Journal of Economic Theory and Econometrics, Vol. 23, No. 3, Sep. 2012, 201–212

Semiparametric Estimation of Dynamic Discrete Choice Models of Optimal Stopping with Unobserved Heterogeneity*

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Abstract This paper derives a multinomial logit form for dynamic discrete choice models of optimal stopping and suggests a two-step estimator. The first stage estimates nonparametrically the choice probability of terminal state next period. The second stage is an implementation of modified multinomial logit estimation. This paper also suggests semiparametric conditional choice probability (CCP) estimators for discrete choice dynamic models allowing for unobserved individual heterogeneity. I describe a modified Expectation-Maximization (EM) algorithm that involves nonparametric first stage estimation.

Keywords Dynamic Model, Optimal Stopping, EM algorithm.

JEL Classification C14, C25

*The author gratefully acknowledges helpful discussions with Geert Ridder. Also I am grateful to two anonymous referees for their helpful comments.

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Received January 17, 2012, Revised April 12, 2012, Accepted July 17, 2012

1. INTRODUCTION

This paper considers discrete choice dynamic models of optimal stopping. Optimal stopping models include job search (Wolpin, 1987), patent renewal (Pakes 1986), and engine replacement (Rust, 1987). The semiparametric identification of the discrete choice dynamic model is considered in many studies such as Heckman and Navarro (2007). In this paper I derive a multinomial logit form for dynamic models of optimal stopping and suggest a two-step estimator. The first stage estimates nonparametrically the choice probability of absorbing state for the next period. The second stage is an implementation of multinomial logit estimation.

The derivation of the multinomial logit form is influenced by the conditional choice probability estimator (CCP) developed by Hotz and Miller (1993). They show that there is a one-to-one mapping from the conditional choice probabilities to the conditional valuation functions (or expected value function). Using this mapping, the conditional valuation function can be expressed as the function of choice probabilities and current utility functions. In this paper I show that this representation can be simplified to a large extent in the case of optimal stopping problem. The estimator in this paper does not need either a backward recursion or the discretization of state space in the case of models with continuous state variables.

Also, this paper suggests a modified Expectation-Maximization (EM) algorithm to include individual permanent heterogeneity for discrete choice dynamic models. Aguirregabiria and Mira (2007) suggested an algorithm to estimate finite mixture dynamic model of markets in *infinite horizon* dynamic games under the assumption that there exist the steady state probability distributions of state variables. Arcidiacono and Miller (2011) adapted the EM algorithm to CCP estimation techniques for discrete and continuous choice dynamic models with *discrete state* variables, where the unobserved heterogeneity can either evolves over time or be permanent. The proposed estimator in this paper is relevant whether the state variable is continuous or discrete.

2. DYNAMIC DISCRETE CHOICE MODEL

This section considers a finite-horizon and discrete-choice model without unobserved heterogeneity. Time is discrete and it is indexed by $t \in \{1,2,...,T\}$. At every period an agent chooses an action $j \in A = \{1,2,...,J\}$. Let $d_{tj} = 1$ indicate the agent chooses action j in t and $d_{tj} = 0$ represents the agent chooses something else. Then $d_t = (d_{t1},...,d_{t,J-1})'$ describes the agent's action in period

t.

Conditions at time t are summarized by a vector of state variables $w_t = (x_t, \varepsilon_t) \in W \subset R^{M+1}$. The x_t is a vector of state variable observable by the econometrician. A vector of random preference shocks or private information in each alternative j is denoted by $\varepsilon_t = (\varepsilon_{t1}, .., \varepsilon_{tJ})$. These state variables, ε_{tj}, are assumed to be distributed i.i.d. across time periods, agents, and actions with type 1 extreme value distribution. The states are assumed to follow a Markov process with transition probability $p(w_{t+1}|w_t, d_t)$. Under these assumptions, the transition probability can be written as $p(w_{t+1}|w_t, d_t) = p(x_{t+1}|x_t, d_t)p(\varepsilon_{t+1})$.

