + \frac{\psi}{\delta}\frac{p^*}{k^*}a_{21}\right)\dot{p} + \frac{\psi}{\delta}\frac{p^*}{k^*}(a_{22}-\rho)\dot{k} \approx a_{11}(p-p^*).$$

where

$$\begin{aligned} a_{11} &= \frac{\beta_1(\beta_0+b_0)g(p^*)^{\beta_0+b_0}}{(\alpha_1+a_1)(\beta_0+b_0)-(\alpha_0+a_0)(\beta_1+b_1)} \\ &= \frac{(\beta_0+b_0)(r+\delta)}{-[(\alpha_0+a_0)(\beta_1+b_1)-(\alpha_1+a_1)(\beta_0+b_0)]} \end{aligned}$$

Summarizing, our bivariate system is

$$\begin{bmatrix} 1+\frac{\psi}{\delta}\frac{p^*}{k^*}a_{21} & \frac{\psi}{\delta}\frac{p^*}{k^*}(a_{22}-\rho) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} p-p^* \\ k-k^* \end{bmatrix}.$$

Changing the order of variables and expressing them in terms of log deviations, we have

$$\begin{bmatrix} 1 & 0 \\ \frac{\psi}{\delta}(\pi_{11}-\rho) & 1+\frac{\psi}{\delta}\pi_{12} \end{bmatrix} \begin{bmatrix} \dot{k}/k^* \\ \dot{p}/p^* \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ 0 & \pi_{22} \end{bmatrix} \begin{bmatrix} (k-k^*)/k^* \\ (p-p^*)/p^* \end{bmatrix}.$$

The determinant of the Jacobian is

$$\begin{aligned} \text{Det} &= \begin{vmatrix} 1 & 0 \\ \frac{\psi}{\delta}(\pi_{11}-\rho) & 1+\frac{\psi}{\delta}\pi_{12} \end{vmatrix}^{-1} \begin{vmatrix} \pi_{11} & \pi_{12} \\ 0 & \pi_{22} \end{vmatrix} \\ &= \left(1+\frac{\psi}{\delta}\pi_{12}\right)^{-1}\pi_{11}\pi_{22}. \end{aligned}$$

Since the Jacobian is

$$\begin{bmatrix} 1 & 0 \\ \frac{\psi}{\delta}(\pi_{11}-\rho) & 1+\frac{\psi}{\delta}\pi_{12} \end{bmatrix}^{-1} \begin{bmatrix} \pi_{11} & \pi_{12} \\ 0 & \pi_{22} \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \frac{-\frac{\psi}{\delta}(\pi_{11}-\rho)\pi_{11}}{1+\frac{\psi}{\delta}\pi_{12}} & \frac{\pi_{22}-\frac{\psi}{\delta}(\pi_{11}-\rho)\pi_{12}}{1+\frac{\psi}{\delta}\pi_{12}} \end{bmatrix},$$

its trace is

$$\text{Tr} = \left(1+\frac{\psi}{\delta}\pi_{12}\right)^{-1}\left(\pi_{11}+\pi_{22}+\frac{\psi}{\delta}\rho\pi_{12}\right).$$