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Standard Auctions with Security Bids

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Abstract Peter DeMarzo, Ilan Kremer and Andrzej Skrzypacz (2005) study auctions where bidders compete in securities. As the main results, they show that: a steeper security generates a higher revenue for the seller; the revenue ranking between the first-price and second-price auction depends on the steepness of security; and the first-price auction combined with call option achieves the highest revenue among a general set of auction mechanisms. As pointed out by Che and Kim (2010), using steeper securities can cause the adverse selection problem and reduce the seller's revenue, in case a bidder who expects a higher return must incur a higher investment cost. While the analysis of Che and Kim (2010) focuses on the second-price auction, this paper analyzes the first-price auction to show that it is plagued by the same problem and in fact more vulnerable than the second-price auction to the adverse selection problem, which may lead to a reversal of the DKS's revenue ranking between the two auctions.

Keywords Auctions, Security design, Adverse selection

JEL Classification D44, G32, G34

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1. INTRODUCTION

For many auctioned objects, their final value or return is realized with some time lag after an auction is completed. This means bidders' bids in the auction cannot be contingent on their future values of the object. However, if these values are verifiable ex-post, then an auction rule can be designed such that bidders' bids determine their payments later according to the ex-post value realizations. In fact, many real-world auctions involve such payment rules: in auctions for the sale of oil leases, bidders often compete in both up-front payment and royalty rate, i.e. the portion of ex-post revenue accruing to the seller; in sponsored search auctions, bidders' payments are proportional to the number of clicks realized ex-post. In these auctions, the bidders' bids take the form of security whose payment depends on the future cash flow. That using security bids helps raise the seller's revenue has been shown by literature in the auction theory such as Robert Hansen (1985), John Riley (1988), Jacques Crémer (1987), and Matthew Rhodes-Kropf and S. Viswanathan (2000).

Recently, Peter DeMarzo, Ilan Kremer and Andrzej Skrzypacz (2005, henceforth DKS) took the issue of security design in auctions to more general level by showing that the steepness of securities plays a critical role in determining the seller's revenue. Their main finding is that in standard auctions—first-price, second-price, and their dynamic counterparts—the seller's revenue is higher with a 'steeper' security—that is, a security whose payment increases faster with the realized value. Intuitively, a steeper security is more effective for reducing the competitive gap among bidders expecting different returns from the auctioned object. An important implication of this result regarding the design of optimal security is that the call option, which is the steepest security, achieves the highest revenue among all feasible securities. Another important implication is regarding the revenue ranking between the first-price and second-price auctions: if call option (debt, resp.) is used, then the first-price (second-price, resp.) auction is superior. These results imply that the first-price auction with call option (debt, resp.) generates the highest (lowest, resp.) revenue among a general set of auction mechanisms.

The current paper aims to show that the DKS's revenue ranking should be taken with some caution, since the first-price auction is more vulnerable to the adverse selection problem, which refers to a phenomenon where a bidder with the lowest return becomes a winner. The adverse selection problem associated with security bids has been first pointed out by Che and Kim (2010, henceforth

CK).[1] Their analysis of the second-price auction shows that in case a bidder who expects a higher return must incur a higher investment cost, using steeper securities can cause the adverse selection and thereby reduce the seller's revenue. The current paper shows that the first-price auction is plagued by the same problem. More importantly, the first-price auction is more vulnerable than the second-price auction in the sense that whenever a security bid causes the adverse selection in the latter, it does the same in the former. When the adverse selection occurs, DKS's revenue ranking is reversed: with call option, the first-price auction generates a lower revenue than the second-price auction. It is also shown that the occurrence of the adverse selection is closely related to how fast the investment cost increases relative to the return from an auctioned object. Thus, if the investment cost is increasing relatively faster than the value, the first-price auction with call option yields lower expected revenue than the second-price auction with any security.

The results of this paper and CK (2010) suggest that any seller who considers using securities as a means of payment in auctions, should pay a close attention to the underlying technology or cost structure faced by bidders. Note that this observation is also important for a seller who cares about the total surplus, since the adverse selection means that the object is sold to a bidder with a lower—in fact lowest!—value.

The next section introduces a model of auction with security following DKS. Section 3 provides the revenue ranking of securities with the auction format fixed for both second-price and first-price auctions. Section 4 compares the two auctions in terms of what auction is more vulnerable to the adverse selection and how it affects their revenue ranking.

2. MODEL

Suppose that the seller has an object to sell to n bidders via an auction. With the object at hand, each bidder i can run a project indexed by v_i that yields a stochastic future (gross) return denoted as Z_i. Suppose that v_i is bidder i's private information. Assuming that v_i and Z_i are i.i.d. across bidders, $F(\cdot)$ and $G(\cdot|v)$ denote the c.d.f of v and Z conditional on v with support $[\underline{v},\overline{v}]$ and $[\underline{Z},\overline{Z}]$, respectively. A project with higher index v generates a higher return, i.e. $\mathbb{E}[Z|v]$ is increasing with v. Running a project v requires some investment cost equal to

[1] Authors such as Byoung Heon Jun and Elmar Wolfstetter (2012) and John Morgan and Shimon Kogan (2010) show that a steeper security can cause adverse selection for a different reason—moral hazard—when bidders can choose how much to invest to increase the value of the object posterior to the auction.