The agent's objective is to maximize the expected value of a sum of period-specific utilities. Let $\widetilde{u}_{tj}(\cdot)$ denote the period-specific utility associated with choice j in period t. I assume that the period-specific utility functions are additively separable and linear in parameters:

$$\begin{aligned} \widetilde{u}_{tj}(x_t, \varepsilon_t, d_t) &= u_{tj}(x_t) + \varepsilon_{tj} \\ &= \alpha_j + x_t \theta_j + \varepsilon_{tj} \end{aligned}$$

where (α_j, θ_j) is a vector of parameters. For the identification purpose, I normalize $\alpha_J = \theta_J = 0$.

The agent sequentially chooses $\{d_t\}_{t=1}^T$ to maximize the expected discount sum of payoffs

$$E\left[\sum_{s=t}^{T}\sum_{j=1}^{J} \beta^{s-t} d_{sj}\left[u_{sj}(x_s) + \varepsilon_{sj}\right] \mid w_t\right]$$

where β is the discount factor. The expectation is over the agent's private shock and actions in the current period as well as future values of the state variables, actions, and private shocks.

Let $d_s^0 = (d_{s1}^0, ..., d_{s,J-1}^0)'$ denote the agent's optimal choice in period s. Define the conditional valuation function associated with choosing j as:

$$v_{tj}(x_t) = E\left[\sum_{s=t+1}^{T}\sum_{j=1}^{J} \beta^{s-t} d_{sj}^0\left[u_{sj}(x_s) + \varepsilon_{sj}\right] |x_t, d_{tj} = 1\right] \tag{1}$$

This implies that the value function in period t can be written as:

$$V_t(x_t, \varepsilon_t) = \max_{j \in A}\ \left[u_{tj}(x_t) + \varepsilon_{tj} + v_{tj}(x_t)\right]$$

3. PROBLEMS OF OPTIMAL STOPPING

3.1. DERIVATION OF CONDITIONAL CHOICE PROBABILITY

Consider an optimal stopping problem in which the state J is an absorbing state. This means $d_{tJ} = 1$ implies $d_{sJ} = 1$ for all periods $s \in \{t+1, ..., T\}$. Under the assumption on the distribution of $\varepsilon' s$, the choice probability that the agent chooses action j is[1]:

$$\begin{aligned} p_{tj}(x_t) &= \Pr\left(k = \underset{j \in A}{\arg\max}\left[u_{tj} + \varepsilon_{tj} + v_{tj}\right]\right) \\ &= \frac{\exp\left[u_{tj} + v_{tj} - v_{tJ}\right]}{1 + \sum_{k=1}^{J-1} \exp\left[u_{tk} + v_{tk} - v_{tJ}\right]} \end{aligned} \tag{2}$$

where $u_{tj} \equiv u_{tj}(x_t)$ and $v_{tj} \equiv v_{tj}(x_t)$ are used for notational simplicity. Similarly, I adopt $p_{tj} \equiv p_{tj}(x_t)$. From (2), we have:

$$\ln p_{tj} - \ln p_{tk} = u_{tj} + v_{tj} - u_{tk} - v_{tk} \tag{3}$$

for any k.

A key step is to derive the conditional valuation function of choosing j as a function of choice probabilities and utility functions. Since the state J is an absorbing state, the conditional valuation function of choosing J at t is:

$$\begin{aligned} v_{tJ}(x_t) &= E\left[\sum_{s=t+1}^{T} \beta^{s-t}\left[u_{sJ} + \varepsilon_{sJ}\right]\right] \\ &= \frac{\beta - \beta^{T-t}}{1-\beta}\gamma \end{aligned}$$

where $E[\varepsilon] = \gamma$ and γ is the Euler's constant ($\gamma \approx 0.57722$).

The next step is to find an expression of the conditional valuation function associated with choosing j. The conditional valuation function (1) can be express

[1] Like static discrete choice modes, the choice probability in the dynamic discrete choice models depends only on the difference in value funcion, not its absolute level. This paper uses the aborbing state J as an alternative for other actions, because the conditional valuation function of choosing J has a simple anlytical form in the optimal stopping problem, as shown in this paper.

as:

$$\begin{aligned} v_{tj} &= E\left[\max_{k\in A}\ u_{t+1,k}(x_{t+1}) + \varepsilon_{t+1,k} + v_{t+1,k}(x_{t+1})|x_t, d_{tj} = 1\right] \\ &= \sum_{k=1}^{J} p_{t+1,k}(x_{t+1}) \times \\ &\quad \times E\left[u_{t+1,k}(x_{t+1}) + \varepsilon_{t+1,k} + v_{t+1,k}(x_{t+1})|x_{t+1}, x_t, d_{tj} = 1\right] \end{aligned} \tag{4}$$

Note that the expectation of the first line is over ε's and x_{t+1}, while the expectation of the second line is over only x_{t+1}. This expression is already useful since there is no max operator. Under the assumed distribution for $\varepsilon's$, the conditional expectation of $\varepsilon_{t+1,j}$ takes the form:

$$\begin{aligned} &E\left[\varepsilon_{t+1,k}|d_{t+1,j} = 1\right] \\ &= E\left[\varepsilon_{t+1,k}|k = \arg\max\ \{u_{t+1,k} + \varepsilon_{t+1,k} + v_{t+1,k}\}\right] \\ &= \gamma + E\left[\ln\left(\sum_{m=1}^{J} \exp\left[u_{t+1,m} + v_{t+1,m}\right]\right) - (u_{t+1,k} + v_{t+1,k})\right] \\ &= \gamma + E\left[\ln\left(\sum_{m=1}^{J} \frac{p_{t+1,m}}{p_{t+1,k}} \exp(u_{t+1,k} + v_{t+1,k})\right) - (u_{t+1,k} + v_{t+1,k})\right] \\ &= \gamma - E\left[\ln p_{t+1,k}\right] \end{aligned} \tag{5}$$

for $k = 1, ..., J$. Substituting (5) into (4) yields:

$$v_{tj}(x_t) = \gamma + \sum_{k=1}^{J} p_{t+1,k} E\left[-\ln p_{t+1,k} + u_{t+1,k} + v_{t+1,k}\right]$$

Equation (3) implies

$$\ln p_{t+1,k} - \ln p_{t+1,J} = u_{t+1,k} + v_{t+1,k} - v_{t+1,J}$$

Using this, above equation can be written as:

$$\begin{aligned} v_{tj}(x_t) &= \gamma + \sum_{k=1}^{J} p_{t+1,k} E\left[-\ln p_{t+1,J} + v_{t+1,J}|x_t, d_{tj} = 1\right] \\ &= \gamma - E\left[\ln p_{t+1,J}|x_t, d_{tj} = 1\right] + E\left[v_{t+1,J}|x_t, d_{tj} = 1\right] \end{aligned} \tag{6}$$

because $\sum_{k=1}^{J} p_{t+1,k} = 1$. Since J is an absorbing state, the expectation of $v_{t+1,J}$

can be expressed as:

$$\begin{aligned} E\left[v_{t+1,J}|x_t, d_{tj}=1\right] &= E\left[\sum_{s=t+2}^{T} \beta^{s-t}\left[u_{sJ}+\varepsilon_{sj}\right]\right] \\ &= \frac{\beta^2-\beta^{T-t}}{1-\beta}\gamma \end{aligned}$$

Substituting this into (6), we finally have:

$$v_{tj}(x_t) = \gamma + \frac{\beta^2-\beta^{T-t}}{1-\beta}\gamma - E\left[\ln p_{t+1,J}|x_t, d_{tj}=1\right]$$

Therefore, the conditional valuation function can be expressed as a function of choice probability of choosing absorbing state for the next period ($p_{t+1,J}$) only.

Using the above results, the conditional choice probability (2) becomes:

$$p_j(x_t) = \frac{\exp\left(\delta_j + x_t\theta_j - E\left[\ln p_{t+1,J}|x_t, d_{tj}=1\right]\right)}{1+\sum_{k=1}^{J-1}\exp\left(\delta_k + x_t\theta_j - E\left[\ln p_{t+1,J}|x_t, d_{tj}=1\right]\right)} \tag{7}$$

where $\delta_j = \alpha_j + (1-\beta)\gamma$. Note that a static logit model does not have the expectation term $E\left[\ln p_{t+1,J}|x_t, d_{tj}=1\right]$.