$x(v)$, where $x(\underline{v}) \geq 0$ and $x'(\cdot) \geq 0$. We assume that $\mathbb{E}[Z|v] - x(v)$ is increasing in v. Then, without loss, we can redefine $v = \mathbb{E}[Z|v] - x(v)$ to be the *net* expected return from the project. Note that the efficient allocation then requires the bidder with the highest v to win the object.

Following DKS, let us consider an *ordered* set of feasible[2] securities $\mathscr{S} = \{S(s,Z) : s \in [0,1]\}$: for all v, $\frac{\partial \mathbb{E}[S(s,Z)|v]}{\partial s} > 0$. Some notable examples are

1. Debt: $S(d,Z) = \min\{d,Z\}$, where d is the face value promised to the seller (unless the return falls short of it);

2. Equity: $S(e,Z) = eZ$, where e is the fraction of return going to the seller;

3. Call option: $S(c,Z) = \max\{Z - c, 0\}$, where c is a strike price at which the seller can take over the project (and its return). Note that in contrast to debt or equity, a lower c corresponds to a higher security.

In auctions with a set of securities $\mathscr{S} = \{S(s,Z) : s \in [0,1]\}$, each bidder's strategy is to choose a security within $\mathscr{S}$, which corresponds to choosing s. Letting $s_{(r)}$ denote the r-th highest bid, a bidder who bids $s_{(1)}$ wins the object while paying to the seller $S(s_{(1)},Z)$ in the first-price auction and $S(s_{(2)},Z)$ in the second-price auction upon a return Z being realized. To simplify notations, write $\mathbb{E}[S(s,Z)|v]$ as $ES(s,v)$ and its derivatives as $ES_v(s,v)$ and $ES_s(s,v)$. Then, two ordered sets of securities, $\mathscr{S}^1$ and $\mathscr{S}^2$, can be compared in terms of steepness: $\mathscr{S}^1$ is *steeper* than $\mathscr{S}^2$ if for all $S^1 \in \mathscr{S}^1$ and $S^2 \in \mathscr{S}^2$, $ES^1_v(s^1,v) > ES^2_v(s^2,v)$ whenever $ES^1(s^1,v) = ES^2(s^2,v)$.

The current model differs from DKS' only in that the investment cost $x(v)$ increases with the project type v. This assumption is reasonable: for example, a firm that can generate a greater value from a under-performing target tends to have a higher opportunity cost (e.g., of pursuing other target) or stand-alone values. In sponsored search auctions an advertiser whose website attracts more customers (i.e. higher v) will have to incur a higher cost of maintaining the website. The assumption is equally compelling in other contexts, such as the government sale of oil leases.

[2]A security is feasible if (i) the payment is nonnegative and does not exceed the realized return, i.e. $S(s,Z) \in [0,Z]$; (ii) the shares accruing to the payee and payer are both nondecreasing in the realized return, i.e. both $S(s,Z)$ and $Z - S(s,Z)$ are nondecreasing in Z. As DKS observe, all standard financial claims satisfy this monotonicity condition. Though the cash payment is not feasible since it violates (i) above, all subsequent results remain valid with the cash payment, considering that it is flatter than any feasible security.

3. RANKING SECURITY DESIGNS

3.1. SECOND-PRICE AUCTIONS

For the analysis of the second-price auctions, we borrow heavily from Che and Kim (2010) and summarize the results therein. In the second-price auction with an ordered set $\mathscr{S}$ of securities, there is a weakly dominant strategy in which each type v submits a bid $s(v) \in [0,1]$ such that

$$v - ES(s(v), v) = 0. \quad (1)$$

Adverse selection arises when bidders employ *decreasing* bidding strategies. The following result shows that a steeper security is more vulnerable to the adverse selection than a flatter design:[3]

Proposition 1. *Let $s^1(\cdot)$ and $s^2(\cdot)$ denote the equilibrium strategies under securities $\mathscr{S}^1$ and $\mathscr{S}^2$, respectively. Suppose that $\mathscr{S}^1$ is steeper than $\mathscr{S}^2$. Then, if $s^2(\cdot)$ is decreasing, $s^1(\cdot)$ is decreasing also. Also, if $s^1(\cdot)$ is increasing, $s^2(\cdot)$ too is increasing.*

The revenue consequence of adverse selection is provided in the following result.

Proposition 2. *Suppose that $\mathscr{S}^1$ is steeper than $\mathscr{S}^2$. Letting $s^1(\cdot)$ denote the equilibrium bidding strategy under $\mathscr{S}^1$, if $s^1(\cdot)$ is decreasing, then the seller's revenue is lower with $\mathscr{S}^2$ than with $\mathscr{S}^1$.*

The next proposition states that any security design as steep as, or steeper than, standard equity will induce a decreasing equilibrium, if the investment cost $x(v)$ increases at a rate faster than the net return v.

Proposition 3. *the second-price equity auction induces a decreasing (resp. increasing) equilibrium bidding strategy if $x(v)/v$ is increasing (resp. decreasing) in v.*

As noted by DKS, a call option is the steepest, and standard debt is the flattest, among all feasible securities. The cash payment is even flatter than a standard debt. Combining this with Propositions 3 and 2 yields the following result:

Corollary 1. *Suppose $x(v)/v$ is increasing in v. Then, for the second-price auction,*

[3] Refer to Che and Kim (2010) for the proofs of all the results in this section.

(i) cash or debt yields higher expected revenue than equity or any securities steeper than standard equity;

(ii) a call option yields the lowest expected revenue among all feasible securities.