3.2. IDENTIFICATION AND ESTIMATION

The equation (7) shows that the constant term in utility function α_j and the discount factor β can not be separately identified. This is a concrete example of the nonidentification of discrete choice dynamic models addressed in Rust (1994, Lemma 3.2, p3127) and Magnac and Thesmar (2002, Proposition 2, p.806).

Estimation procedure is as follows. The first stage estimates the expectation of choice probability of absorbing state for the next period, $E\left[\ln p_{t+1,J}|x_t, d_{tj}=1\right]$, nonparametrically. Plugging the first stage estimates into (7), the second stage estimates parameters by maximizing likelihood function, which is a simple implementation of multinomial logit estimation. One can recover α_j by $\hat{\alpha}_j = \hat{\delta}_j + \beta\gamma - \gamma$ given the values of β, $\gamma \simeq 0.57722$, and the estimates $\hat{\delta}_j$.

The two-step estimator of this paper is a semiparametric estimator considered by Newey (1994). He shows that, under appropriate regularity conditions, the second stage asymptotic variance will be independent of the particular choice of nonparametric method used to estimate the first stage. Nevertheless, asymptotic variance of the estimator in this paper is much more complicated than that of a semiparametric estimator since one has to consider three corrections: (1) estimation error in the preliminary estimate of transition probability $p(s_{t+1}|s_t, d_t)$,

(2) estimation error in the first stage nonparametric estimate of $p_{t+1,J}$ given $x_t, d_{tj} = 1$, and (3) error in the numerical expectation of $E\left[\ln p_{t+1,J} | x_t, d_{tj} = 1\right]$.

In practice it is simple to use a numerical method suggested by Rust (1994, p3109). Denote $\hat{\phi}_1, \hat{\phi}_2 = (\alpha, \theta)$ be the consistent estimates from preliminary stage for the transition probabilities and the second stage respectively. Then, the numerical method is to use $\hat{\phi}_1, \hat{\phi}_2$ as starting values for obtaining a numerical information matrix. The covariance matrix is formed by inverting the information matrix. Alternatively, it will typically be easier to use the bootstrap to estimate standard errors.

3.3. MONTE CARLO EVIDENCE OF THE ESTIMATOR

I apply the estimation approach to an example of models of optimal stopping problem. For Monte Carlo experiments, I set $J = 2$, $T = 4$ and $u_{t1} = -1.5 + 0.5x_t$. $\ln x_t$ is assumed to evolves according to an AR(1) process; $\ln x_t = 0.95 \ln x_{t-1} + \eta_t$ where $\eta_t \sim N(0, 0.2^2)$. Data with 5000 observations are generated using the method described in Rust (1987). I adopt the logit series estimator for the first step estimation and Gauss-Hermite method for the calculation of the expectation. The results after 100 simulations show that the mean of the estimates are $(-1.5163, 0.4960)$ with the standard errors of $(0.070, 0.019)$.

4. ALLOWING FOR UNOBSERVED INDIVIDUAL HETEROGENEITY

For the illustration of our algorithm, I consider a version of a dynamic labor supply model of older males. The model is a finite horizon model where individuals start at age $t = 1$ and end their lives at time T. A state variable is the wage income x_t, which is assumed to be continuous. At date t, individuals choose an action among two alternatives: working $(j = 1)$ and not working $(j = 2)$. I assume that $j = 2$ (not working) is the absorbing state. Therefore the model considered here is a discrete choice model with a continuous state variable. Let $d_{tj} = 1$ if alternative j is chosen at age t and zero otherwise. The state variable evolves according to

$$x_t = d_{t1} \rho x_{t-1} + d_{t2} \rho x_{t-1} + \eta_t$$

where $\eta_t \sim N(0, \sigma^2)$ is a shock. A discrete probability distribution of individual heterogeneity is assumed to has G points of support $\theta_1, ..., \theta_G$ and corresponding masses, $\pi_1, ..., \pi_G$. The mixing proportion, π_g, denotes the probability that an individual belongs to a type g.