This result is in contrast with DKS that show the call option achieves the highest revenue among all feasible securities. The difference is that in DKS, there arises no adverse selection since all project types are assumed to incur the same cost.

3.2. FIRST-PRICE AUCTIONS

We now establish analogous results about security designs under a first-price auction. To this end, we assume that v is symmetrically distributed across bidders, following a distribution $F : [\underline{v}, \overline{v}] \to [0,1]$ with a density $f(v) > 0$ for all $v \in (\underline{v}, \overline{v})$. From now, we focus on a symmetric equilibrium bidding strategy that is differentiable. Let s_F^i denote the equilibrium bidding strategy for the first-price auction with security $\mathscr{S}^i$.

The first result shows that like the second-price auction, the first-price auction with a steeper security is more vulnerable than that with a flatter security to the adverse selection. (All proofs are contained in Appendix unless provided in the text.)

Proposition 4. *Suppose that $\mathscr{S}^1$ is steeper than $\mathscr{S}^2$. If s_F^2 is decreasing, then s_F^1 must also be decreasing.*

Using Proposition 4, one can establish that when adverse selection arises, a steeper security yields a lower revenue than a flatter one.

Proposition 5. *Suppose that $\mathscr{S}^1$ is steeper than $\mathscr{S}^2$. If both s_F^1 and s_F^2 are decreasing, then the seller's expected revenue is lower in the first-price auction with $\mathscr{S}^1$ than with $\mathscr{S}^2$.*

This result is in contrast with DKS's revenue ranking that shows a steeper security is superior if there is no adverse selection. Notice some similarity between Proposition 5 and Proposition 2. In the latter, however, the revenue comparison is available whenever a steeper security causes adverse selection. In the above result, we require that adverse selection arises with both securities. This is because the revenue comparison in the first-price auction is done indirectly by comparing bidders' equilibrium utilities, which only results in the desired revenue ranking if equilibrium bidding strategies under the two securities exhibit the same monotonicity. Though there is no general answer for how revenue compares in case adverse selection arises with a steeper security but not with a flatter one, a numerical example, Example 1, is later provided where debt, which is the flattest security, does not cause adverse selection while call option or equity does, and debt raises a higher revenue than the other two. I suspect that this ranking holds in general, which is beyond the scope of the current analysis, however.

In case of equity, whether the adverse selection arises is closely related to how monotonic $x(v)/v$ is, as was the case with the second-price auction.

Proposition 6. *Let $e_F : [\underline{v}, \overline{v}]$ denote a unique symmetric equilibrium strategy for the first-price auction with equity. (i) If $x(v)/v$ is increasing, then*

$$e_F(v) = \frac{\int_v^{\overline{v}} (n-1)(1-F(t))^{n-2} f(t) \, [1-(x(t)/t)] \, dt}{(1-F(v))^{n-1}}, \tag{2}$$

which is decreasing. (ii) If $x(v)/v$ is decreasing, then

$$e_F(v) = \frac{\int_{\underline{v}}^{v} (n-1) F(t)^{n-2} f(t) [1-(x(t)/t)] dt}{F(v)^{n-1}}, \tag{3}$$

which is increasing.

This equilibrium characterization for the first-price auction with equity under general value distribution is a novel part of the analysis in this paper. It is used for numerical analysis later and may be useful for some future theoretical or empirical analysis.

Combining Proposition 5 and 6 with the fact that call option is steeper than equity, provides a sufficient condition for the DKS's revenue ranking between equity and call option to be reversed due to adverse selection:

Corollary 2. *Suppose $x(v)/v$ is increasing in v. Then, the seller's expected revenue is higher in the first-price auction with equity than with call option.*

4. RANKING AUCTION FORMATS

In this section, we investigate how adverse selection affects the revenue ranking of standard auction formats. To this end, we first establish a sense in which the first-price auction is more prone to adverse selection than the second-price auction:

Proposition 7. *Fix a security and suppose that the equilibrium bidding strategy of the second-price auction is decreasing. Then, any (differentiable) equilibrium bidding strategy of the first-price auction must also be decreasing.*

This difference in the two standard auction formats has consequences on the seller's expected revenue. We show that, much in contrast to DKS's finding, whenever adverse selection plagues both formats with a given security, the second-price auction yields a higher expected revenue than the first-price auction.

To this end, following DKS, we call an ordered set, $\mathscr{S}$, of securities *super-convex* if every $S \in \mathscr{S}$ is steeper than any security obtained from a (nontrivial) convex combination of securities in $\mathscr{S}$, and $\mathscr{S}$ is *convex* if it is equal to its convex hull.

Proposition 8. *Suppose that $\mathscr{S}$ is super-convex (resp. convex) and the second-price auction induces a decreasing equilibrium bidding strategy. Then, any (differentiable) equilibrium bidding strategy of the first-price auction generates a lower (resp. the same) expected revenue than the second-price auction.*

As noted in DKS, call option is super-convex, and equity is convex. Combining the results established thus far yields the following implications.

Corollary 3. *Suppose $x(v)/v$ is increasing in v.*

(i) the second-price auction with debt yields a higher expected revenue than does any standard (i.e. first- or second-price) auction with equity or any securities steeper than equity.

(ii) the first-price auction with call option yields lower expected revenue than does the second-price auction with any securities.

Proof. Given the condition, the second-price equity auction induces a decreasing equilibrium (Proposition 3). Since equity is convex, by Proposition 8, the first- and second-price auctions are revenue equivalent under equity. Part (i) then follows from Propositions 2 and 8.