In the model considered here, the per-period utility functions are, for $g = 1,2$:

$$u_1(x_t, d_t; \theta_g) = \alpha_g + \delta x_t \ \text{ if } d_{t1} = 1$$

where α_g is a type-specific constant term and δ is a type-common coefficient, and $\theta_g = (\alpha_1, \alpha_2, \delta)$. For the identification purpose, the utility function for $j = 2$ is set to be zero, that is $u_2(x_t, d_t; \theta_g) = 0$, as in the static discrete choice model. The agent sequentially chooses $\{d_t\}_{t=1}^T$ to maximize the objective function:

$$E\left[\sum_{s=t}^{T}\sum_{j=1}^{2}\beta^{s-t} d_{sj}\left[u_j(x_s, d_s; \theta_g) + \varepsilon_{sj}\right] \mid w_t\right]$$

where β is the discount factor. The private shocks, ε_{tj}, are distributed i.i.d. across time periods, agents, and actions with type 1 extreme value distribution. For the estimation purpose, I will treat the discount factor β as known. The set of parameters to be estimated is θ_g along with mixing proportion π_1.

Let $d_s^0 = (d_{s1}^0, ..., d_{s,J-1}^0)'$ denote the agent's optimal choice in period s. Define the conditional valuation function associated with choosing j as:

$$v_j(x_t; \theta_g) = E\left[\sum_{s=t+1}^{T}\sum_{j=1}^{2}\beta^{s-t} d_{sj}^0\left[u_j(x_s; \theta_g) + \varepsilon_{sj}\right] | x_t, d_{tj} = 1\right]$$

Under the assumption on the distribution of $\varepsilon' s$, from (7), the choice probability that the agent chooses action $j = 1$ becomes:

$$p_1(x_t; \theta_g) = \frac{\exp\left(\phi_g + \delta x_t - E\left[\ln p_2(x_{t+1}; \theta_g) | x_t, d_{t1} = 1\right]\right)}{1 + \exp\left(\phi_g + \delta x_t - E\left[\ln p_2(x_{t+1}; \theta_g) | x_t, d_{t1} = 1\right]\right)}$$

where $\phi_g = \alpha_g + (1-\beta)\gamma$. We could estimate (ϕ_g, δ) in the second step using the nonparametric estimates of $p_2(x_{t+1}; \theta_g)$ for each type of the first step. One can recover α_g by $\hat{\alpha}_g = \hat{\phi}_g + \beta\gamma - \gamma$ given the values of β, γ, and the estimates $\hat{\phi}_g$. However it is not obvious how to estimate the choice probability for each type, because $p_2(x_{t+1}; \theta_g)$ are different across unobserved types and not observable.

4.1. ESTIMATION OF FINITE MIXTURE

Consider the panel data $\{x_{nt}, d_{nt}\}$ on observed states and decisions of a collection of agents, $t = 1, ..., T$ and $n = 1, ..., N$. The log-likelihood function is:

$$L(d_n, x_n; \pi, \theta) = \log \prod_{n=1}^{N}\prod_{t=1}^{T}\sum_{g=1}^{G}\pi_g f(d_{nt} | x_{nt}; \theta_g)$$

where $d_n = (d'_{n1},...,d'_{nT})', x_n = (x'_{n1},...,x'_{nT})', \pi = (\pi_1,...,\pi_G)'$, $\theta = (\theta'_1,...,\theta'_G)'$, and $f(d_{nt}|x_{nt};\theta_g) = p_j(x_{nt};\theta_g)^{d_{tj}}$. The maximum likelihood (ML) estimate of the finite mixtures cannot be found analytically.