We next prove part (ii). Note first that Propositions 1 and 3 imply that the second-price auction in call option induces a decreasing equilibrium. Then, the same holds for the first-price auction, by Proposition 7. Then, since call option is super-convex (as noted by DKS), Proposition 8 implies that the first-price auction with call option yields lower revenue than the second-price auction with call option, which, by Corollary 2, in turn yields lower revenue than the second-price auction with any securities. □

The following numerical examples illustrate in what magnitude the adverse selection problem affects the revenue/efficiency performance of the standard auction formats as well as different securities.

Example 1. Suppose that there are two bidders, the distribution of v_i is uniform on $[0,1]$, and the distribution of Z_i given v_i is uniform on $[0,2v_i]$. The cost of project with value v is given by $x(v) = \frac{v^2}{2}$. We consider three securities: debt, equity, and call-option. Let $d_A(v)$, $e_A(v)$, and $c_A(v)$ denote the equilibrium bidding strategy for debt, equity share, and strike price, respectively, under the second-price auction format $A = S$ and first-price auction $A = F$.

For the second-price auction, it is straightforward to check

$$\begin{aligned} d_S(v) &= 2v + \sqrt{2}v^{\frac{2}{3}} \\ e_S(v) &= 1 - \frac{v}{2} \\ c_S(v) &= 2v - \sqrt{4v^2 - 2v^3}. \end{aligned}$$

Note that the debt auction efficiently allocates the object since $d_S(\cdot)$ is increasing. In contrast, the adverse selection arises in the second-price auction with debt or call option since $e_S(\cdot)$ is decreasing and $c_S(\cdot)$ is increasing.

To analyze the first-price auction, we let $\pi(s(v'),v)$ denote the payoff for each type v bidding a security $s(v')$, i.e. mimicking type v'. To start with the case of equity, since $x(v)/v = v/2$ is increasing, we can apply (i) of Proposition 6 with $F(v) = v$, which results in a decreasing equilibrium bidding strategy, $e_F(v) = 3/4 - v/4$, causing the adverse selection.

Since call option is steeper than equity, Proposition 4 implies that the first-price auction with call option is also subject to the adverse selection, meaning that any differentiable equilibrium bidding strategy $c_F(\cdot)$ has to be increasing. Then, bidder i with type $v_i = v$ must maximize his payoff by choosing

$$\begin{aligned} v &= \arg\max_{v' \in [0,1]} \pi(c_F(v'),v) \\ &= \big(v - \mathbb{E}[\max\{0, Z_i - c_F(v')\}|v]\big)\,\mathrm{Prob}[c_F(v_j) \le c_F(v'), j \ne i] \\ &= \Big(\mathbb{E}[Z_i - \max\{0, Z_i - c_F(v')\}|v_i = v] - x(v)\Big) \times \\ &\qquad \mathrm{Prob}[c_F(v_j) \le c_F(v'), j \ne i] \\ &= \Big(v - \frac{(2v - c_F(v'))^2}{4v} - v^2/2\Big)(1 - v'). \end{aligned} \tag{4}$$

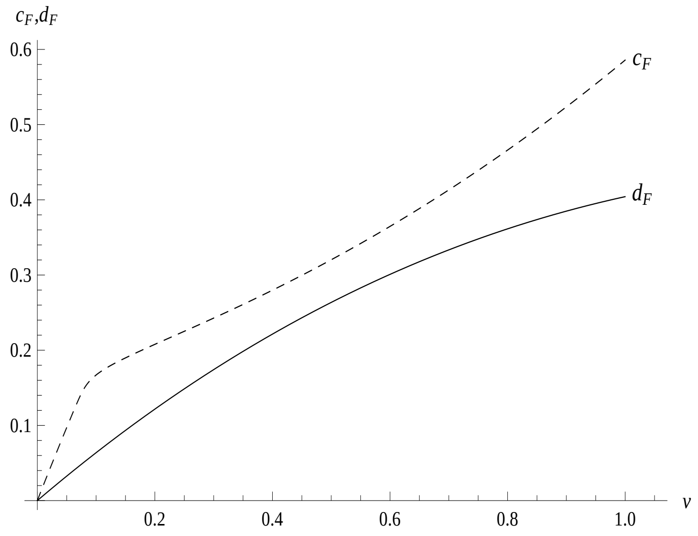

Figure 1: Equilibrium bidding strategy for the first-price auction with call option and debt

The first-order condition for this problem gives us a differential equation, which cannot be solved analytically, however. So we rely on a numerical method to approximate the solution of (4), $c_F(\cdot)$, with the dashed curve in the Figure 1.

For the debt auction, it is unclear whether the equilibrium bidding strategy is increasing or decreasing or even non-monotonic. To figure this out, write the objective function for bidder i with type $v_i = v$ as

$$\begin{aligned}\pi(d_F(v'),v) &= \Big(\mathbb{E}[Z-\min\{Z,d_F(v')\}|v]-x(v)\Big)\mathrm{Prob}[d_F(v_j)\leq d_F(v'),j\neq i]\\ &= \Big(\frac{(2v-d_F(v'))^2}{4v}-v^2/2\Big)\mathrm{Prob}[d_F(v_j)\leq d_F(v'),j\neq i]. \qquad (5)\end{aligned}$$

Note that maximizing $\pi(d_F(v'),v)$ is equivalent to maximizing its logarithm, $\log\pi(d_F(v'),v)$. One can verify that $\log\pi(d_F(v'),v)$ is supermodular with respect to v and d_F, so by Topkis (1978), the maximizer d_F has to be weakly increasing in v. Further, d_F must be strictly increasing since if it were constant on some interval, one of the types in the interval could slightly overbid and increase its payoff by a discrete amount. Thus, bidder i with type $v_i = v$ maximizes his payoff by choosing

$$v = \arg \max_{v' \in [0,1]} \Big(\frac{(2v - d_F(v'))^2}{4v} - v^2/2 \Big) v',$$

which gives us a differential equation that can only be solved numerically. The numerical solution of the differential equation is depicted by the solid curve in the Figure 1.

The following table reports the measures of surplus and revenue resulting from the equilibrium strategy described so far:

	First-Price Auction		Second-Price Auction	
	Surplus	Revenue	Surplus	Revenue
Debt	5/12	0.273	5/12	0.250
Equity	1/4	0.208	1/4	0.208
Call-option	1/4	0.136	1/4	0.146

Note that the surplus here is the expected *net* surplus that accounts for the cost of chosen project as well as its value. In both auction formats, the debt security is free from the adverse selection problem, which leads to a significantly higher revenue and surplus compared to the other securities. To see the effect of auction format on revenue, observe that with debt, the revenue of the first-price auction is about 9 % higher than that of the second-price auction while with call option, the latter is about 7 % higher than the former. Overall, however, the effect of security is more pronounced than that of auction format. In both auction formats, the revenue is about twice as large with debt as with call option. □

In the above example, both auction formats generate either *perfect* or *no* adverse selection irrespective of security, thereby exhibiting the same efficiency performance. This need not always be the case as the next example shows.

Example 2. Suppose that there are two bidders, the distribution of V_i is uniform on $[0,1]$ as in Example 1, and the distribution of Z_i given V_i is uniform on $[0, 2V_i + 1]$. The cost of project with value v is given by $x(v) = 0.1v + 0.35$. The same numerical approach as in Example 1 gives us the equilibrium bidding strategy for the two auction formats with call option, which is depicted in Figure 2.

In the first-price auction, adverse selection occurs to its full extent while it does not in the second-price auction, The resulting level of expected surplus is 0.45 for the first-price auction and 0.463 for the second-price auction, about 3 % higher than the first-price auction. □

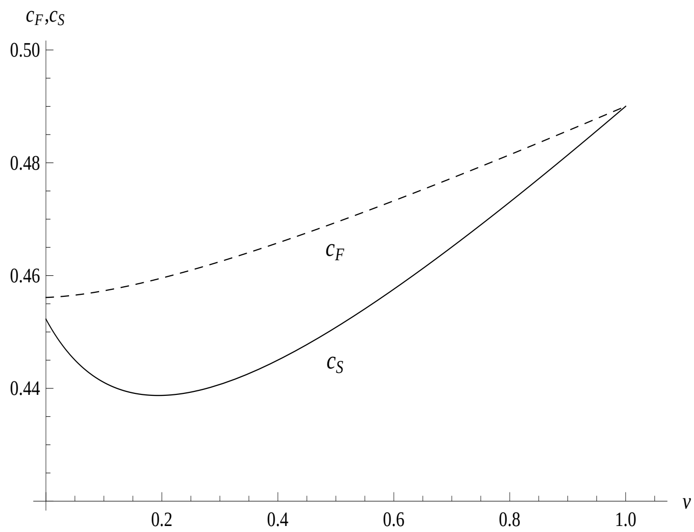

Figure 2: Equilibrium bidding strategy for first-price and second-price auction with call option

APPENDIX

Proof of Proposition 4. Let $s_F^i(\cdot)$ denote the equilibrium strategy of the first-price auction with $\mathscr{S}^i$ and $s^i(\cdot)$ that of the second-price auction so that $s^i(\cdot)$ is a solution of (1) with $\mathscr{S}^i$. As assumed, $s_F^2(\cdot)$ is decreasing. The standard argument can be used to show that there is no atom in the support of $s_F^i(\cdot)$, $i = 1, 2$.

Step 1. *$\overline{v}$ is the unique minimizer of $s_F^1(\cdot)$.*

Proof. Suppose for a contradiction that $s_F^1(\cdot)$ is minimized at some $v' < \overline{v}$ so that v' has to obtain zero payoff at equilibrium under $\mathscr{S}^1$ since there is no atom. Since $s_F^2(\cdot)$ is decreasing and thus $\overline{v}$ obtains zero equilibrium payoff with $\mathscr{S}^2$, it must be that $ES^2(s_F^2(\overline{v}), \overline{v}) = \overline{v} = ES^1(s^1(\overline{v}), \overline{v})$. (Recall $s^i(\cdot)$ denotes the equilibrium strategy of the second-price auction with security $\mathscr{S}^i$, satisfying (1).) Given this, $\mathscr{S}^1$ being steeper than $\mathscr{S}^2$ implies that

$$ES^2(s_F^2(\overline{v}), v) > ES^1(s^1(\overline{v}), v) \text{ for all } v < \overline{v}. \tag{6}$$