Estimating the mixtures is an incomplete data problem where the type labels of each observation are missing and the EM algorithm can be adopted. Here, the missing part is a label z_n, associated to each observation (x_n, d_n), indicates the component of that observation. Each label is a binary vector $z_n = [z_{n1},...,z_{nG}]'$ such that

$$z_{ng} = 1 \text{ and } z_{nl} = 0 \text{ for } l \neq g$$

indicates that (x_n, d_n) is generated by component g. If the missing data $z = \{z_1,...,z_N\}$ was observed, we could write the complete log-likelihood function:

$$L_c(z_{ng}, d_n, x_n; \pi, \theta) = \sum_{n=1}^{N}\sum_{t=1}^{T}\sum_{g=1}^{G} z_{ng} \log\left[\pi_g f_g(d_{nt}|x_{nt};\theta_g)\right]$$

The ML estimates are, for $g = 1,...,G$:

$$\hat{\pi}_g = \frac{1}{N}\sum_{n=1}^{N} z_{ng} \tag{8}$$

$$\hat{\theta}_g = \arg\max \sum_{n=1}^{N}\sum_{t=1}^{T} z_{ng} \log f_g(d_{nt}|x_{nt};\theta_g) \tag{9}$$

The EM algorithm proceeds by applying two steps. The E-Step computes the conditional expectation of L_c given current m-step parameter estimates ; $\hat{\pi}^{(m)}$ and $\hat{\theta}^{(m)}$,

$$\begin{aligned} Q(\pi,\theta|;\hat{\pi}^{(m)},\hat{\theta}^{(m)}) &\equiv E\left[L_c(d,x,z;\pi,\theta)|d,x;\hat{\pi}^{(m)},\hat{\theta}^{(m)}\right] \\ &= L_c\left(d,x;\pi,\theta,E\left[z|x;\hat{\pi}^{(m)},\hat{\theta}^{(m)}\right]\right) \end{aligned}$$

because L_c is linear in the z_{ng}'s. In other words, the E-step reduces to the computation of the expected value of the missing data, which is then plugged into the complete log-likelihood function. Since the z_{ng}'s are binary,

$$E\left[z_{ng}|d,x;\pi^{(m)},\theta^{(m)}\right] = \frac{\hat{\pi}_g^{(m)}\sum_{t=1}^{T} f(d_{nt}|x_{nt};\hat{\theta}_g^{(m)})}{\sum_{r=1}^{K}\hat{\pi}_r^{(m)}\sum_{t=1}^{T} f(d_{nt}|x_{nt};\hat{\theta}_r^{(m)})} \equiv \tau_g(x_n,d_n;\widehat{\theta}^{(m)}) \tag{10}$$

which is the posterior probability that (d_n, x_n) was generated by type g. The M-step solves equations (8) and (9) with $\tau_g(x_n, d_n; \widehat{\theta}^{(m)})$ replacing z_{ng}.

Based on the above, I propose the following algorithm:

1. The current estimates are $\hat{\pi}_g^{(m)}, \hat{\theta}_g^{(m)}$, and nonparametric estimates $\hat{p}^{(m)}(x_{t+1})$ for each type g.

2. Use these estimates to obtain conditional valuation function $v_j(x_t;\theta_g^{(m)})$. Calculate also $\tau_g(x_n,d_n;\hat{\theta}_g^{(m)})$ by (10).

3. Estimate $(\hat{\pi}_g^{(m+1)}, \hat{\theta}_g^{(m+1)})$ by

$$\begin{aligned} \hat{\theta}_g^{(m+1)} &= \arg\max_{\theta\in\Theta} \sum_{n=1}^{N}\sum_{t=1}^{T} \tau_g(x_n,d_n;\hat{\theta}_g^{(m)}) \log f_g(d_n|x_n;\theta) \qquad (11) \\ \hat{\pi}_g^{(m+1)} &= \frac{1}{N}\sum_{n=1}^{N} \tau_g(x_n,d_n;\hat{\theta}_g^{(m)}) \end{aligned}$$

4. For each observation n, estimate the component label vector z_n by

$$\begin{aligned} \hat{z}_{ng}^{(m+1)} &= 1 \text{ if } g = \arg\max_h \ \tau_h(x_n,d_n;\hat{\theta}_h^{(m)}) \\ &= 0 \text{ otherwise} \end{aligned}$$

   That is, update the allocation rule by the empirical Bayes modal predictor.