So, we have

$$\begin{aligned} v' - ES^1(s_F^1(\overline{v}), v') &\geq v' - ES^1(s^1(\overline{v}), v') \\ &> v' - ES^2(s_F^2(\overline{v}), v') \\ &> v' - ES^2(s_F^2(v'), v') \\ &\geq 0 = v' - ES^1(s_F^1(v'), v'), \end{aligned}$$

where the first inequality follows from $s_F^1(\overline{v}) \leq s^1(\overline{v})$, the second from (6), the third from $s_F^2(\cdot)$ decreasing, the fourth from $s_F^2(v')$ being the equilibrium bid for v' with $\mathscr{S}^2$, and the last equality from v' earning zero equilibrium payoff with $\mathscr{S}^1$. The above inequality results in $ES^1(s_F^1(\overline{v}), v') < ES^1(s_F^1(v'), v')$ or $s_F^1(\overline{v}) < s_F^1(v')$, contradicting that $s_F^1(\cdot)$ is minimized at v'. $\|$

To simplify notation, $s_F^1(\cdot)$ is denoted as $s_F(\cdot)$ from now on. Then, Step 1 implies $s_F(\overline{v}) < s_F(\underline{v})$, so the proof will be complete if it can be shown that $s_F'(v) \neq 0$ for all $v \in (\underline{v}, \overline{v})$, which is established in the next two steps.

Step 2. *Let* $V_0 = \{v \in (\underline{v}, \overline{v}) : s_F'(v) = 0$ *and* v *is a local minimum*$\}$. *Then,* $V_0 = \emptyset$.

Proof. Suppose that V_0 is not empty. One can then find $v_0 \in V_0$ such that $s_F(v_0) \leq s_F(v')$ for all $v' \in V_0$. One can also find some $v_1 \in (v_0, \overline{v})$ such that $s_F(v_1) = s_F(v_0)$ and $s_F'(v_1) < 0$.[4] Consider a downward deviation by v_1 to slightly lower s and mimic $v_1 + \varepsilon$ with small $\varepsilon > 0$. It is straightforward that as $\varepsilon \to 0$, the marginal cost from decrease in the winning probability is

$$d_- := (n-1) f(v_1) F^{n-2}(v_1)[v_1 - ES(s_F(v_1), v_1)]$$

while the marginal benefit from decrease in the expected payment is

$$d_+ := -(1 - F(v_1))^{n-1} s_F'(v_1) ES_s(s_F(v_1), v_1).$$

For this deviation to be unprofitable, we must have $d_- \geq d_+$. Consider now an upward deviation by v_1 to slightly raise s and mimic $v_1 - \varepsilon$ for small $\varepsilon > 0$. We will then be able to find some $\varepsilon_1(\varepsilon), \varepsilon_2(\varepsilon) > 0$ such that $s_F(v_0 - \varepsilon_1(\varepsilon)) = s_F(v_0 + \varepsilon_2(\varepsilon)) = s_F(v_1 - \varepsilon)$. With this deviation, the winning probability is at least

$$w(\varepsilon) := \sum_{k=0}^{n-1} \binom{n-1}{k} (F(v_0 + \varepsilon_2(\varepsilon)) - F(v_0 - \varepsilon_1(\varepsilon)))^k (1 - F(v_1 - \varepsilon))^{n-1-k}.$$

[4]This is possible since $\overline{v}$ is a unique minimizer of $s_F(\cdot)$

Thus,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{w(\varepsilon)-w(0)}{\varepsilon}\\
\geq &\lim_{\varepsilon\to 0}\frac{(1-F(v_1-\varepsilon))^{n-1}-(1-F(v_1))^{n-1}}{\varepsilon}\\
&+\lim_{\varepsilon\to 0}\frac{(n-1)(F(v_0+\varepsilon_2(\varepsilon))-F(v_0-\varepsilon_1(\varepsilon)))(1-F(v_1-\varepsilon))^{n-2}}{\varepsilon}\\
> &(n-1)f(v_1)F^{n-2}(v_1).
\end{aligned} \tag{7}$$

Here, the strict inequality holds true since

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{(n-1)(F(v_0+\varepsilon_2(\varepsilon))-F(v_0-\varepsilon_1(\varepsilon)))(1-F(v_1-\varepsilon))^{n-2}}{\varepsilon}\\
= &(n-1)f(v_0)(\varepsilon_1'(0)+\varepsilon_2'(0))(1-F(v_1))^{n-2}>0,
\end{aligned}$$

which is in turn due to the fact that $\varepsilon_1'(0)$ and $\varepsilon_2'(0)$ are positive since v_0 is a local minimizer of $s_F(\cdot)$.[5] By (7), the marginal increase in the winning probability from the upward deviation is greater than $(n-1)f(v_1)F^{n-2}(v_1)$, which means that the associated marginal benefit is greater than d_-. Clearly, the marginal cost from increase in the expected payment is equal to d_+. Considering $d_- \geq d_+$, however, this implies that the upward deviation is profitable. ‖

Given Step 2, that $s_F'(v)\neq 0$ for all $v\in(\underline{v},\overline{v})$ can be established if we rule out the case in which $s_F(\cdot)$ is hump-shaped, which leads to Step 3.

Step 3. *There is no $v\in(\underline{v},\overline{v})$ at which $s_F(\cdot)$ achieves a (global) maximum.*

Proof. Suppose to the contrary that there is a global maximizer $v_m\in(\underline{v},\overline{v})$. One must be then able to find some $v_1\in(v_m,\overline{v})$ such that $s_F(v_1)=s_F(\underline{v})$ and $s_F'(v_1)<0$. Considering a downward deviation by v_1, one can obtain the expressions for associated marginal benefit and cost that are the same as d_+ and d_- defined in Step 2. Also, it must be that $d_-\geq d_+$. Then, a similar argument to the one that resulted in (7) above, can be used to show that an upward deviation will lead to the marginal increase in winning probability, which is greater than $(n-1)f(v_1)F^{n-2}(v_1)$. So, the marginal benefit from the upward deviation is greater than d_- while the marginal cost is equal to d_+, which implies that the upward deviation is profitable. ‖ □

[5] More precisely, $\varepsilon_1(\varepsilon)=v_0-s_F^{-1}(s_F(v_1-\varepsilon))$ and thus $\varepsilon_1'(0)=\frac{-s_F'(v_1)}{s_F'(v_0)}=\infty$, and similarly for $\varepsilon_2'(0)$.

Proof of Proposition 5. The proof below is analogous to the proof of Proposition 1 in DKS. Let $s^i_F(v)$ and $U^i(v)$ denote the type v's equilibrium bid and equilibrium payoff, respectively, in the first-price auction with security $\mathscr{S}^i$.

Claim 1. *If $U^2(v) < U^1(v)$ for some $v < \overline{v}$, then $U^2(v') < U^1(v')$ for all $v' < v$.*

Proof. Suppose not. Then, there must be some $\hat{v} < \overline{v}$ such that $U^1(\hat{v}) = U^2(\hat{v})$ and $\frac{dU^1(\hat{v})}{dv} \geq \frac{dU^2(\hat{v})}{dv}$. Since $U^i(v) = [v - ES^i(s^i_F(v), v)](1-F(v))^n$, the fact that $U^1(\hat{v}) = U^2(\hat{v})$ implies that $ES^1(s^1_F(\hat{v}), \hat{v}) = ES^2(s^2_F(\hat{v}), \hat{v})$. Also, by the envelope theorem,

$$\begin{aligned}\frac{dU^1(\hat{v})}{dv} &= (1-F(\hat{v}))^n[1 - \frac{\partial ES^1(s^1_F(\hat{v}), \hat{v})}{\partial v}] \\ &\geq (1-F(\hat{v}))^n[1 - \frac{\partial ES^2(s^2_F(\hat{v}), \hat{v})}{\partial v}] = \frac{dU^2(\hat{v})}{dv}\end{aligned}$$

or $\frac{\partial ES^1(s^1_F(\hat{v}), \hat{v})}{\partial v} \leq \frac{\partial ES^2(s^2_F(\hat{v}), \hat{v})}{\partial v}$, which is a contradiction since $\mathscr{S}^1$ is steeper than $\mathscr{S}^2$. ||

Now consider the equilibrium of the first-price auction under $\mathscr{S}^2$ with a modified distribution which is the same as $F(\cdot)$ except that the support is truncated at $\overline{v} - \varepsilon$ for small ε and there is a mass equal to $1 - F(\overline{v} - \varepsilon)$ at $\overline{v} - \varepsilon$. Letting $U^2_\varepsilon(\cdot)$ denote the payoff for this equilibrium, we have $U^2_\varepsilon(\overline{v} - \varepsilon) = 0 < U^1(\overline{v} - \varepsilon)$. Note that the above claim still holds between $U^2_\varepsilon(\cdot)$ and $U^1(\cdot)$ with $\overline{v}$ being replaced by $\overline{v} - \varepsilon$. Thus, we have $U^2_\varepsilon(v) < U^1(v)$ for all $v \leq \overline{v} - \varepsilon$. By making ε converge to zero, we conclude that the bidders' payoffs are higher with $\mathscr{S}^1$. This implies that the seller's revenue is higher with $\mathscr{S}^2$, since the total surplus generated under the two securities is identical while bidders get a higher surplus with $\mathscr{S}^1$. □

Proof of Proposition 6. We provide a proof for (i) since (ii) can be proved analogously. Given the equilibrium strategy $e_F(\cdot)$, letting $\pi(e_F(v'), v)$ denote the payoff for type v bidding $e_F(v')$ (i.e. mimicking type v'), bidder i with type $v_i = v$ must maximizes his payoff by choosing

$$\begin{aligned} v &= \arg\max_{v'} \ \pi(e_F(v'), v) \\ &= \Big(v - \mathbb{E}[e_F(v')Z|v]\Big) \\ &= \Big(\mathbb{E}[(1-e_F(v'))Z|v] - x(v)\Big)\text{Prob}[e_F(v_j) \leq e_F(v'), \forall j \neq i] \\ &= \Big((1-e_F(v'))v - x(v)\Big)\text{Prob}[e_F(v_j) \leq e_F(v'), \forall j \neq i].\end{aligned}$$

Note that maximizing $\pi(e_F(v'), v)$ is equivalent to maximizing its logarithm,

$$\begin{aligned}\Pi(e_F(v'), v) &:= \log \pi(e_F(v'), v) \\ &= \log[(1 - e_F(v'))v - x(v)] + \log \operatorname{Prob}[e_F(v_j) \leq e_F(v'), \forall j \neq i].\end{aligned}$$