5. Use the sample (x_n, d_n, z_n) to estimate the choice probability $\hat{p}_j^{(m+1)}(x_{t+1})$ for each type nonparametrically and go to step 2 and iterate the procedure until some criterion is met.

In the Step 4, we assign each observation to the type of the mixture to which it has the highest posterior probability of belonging. Given the estimate of the type label vector $(\hat{z})$, the choice probabilities $(\hat{p}_j^{(m+1)}(x_{t+1}))$ for each type can be estimated by nonparametric smoothing methods using kernel, sieve, or parametric smoothing methods such as linear probability method. Therefore, the proposed updating steps 4 and 5 work directly even when the state variable is continuous.

The algorithms proposed in this paper and by Arcidiacono and Miller (2011) are different in the updating rule. Arcidiacono and Miller suggest a way to update $\hat{p}_j^{(m+1)}(x_{t+1})$ using the weighted empirical likelihood:

$$\hat{p}_j^{(m+1)}(x_{nt}=x;\hat{\theta}_g^{(m+1)}) = \frac{\sum_{n=1}^{N} d_{njt}\tau_g(x_n,d_n;\hat{\theta}_g^{(m+1)}) I(x_{nt}=x)}{\sum_{n=1}^{N} \tau_g(x_n,d_n;\hat{\theta}_g^{(m+1)}) I(x_{nt}=x)}$$

This updating method is applied on a discrete point x of x_{nt}, and hence pointwise. So one has to use another approximation methods such as interpolation or extrapolation in the case of continuous state variable model. On the contrary, the proposed updating step 4 is relevant even when the state variable is continuous. The Arcidiacono and Miller's algorithm can accommodate very flexible forms of unobserved heterogeneities, whereas the suggested algorithm in this paper can allow for only a small number of time-constant unobserved types.

In this paper I do not examine a consistent estimate of standard errors of the suggested estimator. In practice one can use a numerical method suggested by Rust (1994, p3109), mentioned in Section 3.2, or use the bootstrap to estimate standard errors.

4.2. MONTE CARLO EVIDENCE OF THE ESTIMATOR

For the Monte Carlo experiment, I set $(\alpha_1, \alpha_2, \delta_1, \pi_1) = (-1.5, -0.5, 0.5, 0.6)$. I also set $(\beta, \rho, \sigma) = (0.95, 0.95, 0.2)$. Using this parameterized model, data with 2000 observations were generated. I adopt the logit series estimator for this first step estimation and Gauss-Hermite method for the calculation of the expectation. The results from 100 simulations are shown in column (1) of Table 1. The estimators are close to unbiased. Around 30 iterations are needed for the algorithm converges. In the second experiment (Experiment 2), I set $\rho = 0.90$ and the remaining parameters same as the first experiment. In the third experiment (Experiment 3), I set $(\alpha_1, \alpha_2, \delta_1, \pi_1) = (-1.0, -0.5, 0.7, 0.55)$ and $(\beta, \rho, \sigma) = (0.95, 0.95, 0.2)$. The results from the both experiments are shown in column (2) and (3) respectively. The experimental results show that the estimators are close to unbiased.

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Table 1: Monte Carlo Simulation Results

	Experiment 1		Experiment 2		Experiment 3	
(β,ρ,σ)	(0.95,0.95,0.2)		(0.95,0.90,0.2)		(0.95,0.95,0.2)	
	True	Result (1)	True	Result (2)	True	Result (3)
α_1	-1.5	-1.509	-1.5	-1.490	-1.0	-1.052
		(0.089)		(0.160)		(0.160)
α_2	-0.5	-0.514	-0.5	-0.518	-0.5	-0.511
		(0.060)		(0.106)		(0.130)
δ	0.5	0.497	0.5	0.497	0.7	0.704
		(0.019)		(0.038)		(0.043)
π_1	0.6	0.592	0.6	0.594	0.55	0.552
		(0.014)		(0.015)		(0.034)
Iteration		29.48		30.96		40.62
		(10.45)		(9.94)		(16.20)

(note) Standard deviations are in parentheses.

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