According to Topkis (1978), the maximizer $e_F(v)$ must be non-increasing in v if $\Pi(e_F(v'), v)$ is submodular in v and $e_F(v')$, which can be shown as follows:

$$\begin{aligned}\frac{\partial^2}{\partial v \partial e_F(v')} \Pi(e_F(v'), v) &= \frac{\partial^2}{\partial v \partial e_F} \log[(1 - e_F(v'))v - x(v)] \\ &= \frac{\partial}{\partial v} \frac{-v}{(1 - e_F(v'))v - x(v)} \\ &= \frac{x(v) - vx'(v)}{[(1 - e_F(v'))v - x(v)]^2} < 0,\end{aligned}$$

since $x(v) - vx'(v)$ is negative if $x(v)/v$ is increasing. Further, $e_F(\cdot)$ cannot be constant over any interval of types since then those types would be tied so one of them can slightly overbid and increase its payoff by a discrete amount. In sum, any symmetric equilibrium must be (strictly) decreasing, which implies that the maximization problem in (5) turns into

$$\begin{aligned}v = \max_{v' \in [\underline{v}, \overline{v}]} &\Big((1 - e_F(v'))v - x(v)\Big) \operatorname{Prob}[v_j > v', \forall j \neq i] \\ &= \Big((1 - e_F(v'))v - x(v)\Big)(1 - F(v'))^{n-1}]. \qquad (8)\end{aligned}$$

The first-order condition of this problem gives us the following differential equation:

$$-e_F'(v)v(1 - F(v))^{n-1} + [(1 - e_F(v))v - x(v)] \frac{d}{dv}(1 - F(v))^{n-1} = 0$$

with boundary condition $(1 - e_F(\overline{v}))\overline{v} - x(\overline{v}) = 0$ since type $\overline{v}$ must obtain zero payoff. This equation can be rearranged to

$$\frac{d}{dv}\left[(1 - F(v))^{n-1} e_F(v)\right] = -(n-1)(1 - F(v))^{n-2} f(v)[1 - (x(v)/v)]$$

or

$$(1 - F(v))^{n-1} e_F(v) = K + \int_v^{\overline{v}} (n-1)(1 - F(t))^{n-2} f(t)\,[1 - (x(t)/t)]\,dt \qquad (9)$$

for some constant K. Here we must have $K = 0$ for (9) to hold at $v = \overline{v}$, which results in the expression in (2). The fact that $e_F(\cdot)$ in (2) is decreasing can be shown as follows:

$$\begin{aligned}\frac{e'_F(v)}{(n-1)(1-F(v))^{-n}f(v)} &= \int_v^{\overline{v}} (n-1)(1-F(t))^{n-2}f(t)\left[1-\frac{x(t)}{t}\right]dt \\ &\quad -(1-F(v))^{n-1}\left[1-\frac{x(v)}{v}\right] \\ &< \left[1-\frac{x(v)}{v}\right]\int_v^{\overline{v}}(n-1)(1-F(t))^{n-2}f(t)dt \\ &\quad -(1-F(v))^{n-1}\left[1-\frac{x(v)}{v}\right] \\ &= 0,\end{aligned}$$

where the inequality holds since $x(v)/v$ is increasing.

Lastly, we check the "second-order condition" by verifying that the function $e_F(\cdot)$ satisfying the first-order condition is indeed a global maximizer. To see this, differentiate $\Pi(e_F(v'), v)$ with v' to obtain

$$\frac{\partial \Pi(e_F(v'),v)}{\partial v'} = e'_F(v')\frac{\partial \Pi(e_F(v'),v)}{\partial e_F(v')} \gtreqless 0 \text{ if } v' \lesseqgtr v, \tag{10}$$

where the inequalities hold since $e'_F(v') < 0$ and $\Pi(e_F(v'), v)$ is submodular. Note that the inequalities in (10) mean that $\Pi(e_F(v'), v)$ is maximized at $e_F(v') = e_F(v)$ as desired. □

Proof of Proposition 7. Let $s(\cdot)$ denote the solution of (1), that is equilibrium bidding strategy for the second-price auction. Clearly, $s_F(v) \le s(v)$ for all v.

As in Step 1 of Proposition 4, we first prove that the highest type $\overline{v}$ must be the unique minimizer of $s_F(\cdot)$. Suppose to the contrary that there is some $v' < \overline{v}$ at which $s_F(\cdot)$ is minimized and that the interim equilibrium payoff is zero or $v' = ES(s_F(v'), v')$ so $s_F(v') = s(v')$. However, since $v - ES(s_F(v'), v)$ is (strictly) increasing in v, we have for some $v < v'$,

$$v - ES(s_F(v'), v) < v' - ES(s_F(v'), v') = 0,$$

which leads to a contradiction that

$$v - ES(s(v), v) \le v - ES(s_F(v), v) \le v - ES(s_F(v'), v) < 0,$$

since $s(v) \ge s_F(v) \ge s_F(v')$. Thus, we conclude that $\overline{v}$ is the unique minimizer of $s_F(\cdot)$. The rest of proof then follows the same line of argument as that in Step 2 and Step 3 of the proof of Proposition 4 and is thus omitted. □

Proof of Proposition 8. Note that according to Proposition 7, any equilibrium bidding strategy of the first-price auction must be decreasing. Then, the proof follows the same line of argument as the proof of Proposition 5, for the super-convexity of $\mathscr{S}$ implies that

$$\frac{\partial ES(s_F(v),v)}{\partial v} > \frac{\partial \mathbb{E}[ES(s(v'),v)|v' > v]}{\partial v} \tag{11}$$

if $ES(s_F(v),v) = \mathbb{E}[ES(s(v'),v)|v' > v]$.

The proof of revenue equivalence between the first- and second-price auctions with convex securities follows from the observation that if $\mathscr{S}$ is convex, then the inequality in (11) becomes an equality, which makes all the inequalities in the proof of Proposition 5 into equalities. □

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